University of Toronto

Faculty of Applied Science and Engineering

Solutions to Final Examination, December 2015

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT186H1F - Calculus I

Examiners: G. Benyamini, D. Burbulla, S. Cohen, D. Fusca, L. Shorser, M. Matviichuk, A. Zaman

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

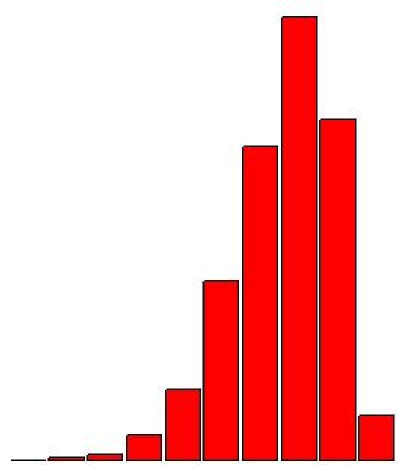
This exam consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

1. Questions 1 to 7 were well done! No surprise: many were very similar to previous exam questions.
2. Question 8 was the only question that posed difficulty. Many students explicitly wondered what this question covered. It consisted of three parts: finding a cubic function that satisfied four given conditions; a related rates problem, relating vertical motion to horizontal motion; and a max/min problem, to find the maximum vertical acceleration. Very few students got this question completely perfect; most students couldn't really get started.

Breakdown of Results: 877 students wrote this exam. The marks ranged from 13.75% to 98.75%, and the average was 70.2%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	3.2%
A	27.1%	80-89%	23.9%
B	31.0%	70-79%	31.0%
C	21.9%	60-69%	21.9%
D	12.6%	50-59%	12.6%
F	7.4%	40-49%	4.9%
		30-39%	1.8%
		20-29%	0.5%
		10-19%	0.2%
		0-9%	0.0 %



1. [avg: 8.08/10] Find the following:

(a) [5 marks] $\int_0^4 x\sqrt{16-x^2}\,dx$

Solution: let $u = 16 - x^2$. Then $du = -2x\,dx$ and

$$\begin{aligned}\int_0^4 x\sqrt{16-x^2}\,dx &= -\frac{1}{2}\int_{16}^0 \sqrt{u}\,du \\ &= \frac{1}{2}\int_0^{16} \sqrt{u}\,du \\ &= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^{16} \\ &= \frac{64}{3}\end{aligned}$$

(b) [5 marks] $F'(2)$, if $F(x) = \int_{-x}^{x} \sqrt{32+t^2}\,dt$

Solution: the integrand is even, so

$$F(x) = 2\int_0^x \sqrt{32+t^2}\,dt.$$

Use the Fundamental Theorem of Calculus, Part I:

$$F'(x) = 2\sqrt{32+x^2}.$$

In particular,

$$F'(2) = 2\sqrt{36} = 12.$$

Otherwise:

$$F(x) = \int_{-x}^0 \sqrt{32+t^2}\,dt + \int_0^x \sqrt{32+t^2}\,dt = -\int_0^{-x} \sqrt{32+t^2}\,dt + \int_0^x \sqrt{32+t^2}\,dt$$

$$\Rightarrow F'(x) = -\sqrt{32+(-x)^2}(-1) + \sqrt{32+x^2} = 2\sqrt{32+x^2};$$

so

$$F'(2) = 2\sqrt{36} = 12,$$

as before.

2. [avg: 9.47/10] Let $v = t^2 - 4t + 3$ be the velocity of a particle at time t, for $0 \le t \le 3$. Find:

(a) [4 marks] the average velocity of the particle.

Solution:

$$v_{\text{avg}} = \frac{1}{3}\int_0^3 v\,dt = \frac{1}{3}\int_0^3 (t^2 - 4t + 3)\,dt = \frac{1}{3}\left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^3 = \frac{1}{3}(9 - 18 + 9) = 0.$$

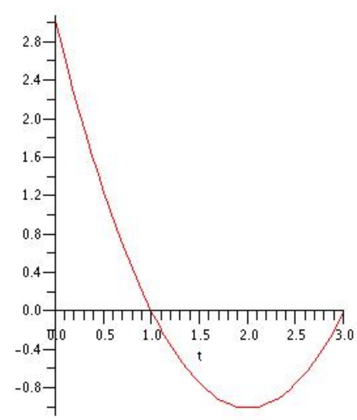
(b) [6 marks] the average speed of the particle.

Solution: speed is $|v|$. We have $v = (t-1)(t-3)$; so

$$v > 0 \text{ if } 0 < t < 1, \text{ but } v < 0 \text{ if } 1 < t < 3.$$

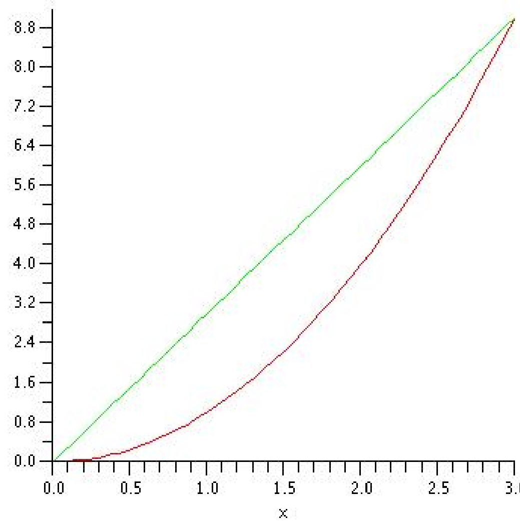
Thus

$$\begin{aligned}
\text{speed}_{\text{avg}} &= \frac{1}{3}\int_0^3 |v|\,dt \\
&= \frac{1}{3}\left(\int_0^1 v\,dt + \int_1^3 (-v)\,dt\right) \\
&= \frac{1}{3}\left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^1 - \frac{1}{3}\left[\frac{t^3}{3} - 2t^2 + 3t\right]_1^3 \\
&= \frac{1}{3}\left(\frac{4}{3}\right) - \frac{1}{3}\left(-\frac{4}{3}\right) \\
&= \frac{8}{9}
\end{aligned}$$



Continued...

3. [avg: 9.48/10] The region bounded by the curves $y = 3\,x$ and $y = x^2$ is revolved about the x-axis. Find the volume of the solid that is generated.



Solution: find the intersection points.

$$x^2 - 3x = 0 \Leftrightarrow x = 0 \text{ or } x = 3.$$

Then, using the method of washers, the volume is given by

$$V = \int_0^3 \pi\left((3x)^2 - (x^2)^2\right)\,dx.$$

So the volume is

$$V = \int_0^3 \pi\left(9x^2 - x^4\right)\,dx = \pi\left[3x^3 - \frac{x^5}{5}\right]_0^3 = \frac{162\,\pi}{5}.$$

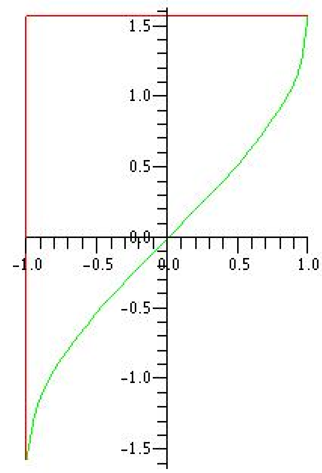
Alternate Solution: use the method of cylindrical shells and integrate with respect to y:

$$\begin{aligned} V &= \int_0^9 2\pi y\left(\sqrt{y} - \frac{y}{3}\right)\,dy \\ &= 2\pi\int_0^9\left(y^{3/2} - \frac{y^2}{3}\right)\,dy \\ &= 2\pi\left[\frac{2}{5}y^{5/2} - \frac{y^3}{9}\right]_0^9 \\ &= 2\pi\left(\frac{486}{5} - \frac{729}{9}\right) \\ &= \frac{162\,\pi}{5} \end{aligned}$$

Continued...

4. [avg: 7.17/10] Let A be the area of the region between $y = \pi/2$ and $y = \sin^{-1} x$ for $-1 \le x \le 1$.

(a) [6 marks] Express the value of A in terms of one or more integrals with respect to x **and** in terms of one or more integrals with respect to y. (Draw a diagram!)



Solution: in terms of an integral with respect to x,

$$A = \int_{-1}^{1} \left(\frac{\pi}{2} - \sin^{-1} x\right) dx.$$

With respect to y, use that $x = \sin y$. Then

$$A = \int_{-\pi/2}^{\pi/2} (\sin y - (-1))\ dy = \int_{-\pi/2}^{\pi/2} (1 + \sin y)\ dy.$$

(b) [4 marks] Find A.

Solution: of the two integrals above, only the second one can be evaluated with the methods of integration we have covered in Chapter 5. That is,

$$A = \int_{-\pi/2}^{\pi/2} (1 + \sin y)\ dy = 2\int_{0}^{\pi/2} 1\, dy + \int_{-\pi/2}^{\pi/2} \sin y\, dy = \pi + 0 = \pi,$$

where we have used symmetry to simplify the calculations. Or, more laboriously,

$$A = \int_{-\pi/2}^{\pi/2} (1 + \sin y)\ dy = [y - \cos y]_{-\pi/2}^{\pi/2} = \frac{\pi}{2} - \cos\frac{\pi}{2} - \left(-\frac{\pi}{2} - \cos\left(-\frac{\pi}{2}\right)\right) = \pi.$$

Alternately, if you know integration by parts, you could evaluate the first integral from part (a) as well. First you would have to use parts to establish that

$$\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C :$$

then

$$A = \int_{-1}^{1} \left(\frac{\pi}{2} - \sin^{-1} x\right) dx = \left[\frac{\pi}{2}x - x\sin^{-1} x - \sqrt{1 - x^2}\right]_{-1}^{1} = \frac{\pi}{2} - \frac{\pi}{2} - 0 + \frac{\pi}{2} + \frac{\pi}{2} + 0 = \pi.$$

Continued...

5. [avg: 5.43/10] Let $r > 0$. Let V be the volume of the solid of revolution generated by revolving the region in the xy-plane bounded by $y = \dfrac{1}{1+x^2}$, $y = 0$, $x = r$ and $x = r + 1$, about the y-axis.

(a) [4 marks] Write down the definite integral that gives the value of V.

Solution: use the method of shells, and integrate with respect to x:

$$V = \int_r^{r+1} \frac{2\pi x}{1+x^2}\,dx$$

(b) [6 marks] Which value of r will maximize the value of V?

Solution: to find the derivative of V, its easiest to use the Fundamental Theorem of Calculus, Part I:

$$\frac{dV}{dr} = \frac{2\pi(r+1)}{1+(r+1)^2} - \frac{2\pi r}{1+r^2}.$$

OR, the long way: evaluate V first, and then differentiate:

$$V = \pi\left[\ln(1+x^2)\right]_r^{r+1} = \pi\left(\ln(1+(r+1)^2) - \ln(1+r^2)\right);$$

from which

$$\frac{dV}{dr} = \pi\left(\frac{2(r+1)}{1+(r+1)^2} - \frac{2r}{1+r^2}\right),$$

as before. Next: find the critical point:

$$\begin{aligned}
\frac{dV}{dr} = 0 \quad &\Rightarrow \quad \frac{r+1}{1+(r+1)^2} = \frac{r}{1+r^2} \\
&\Rightarrow \quad (r+1)(1+r^2) = r(1+1+2r+r^2) \\
&\Rightarrow \quad r^3 + r^2 + r + 1 = r^3 + 2r^2 + 2r \\
&\Rightarrow \quad r^2 + r - 1 = 0 \\
&\Rightarrow \quad r = \frac{-1 \pm \sqrt{5}}{2}
\end{aligned}$$

Since $r > 0$, we must take

$$r = \frac{-1+\sqrt{5}}{2} \approx 0.618.$$

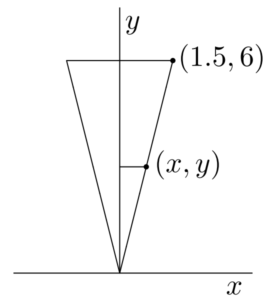
By the first derivative test, V has a maximum value at the critical point, since

$$V'(1/2) = \frac{8\pi}{65} > 0 \text{ and } V'(1) = -\frac{\pi}{5} < 0.$$

Continued...

6. [avg: 7.7/10] A water tank is shaped like an inverted cone with height 6 m and radius 1.5 m at the top. If the tank is full, how much work is required to pump the water to the level of the top of the tank, and out of the tank? (Assume the density of water is $\rho = 1000$ kg/m^3 and that $g = 9.8$ m/sec^2.)

Solution: in the diagram to the right, a side view of the cone is shown.



By similar triangles,

$$\frac{x}{y} = \frac{1.5}{6} \Leftrightarrow x = \frac{1}{4}y.$$

Therefore

$$A(y) = \pi x^2 = \frac{\pi}{16}y^2$$

and

$$\begin{aligned} W &= \int_0^6 \rho g \, \frac{\pi}{16} y^2 \, (6-y)\, dy \\ &= \frac{\pi \rho g}{16} \int_0^6 (6y^2 - y^3)\, dy \\ &= \frac{\pi \rho g}{16} \left[2y^3 - \frac{y^4}{4} \right]_0^6 \\ &= \frac{27\pi\rho g}{4} \\ &= 66,150\,\pi \\ &\approx 207,816.354 \text{ (Joules)} \end{aligned}$$

 Continued...

7. [avg: 6.97/10] Consider the curve $y = \frac{2}{3}x^{3/2}$, for $0 \le x \le 8$.

(a) [5 marks] Find the length of the curve.

Solution:

$$\frac{dy}{dx} = \sqrt{x} \Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1+x},$$

and the length of the curve is

$$\int_0^8 \sqrt{1+x}\, dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^8 = \frac{2}{3}\left(9^{3/2} - 1\right) = \frac{52}{3}$$

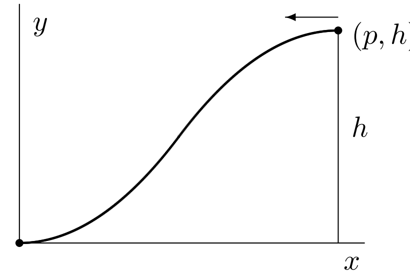
(b) [5 marks] Find the area of the surface generated by revolving the curve about the y-axis.

Solution: revolving the curve about the y-axis makes x the radius of each surface strip.

$$\begin{aligned} \text{SA} &= \int_0^8 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_0^8 2\pi x \sqrt{1+x}\, dx, \text{ using calculation from part (a)} \\ (\text{let } u = 1+x) \quad &= \int_1^9 2\pi(u-1)\sqrt{u}\, du = 2\pi \int_1^9 \left(u^{3/2} - \sqrt{u}\right) du \\ &= 2\pi \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^9 \\ &= 2\pi \left(\frac{2}{5}(243) - \frac{2}{3}(27) - \frac{2}{5} + \frac{2}{3}\right) = \frac{2384\,\pi}{15} \approx 499.3 \end{aligned}$$

Solution 2: solve for $x = g(y)$ and integrate with respect to y. See page 10 for the details of these calculations.

8. [avg: 1.87/10] Suppose that when an airplane approaches an airport for landing its path must satisfy the following five conditions:



(*i*) Altitude must be $y = h$ meters when descent begins.

(*ii*) Smooth touchdown occurs at $x = 0$.

(*iii*) Constant horizontal speed u m/sec must be maintained throughout.

(*iv*) At no time must the vertical acceleration in absolute value exceed a fixed positive constant k.

(*v*) The path must be a cubic polynomial with zero slope at the beginning and end of the descent.

How far from the landing strip must the descent begin? Your answer will be in terms of u, h and k.

Solution: let $y = f(x) = ax^3 + bx^2 + cx + d$ be the path of descent. Then $f'(x) = 3ax^2 + 2bx + c$. Since $f(0) = 0$ and $f'(0) = 0$, we know immediately that $c = d = 0$. Thus

$$f(x) = ax^3 + bx^2 \text{ and } f'(x) = 3ax^2 + 2bx.$$

Let the descent begin at $x = p$; then $f(p) = h$ and $f'(p) = 0$. Use these two equations to solve for a and b:

$$\begin{cases} ap^3 & + & bp^2 & = & h \\ 3ap^2 & + & 2bp & = & 0 \end{cases} \Leftrightarrow \begin{cases} 3ap^3 & + & 3bp^2 & = & 3h \\ 3ap^3 & + & 2bp^2 & = & 0 \end{cases} \Leftrightarrow b = \frac{3h}{p^2} \text{ and } a = -\frac{2h}{p^3}.$$

So, in terms of p and h, the path of the descent is

$$y = f(x) = -\frac{2hx^3}{p^3} + \frac{3hx^2}{p^2}.$$

Since the horizontal speed is constant u, the horizontal velocity is $-u$, and the vertical velocity is

$$\frac{dy}{dt} = -\frac{6hx^2}{p^3}\frac{dx}{dt} + \frac{6hx}{p^2}\frac{dx}{dt} = \frac{6hx^2}{p^3}u - \frac{6hx}{p^2}u = \frac{6hu}{p^3}\left(x^2 - px\right).$$

Then the vertical acceleration is

$$\frac{d^2y}{dt^2} = \frac{6hu}{p^3}(2x - p)\frac{dx}{dt} = \frac{6hu^2}{p^3}(p - 2x)$$

The maximum value of $\left|\frac{d^2y}{dt^2}\right|$ is at either end point, $x = 0$ or $x = p$. Finally, to satisfy condition *(iv)*, set

$$\left|\frac{d^2y}{dt^2}\right| \leq k \Leftrightarrow \frac{6hu^2}{p^2} \leq k \Leftrightarrow p \geq u\sqrt{\frac{6h}{k}}.$$

Conclusion: the descent must begin at least $u\sqrt{\frac{6h}{k}}$ meters from the landing strip.

Continued...

Some Alternate Calculations:

Quesiton 7(b), Solution 2: solve for $x = g(y)$ and integrate with respect to y. This will work, but it is much, much messier, algebraically, and trickier in terms of calculus.

$$y = \frac{2}{3}x^{3/2} \Rightarrow x = \left(\frac{9}{4}\right)^{1/3} y^{2/3},$$

so

$$g(y) = \left(\frac{9}{4}\right)^{1/3} y^{2/3} \text{ and } g'(y) = \left(\frac{9}{4}\right)^{1/3}\left(\frac{2}{3\,y^{1/3}}\right).$$

Then

$$\begin{aligned}
SA &= \int_0^{32\sqrt{2}/3} 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy \\
&= 2\pi\int_0^{32\sqrt{2}/3}\left(\frac{9}{4}\right)^{1/3} y^{2/3}\sqrt{1+\left(\frac{4}{9}\right)^{1/3}\left(\frac{1}{y^{2/3}}\right)}\,dy \\
&= 2\pi\int_0^{32\sqrt{2}/3}\left(\frac{9}{4}\right)^{1/3} y^{1/3}\sqrt{y^{2/3}+\left(\frac{4}{9}\right)^{1/3}}\,dy \\
\left(\text{let } u = y^{2/3}+\left(\frac{4}{9}\right)^{1/3}\right) \quad &= 2\pi\left(\frac{3}{2}\right)\int_{(4/9)^{1/3}}^{9(4/9)^{1/3}}\left(\frac{9}{4}\right)^{1/3}\left(u-\left(\frac{4}{9}\right)^{1/3}\right)\sqrt{u}\,du \\
&= 3\pi\int_{(4/9)^{1/3}}^{9(4/9)^{1/3}}\left(\left(\frac{9}{4}\right)^{1/3} u^{3/2}-\sqrt{u}\right)du \\
&= 3\pi\left[\left(\frac{9}{4}\right)^{1/3}\left(\frac{2}{5}\right)u^{5/2}-\frac{2}{3}u^{3/2}\right]_{(4/9)^{1/3}}^{9(4/9)^{1/3}}
\end{aligned}$$

$$= 3\pi\left(\left(\frac{9}{4}\right)^{1/3}\left(\frac{2}{5}\right)(9(4/9)^{1/3})^{5/2}-\frac{2}{3}(9(4/9)^{1/3})^{3/2}-\left(\frac{9}{4}\right)^{1/3}\left(\frac{2}{5}\right)((4/9)^{1/3})^{5/2}+\frac{2}{3}((4/9)^{1/3})^{3/2}\right)$$

$$= 3\pi\left(\frac{324}{5}-12-\frac{4}{15}+\frac{4}{9}\right) = 3\pi\left(\frac{2384}{45}\right) = \frac{2384\pi}{15}, \text{ as before!!}$$

Question 8, Alternate Answer: if D is the distance from $(x, y) = (p, h)$ to the origin $(x, y) = (0, 0)$, then

$$D^2 = p^2 + h^2 \geq \left(u\sqrt{\frac{6h}{k}}\right)^2 + h^2 = u^2\left(\frac{6h}{k}\right) + h^2 \Rightarrow D \geq \sqrt{u^2\left(\frac{6h}{k}\right)+h^2}.$$

But this approach, to set up the problem in terms of D, is much messier. Part of using math to solve a problem is to try and use the simplest approach!

Continued...

Assuming Constant Vertical Acceleration in Question 8: this is of course incorrect, but if you do assume a constant vertical acceleration of

$$\frac{d^2y}{dt^2} = -k,$$

then

$$y = -\frac{1}{2}kt^2 + h.$$

At the moment of landing, $y = 0$, so

$$t^2 = \frac{2h}{k}.$$

Using the fact that the horizontal speed is constant, u, we have, for p the horizontal distance to the landing strip,

$$t = \frac{p}{u} \Rightarrow t^2 = \frac{p^2}{u^2}.$$

Hence

$$p^2 = \left(\frac{2h}{k}\right) u^2 \Rightarrow p = u\sqrt{\frac{2h}{k}},$$

which 'looks' very close to the actual answer, but is totally wrong. Its totally wrong because if the vertical deceleration is constant, and the horizontal speed is constant, then the path of descent will be a parabola, i.e. a quadratic function, not a cubic function. And physically, a quadratic path of descent would be problematic: there would only be one point on the path at which the tangent line would be horizontal. Either the landing or the initial moment of descent—or both—would not be smooth!

Continued...

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

The end.