University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2014**
Duration: 2 and 1/2 hours
First Year - CHE, CIV, IND, LME, MEC, MMS

MAT186H1S - CALCULUS I
Exam Type: A

General Comments:

1. This exam was almost identical in types of problems to the exam of December 2013. Anybody who studied that exam should have done well on this exam.

2. As with the exam of December 2013, the hardest questions were Questions 6 and 7. Not surprisingly, most people did very poorly on these two questions.

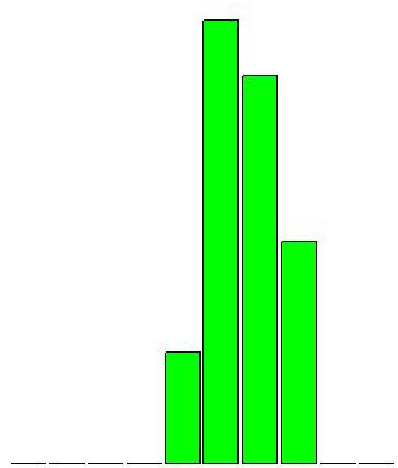
3. In 8(a) most students could set up the integral, but could not simplify it correctly. In 8(b) most students could not set up the integral correctly.

4. In Question 3(b) many students found the correct asymptotes but then did not draw graphs which were actually asymptotic to the asymptotes!

5. Question 5(a) caused problems because many students could not differentiate $\sec^{-1}\sqrt{x}$ correctly. Use the chain rule!

Breakdown of Results: 21 students wrote the exam. The marks ranged from 41% to 77%, and the average was 61.2%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right. Nobody failed the course.

Grade	%	Decade	%
		90-100%	0.0%
A	0.0%	80-89%	0.0%
B	19.1%	70-79%	19.1%
C	33.3%	60-69%	33.3%
D	38.1%	50-59%	38.1%
F	9.5%	40-49%	9.5%
		30-39%	0.0%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



1. [15 marks] Find the following:

(a) [5 marks] $\displaystyle\int \left(e^{-x} + \frac{1}{1+x^2} + \frac{1}{x+2} + \cosh x\right) dx$

Solution:

$$\begin{aligned}&\int \left(e^{-x} + \frac{1}{1+x^2} + \frac{1}{x+2} + \cosh x\right) dx \\ &= \int e^{-x}\,dx + \int \frac{1}{1+x^2}\,dx + \int \frac{1}{x+2}\,dx + \int \cosh x\,dx \\ &= -e^{-x} + \tan^{-1} x + \ln|x+2| + \sinh x + C\end{aligned}$$

(b) [4 marks] $\displaystyle\int_0^{\pi/2} \sin^3 x\, \cos x\, dx.$

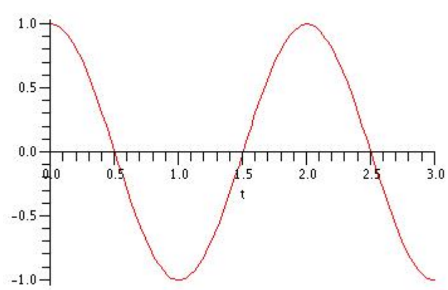
Solution: let $u = \sin x$. Then $du = \cos x\, dx$ and

$$\int_0^{\pi/2} \sin^3 x\, \cos x\, dx = \int_0^1 u^3\, du = \left[\frac{u^4}{4}\right]_0^1 = \frac{1}{4}.$$

(c) [6 marks] the average speed of a particle over the time period $t = 0$ to $t = 3$ if the velocity of the particle at time t is given by $v = \cos(\pi t)$.

Solution: the graph of v is shown below, it includes 1.5 cycles of the graph. Observe that $v > 0$ for $0 < t < 1/2, 3/2 < t < 5/2$ and that $v < 0$ for $1/2 < t < 3/2, 5/2 < t < 3$. In particular,

$$\int_0^3 |v|\, dt = 6\int_0^{1/2} v\, dt.$$



Thus the average speed is given by

$$\begin{aligned}\frac{1}{3}\int_0^3 |v|\, dt &= \frac{6}{3}\int_0^{1/2} \cos(\pi t)\, dt \\ &= 2\left[\frac{\sin(\pi t)}{\pi}\right]_0^{1/2} \\ &= 2\left(\frac{1}{\pi} - 0\right) = \frac{2}{\pi}.\end{aligned}$$

2. [10 marks] Find the following:

(a) [4 marks] $F'(1)$ if $F(x) = \int_0^{x^2} \tan^{-1} t\, dt$.

Soluton: By the Fundamental Theorem of Calculus, and the chain rule,

$$F'(x) = 2x \tan^{-1} x^2.$$

So

$$F'(1) = 2 \tan^{-1} 1 = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

(b) [6 marks] an approximation of the solution to the equation $x^3 + x = 1$, correct to 4 decimal places.

Solution: let $f(x) = x^3 + x - 1$ and use Newton's method to approximate the solution to the equation $f(x) = 0$. Observe that

$$f(0) = -1 < 0 \text{ and } f(1) = 1 > 0.$$

So the solution is in the interval $[0, 1]$. We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} = \frac{2x_n^3 + 1}{3x_n^2 + 1}.$$

Then

$$x_0 = 0.5 \Rightarrow x_1 = 0.7142857143 \cdots \Rightarrow x_2 = 0.6831797236 \ldots$$
$$\Rightarrow x_3 = 0.6823284233 \cdots \Rightarrow x_4 = 0.6823278037 \ldots$$

The solution is $x = 0.6823$, correct to three decimal places.

3. [13 marks] Let $f(x) = \dfrac{x^2}{x-3}$, for which $f'(x) = \dfrac{x(x-6)}{(x-3)^2}$, $f''(x) = \dfrac{18}{(x-3)^3}$.

(a) [4 marks] Find the interval(s) on which f is increasing and the interval(s) on which f is decreasing.

Solution: For increasing: $f'(x) > 0 \Leftrightarrow x(x-6) > 0$ and $x \neq 3 \Leftrightarrow x < 0$ or $x > 6$. For decreasing: $f'(x) < 0 \Leftrightarrow x(x-6) < 0$ and $x \neq 3 \Leftrightarrow 0 < x < 3$ or $3 < x < 6$.

(b) [2 mark] Find the interval(s) on which f is concave up and the interval(s) on which f is concave down.

Solution: For concave up: $f''(x) > 0 \Leftrightarrow (x-3)^3 > 0 \Leftrightarrow x > 3$.
For concave down: $f''(x) < 0 \Leftrightarrow (x-3)^3 < 0 \Leftrightarrow x < 3$.

(c) [3 marks] Find the equations of all asymptotes to the graph of f.

Solution: long division gives

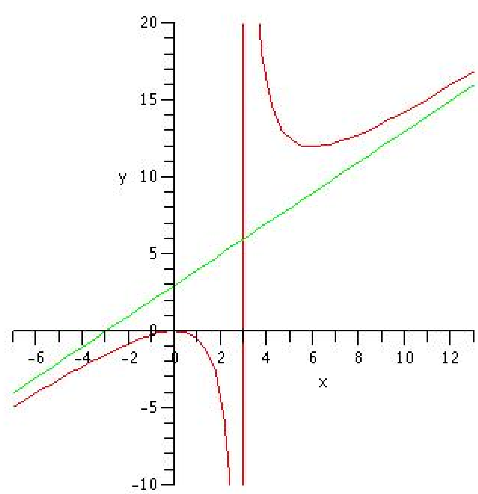
$$\frac{x^2}{x-3} = x + 3 + \frac{9}{x-3},$$

so $y = x + 3$ is a slant (or oblique) asymptote; and $x = 3$ is a vertical tangent because

$$\lim_{x\to 3^-} \frac{x^2}{x-3} = -\infty, \quad \lim_{x\to 3^+} \frac{x^2}{x-3} = \infty.$$

(d) [4 marks] Sketch the graph of f labeling all critical points, inflection points and asymptotes, if any.

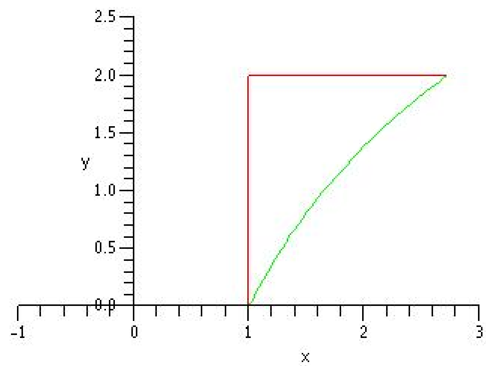
Solution:
The **graph** is to the right.
There is a min at $(6, 12)$ and a max at $(0, 0)$. There are no inflection points.



4. [12 marks] Let A be the area of the region in the xy-plane bounded by the curves $y = 2$ and $y = 2 \ln x$ on the interval $1 \leq x \leq e$.

(a) [8 marks] Write down two integrals, one with respect to x and one with respect to y, that both give the value of A.

Solution: the region A is indicated in the graph below.



With respect to x:

$$A = \int_1^e (2 - 2 \ln x)\ dx.$$

With respect to y:

$$A = \int_0^2 (e^{y/2} - 1)\, dy,$$

since $y = 2 \ln x \Rightarrow x = e^{y/2}$.

(b) [4 marks] Find the value of A.

Solution: integrate with respect to y, since at this stage of the game, we don't know how to integrate $\ln x$ with respect to x.

$$A = \int_0^2 (e^{y/2} - 1)\, dy = \left[2e^{y/2} - y\right]_0^2 = 2e - 2 - 2 + 0 = 2e - 4.$$

Alternate Calculation: for those interested, or for those who somehow already know integration by parts, it can be shown that

$$\int \ln x\, dx = x \ln x - x + C,$$

so

$$\int_1^e (2 - 2 \ln x)\, dx = 2\left[x - x \ln x + x\right]_1^e = 2(e - 2) = 2e - 4$$

5. [15 marks] Find the following limits, if they exist:

(a) [5 marks] $\lim_{x\to 1^+} \frac{\sec^{-1}\sqrt{x}}{\ln x}$

Solution: this limit is in the 0/0 form. Use L'Hopital's Rule:

$$\lim_{x\to 1^+} \frac{\sec^{-1}\sqrt{x}}{\ln x} = \lim_{x\to 1^+} \frac{1/(\sqrt{x}\sqrt{x-1}\cdot 2\sqrt{x})}{1/x} = \lim_{x\to 1^+} \frac{1}{2\sqrt{x-1}} = +\infty.$$

(b) [5 marks] $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

Solution: this limit is in the $\infty - \infty$ form. Rewrite it as 0/0 and use L'Hopital's Rule:

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \frac{\sin x - x}{x\sin x}$$

$$= \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} = \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0.$$

(c) [5 marks] $\lim_{x\to 0^+} (1+\sin x)^{2/x}$

Solution: this limit is in the 1^∞ form, so let the limit be L and calculate $\ln L$.

$$L = \lim_{x\to 0^+} (1+\sin x)^{2/x}$$

$$\Rightarrow \ln L = \lim_{x\to 0^+} \ln(1+\sin x)^{2/x} = \lim_{x\to 0^+} \frac{2\ln(1+\sin x)}{x} = 2\lim_{x\to 0^+} \frac{\cos x}{1+\sin x} = 2$$

$$\Rightarrow L = e^2$$

6. [10 marks] Sketch the graph of the relation $x^3 + y^3 = 3xy$ indicating all critical points, inflection points, horizontal tangents and vertical tangents, if any. BONUS. This graph has a slant asymptote: +2 if you can draw it in reasonably accurately; +3 more if you can find its equation. DON'T waste time if you haven't finished the other questions!

Solution: this graph is symmetric in the line $y = x$ and there are no points in the third quadrant, since $x < 0, y < 0 \Rightarrow x^3 + y^3 < 0$ but $3xy > 0$. To find the derivatives, differentiate implicitly.

$$3x^2 + 3y^2 \frac{dy}{dx} = 3y + 3x \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y - x^2}{y^2 - x},$$

$$\frac{d^2y}{dx^2} = \frac{(y' - 2x)(y^2 - x) - (y - x^2)(2yy' - 1)}{(y^2 - x)^2} = \frac{2xy}{(x - y^2)^3},$$

after substituting for y', simplifying and using the fact that $x^3 + y^3 = 3xy$.

Horizontal Tangents:

$$\frac{dy}{dx} = 0 \Rightarrow y = x^2 \text{ and } x^3 + y^3 = 3xy \Rightarrow x^6 = 2x^3 \Rightarrow x = 0, x = 2^{1/3};$$

so the graph has horizontal tangents at the two points $(0, 0)$ and $(2^{1/3}, 2^{2/3})$.

Vertical Tangents: in this case the derivative must be undefined. Hence

$$x = y^2 \text{ and } x^3 + y^3 = 3xy \Rightarrow y^6 = 2y^3 \Rightarrow y = 0, y = 2^{1/3};$$

so the graph has vertical tangents at the two points $(0, 0)$ and $(2^{2/3}, 2^{1/3})$.

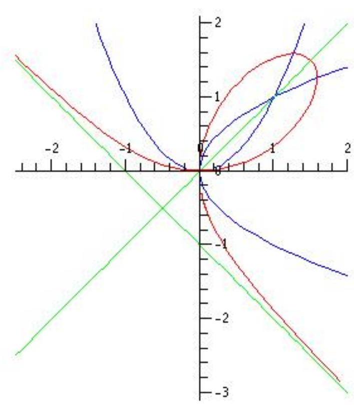
Concavity: there are two inflection points.

$$\frac{d^2y}{dx^2} = 0 \Leftrightarrow (x, y) = (0, 0);$$

$\frac{d^2y}{dx^2}$ is undefined if and only if

$x = y^2 \Leftrightarrow (x, y) = (2^{2/3}, 2^{1/3})$.

The graph is concave up everywhere except on the top part of the loop between $(0, 0)$ and $(2^{2/3}, 2^{1/3})$, where it is concave down.



The diagram to the right shows the graph in red, the asymptote and line $y = x$ in green, and the parabolas $y = x^2, x = y^2$ in blue. **The Slant Asymptote:** its equation is $x + y + 1 = 0$, but its hard to find. If $y' \to m$, so that the slant asymptote is $y = mx + b$, then

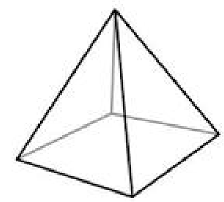
$$m = \lim_{x \to \infty} \frac{dy}{dx} = \lim_{x \to \infty} \frac{mx + b - x^2}{m^2x^2 + 2mbx + b^2 - x} = -\frac{1}{m^2} \Rightarrow m^3 = -1 \Rightarrow m = -1.$$

So the slant asymptote must have equation $x + y = b$. Rewrite the original equation as

$$x^3 + y^3 = 3xy \Leftrightarrow x^3 + 3x^2y + 3xy^2 + y^3 = 3xy + 3x^2y + 3xy^2 \Leftrightarrow (x + y)^3 = 3xy(1 + x + y),$$

which in the limit as $x \to \pm\infty, y \to \mp\infty$, becomes $b^3 = (-\infty)(1 + b)$; only if $b = -1$.

7. [10 marks] The frame of a pyramid, with square base and apex above the centre of the base, is to be constructed out of a piece of wire of length 36. What is the maximum volume of such a pyramid? Recall: volume of a pyramid is $V = \frac{1}{3}a^2h$.



Solution: let the dimensions of the square base be $a \times a$, with $a > 0$. Let x be the length of the wire pieces joining the vertices of the base to the apex. Then $4a + 4x = 36 \Leftrightarrow x = 9 - a$. Since $x > 0$, we must have $a < 9$. The volume of the pyramid is

$$V = \frac{1}{3}a^2h$$

with

$$h^2 + \left(\frac{\sqrt{2}\,a}{2}\right)^2 = x^2 \Leftrightarrow h = \sqrt{x^2 - \frac{a^2}{2}},$$

since the diagonal of the base has length $\sqrt{2}\,a$. Also, since $h > 0$ we have

$$a^2 < 2x^2 \Rightarrow a < \sqrt{2}\,x = \sqrt{2}\,(9-a) = 9\sqrt{2} - \sqrt{2}\,a \Rightarrow (1+\sqrt{2})a < 9\sqrt{2} \Rightarrow a < \frac{9\sqrt{2}}{1+\sqrt{2}}.$$

In terms of a, the problem is: maximize

$$V = \frac{1}{3}a^2h = \frac{1}{3}a^2\sqrt{(9-a)^2 - \frac{a^2}{2}} = \frac{1}{3}a^2\sqrt{81 - 18a + \frac{a^2}{2}}$$

for $0 \le a \le 9\sqrt{2}/(1+\sqrt{2})$. At each endpoint $V = 0$, so the maximum value of V must be at a critical point of V in the interval $(0, 9\sqrt{2}/(1+\sqrt{2}))$. Differentiate V:

$$\frac{dV}{da} = \frac{2}{3}a\sqrt{81 - 18a + \frac{a^2}{2}} + \frac{1}{3}a^2\frac{(-18+a)}{2\sqrt{81 - 18a + \frac{a^2}{2}}} = \frac{4a\left(81 - 18a + \frac{a^2}{2}\right) + a^2(-18+a)}{6\sqrt{81 - 18a + \frac{a^2}{2}}}.$$

$$\frac{dV}{da} = 0 \Rightarrow 324a - 72a^2 + 2a^3 - 18a^2 + a^3 = 0 \Rightarrow a^2 - 30a + 108 = 0,$$

since we are assuming $a > 0$. The roots of the quadratic are

$$a = \frac{30 \pm \sqrt{900 - 432}}{2} = 15 \pm \sqrt{117},$$

only one of which is less than $9\sqrt{2}/(1+\sqrt{2}) \approx 5.3$, namely $a = 15 - \sqrt{117} \approx 4.2$. Then

$$x = \sqrt{117} - 6 \approx 4.8,\ h = \sqrt{3\sqrt{117} - 18} \approx 3.8,\ V = (114 - 10\sqrt{117})\sqrt{3\sqrt{117} - 18} \approx 22.2$$

Note: If you thought that $h^2 + (a/2)^2 = x^2$, then your answer would be $a = 10 - 2\sqrt{7}$ and

$$x = 2\sqrt{7} - 1,\ h = \sqrt{6\sqrt{7} - 3} \text{ and } V = \frac{8}{3}(16 - 5\sqrt{7})\sqrt{6\sqrt{7} - 3};$$

that is $a \approx 4.7,\ x \approx 4.3,\ h \approx 3.6, V \approx 26.5$. This would just cost you one mark.

8. [15 marks] The parts of this question are unrelated.

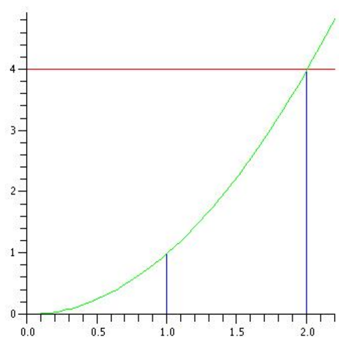
(a) [7 marks] Find the length of the curve $y = \dfrac{x^3}{24} + \dfrac{2}{x}$ for $1 \le x \le 4$.

Solution:

$$\begin{aligned}
L &= \int_1^4 \sqrt{1+(f'(x))^2}\,dx = \int_1^4 \sqrt{1+\left(\frac{x^2}{8}-\frac{2}{x^2}\right)^2}\,dx \\
&= \int_1^4 \sqrt{1+\left(\frac{x^4}{64}-\frac{1}{2}+\frac{4}{x^4}\right)}\,dx = \int_1^4 \sqrt{\left(\frac{x^4}{64}+\frac{1}{2}+\frac{4}{x^4}\right)}\,dx \\
&= \int_1^4 \sqrt{\left(\frac{x^2}{8}+\frac{2}{x^2}\right)^2}\,dx = \int_1^4 \left(\frac{x^2}{8}+\frac{2}{x^2}\right)\,dx \\
&= \left[\frac{x^3}{24}-\frac{2}{x}\right]_1^4 = \frac{64}{24}-\frac{2}{4}-\frac{1}{24}+2 = \frac{33}{8}
\end{aligned}$$

(b) [8 marks] The region bounded by $y = x^2, x = 1, x = 2$ and $y = 0$ is rotated about the line $y = 4$. Find the volume of the resulting solid.

Solution: integrate with respect to x, use the method of washers:



$$\begin{aligned}
V &= \int_1^2 \pi\left(4^2-(4-x^2)^2\right)\,dx \\
&= \pi\int_1^2 \left(8x^2-x^4\right)\,dx \\
&= \pi\left[\frac{8}{3}x^3-\frac{1}{5}x^5\right]_1^2 = \frac{187\,\pi}{15}
\end{aligned}$$

Alternatively: integrate with respect to y and use the method of shells:

$$\begin{aligned}
V &= \int_0^1 2\pi(4-y)(1)\,dy + \int_1^4 2\pi(4-y)(2-\sqrt{y})\,dy \\
&= 2\pi\int_0^1 (4-y)\,dy + 2\pi\int_1^4 (8-4\sqrt{y}-2y+y^{3/2})\,dy \\
&= 2\pi\left[4y-\frac{y^2}{2}\right]_0^1 + 2\pi\left[8y-\frac{8}{3}y^{3/2}-y^2+\frac{2}{5}y^{5/2}\right]_1^4 \\
&= 7\pi + 2\pi\left(\frac{41}{15}\right) = \frac{187\,\pi}{15}
\end{aligned}$$