High School Differential Calculus Test Answers and Solutions:

Answers to the Test:

1. T 2. T 3. F 4. T 5. D 6. C 7. B 8. A 9. B 10. D

Solutions and Comments:

1. By definition, $f'(a) = \lim_{x \to 0} \dfrac{f(a+x) - f(a)}{x}$. This is True.

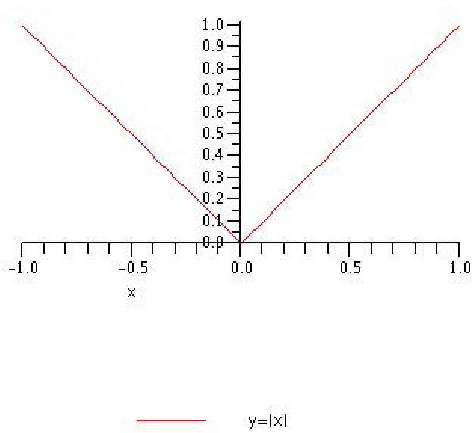
2. If $y = 6e^{-3x}$, then $\dfrac{dy}{dx} = -3y$. This is True.

$$y = 6e^{-3x} \Rightarrow \frac{dy}{dx} = -3\,(6)\,e^{-3x} \Rightarrow \frac{dy}{dx} = -3y$$

3. If $y = \sin x - 2\cos x$, then $\dfrac{d^2y}{dx^2} = y$. This is False.

$$\begin{aligned} y = \sin x - 2\cos x \quad &\Rightarrow \quad \frac{dy}{dx} = \cos x + 2\sin x \\ &\Rightarrow \quad \frac{d^2y}{dx^2} = -\sin x + 2\cos x \\ &\Rightarrow \quad \frac{d^2y}{dx^2} = -y \end{aligned}$$

4. For $x \neq 0$, $\dfrac{d\,|x|}{dx} = \dfrac{|x|}{x}$. This is True.



Recall

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

So

$$\frac{d|x|}{dx} = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

And

$$\frac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1 & \text{if} \quad x > 0 \\ \\ \dfrac{-x}{x} = -1 & \text{if} \quad x < 0 \end{cases}$$

5. Get a common denominator and simplify:

$$
\begin{aligned}
\lim_{t\to 0}\frac{(1+t)^{-2}-1}{t} &= \lim_{t\to 0}\frac{1-(1+t)^2}{(1+t)^2\,t}\\
&= \lim_{t\to 0}\frac{-2t-t^2}{(1+t)^2\,t}\\
&= \lim_{t\to 0}\frac{-2-t}{(1+t)^2}\\
&= -2
\end{aligned}
$$

6. Rationalize and simplify:

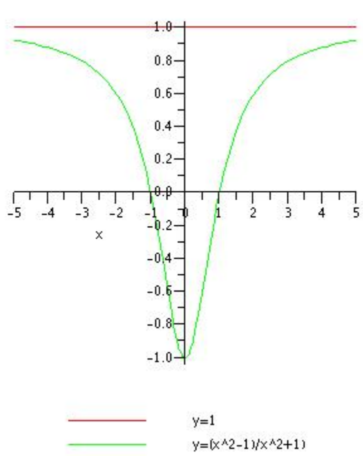
$$
\begin{aligned}
\lim_{y\to 4}\frac{y-4}{\sqrt{2y+1}-3} &= \lim_{y\to 4}\frac{(y-4)(\sqrt{2y+1}+3)}{(\sqrt{2y+1}-3)(\sqrt{2y+1}+3)}\\
&= \lim_{y\to 4}\frac{(y-4)(\sqrt{2y+1}+3)}{2y+1-9}\\
&= \lim_{y\to 4}\frac{(y-4)(\sqrt{2y+1}+3)}{2(y-4)}\\
&= \frac{1}{2}\lim_{y\to 4}\sqrt{2y+1}+3\\
&= \frac{6}{2}=3
\end{aligned}
$$

7. There is only one vertical asymptote to the graph of $f(x)=\dfrac{x^2+5x+6}{x^2-4}$ because

$$
\frac{x^2+5x+6}{x^2-4}=\frac{(x+3)(x+2)}{x-2)(x+2)}=\frac{x+3}{x-2},\ \text{if } x\neq -2.
$$

So the only vertical asymptote is the line $x=2$. (Aside: $\displaystyle\lim_{x\to -2}f(x)=-\frac{1}{4}$.)

8. The range of the function $f(x)=\dfrac{x^2-1}{x^2+1}$ is $1>y\geq -1$



By long division

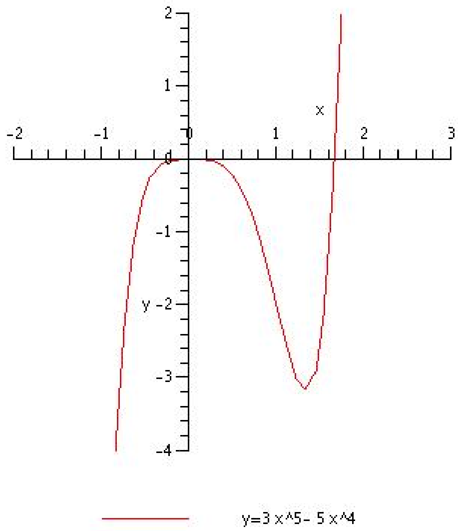
$$
f(x)=1-\frac{2}{x^2+1}.
$$

So

$$
f(x)<1 \text{ and } f'(x)=\frac{2x}{(x^2+1)^2},
$$

whence the only critical point of f is $x=0$. Thus $f(0)=-1$ is the absolute minimum value of f.

9. There is only one inflection point on the graph of $f(x) = 3x^5 - 5x^4$.



$$f'(x) = 15x^4 - 20x^3$$
$$f''(x) = 60x^3 - 60x^2 = 60x^2(x-1)$$

Thus

$$f''(x) > 0 \Leftrightarrow x > 1$$

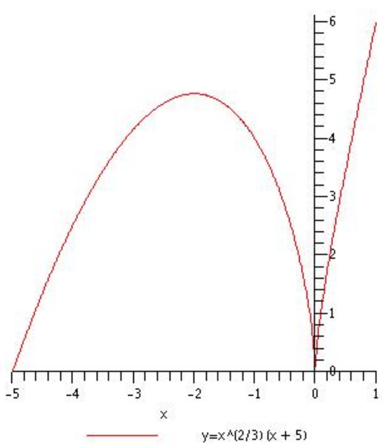
and

$$f''(x) < 0 \Leftrightarrow x < 1, x \neq 0.$$

So the only inflection point is at $(1, -2)$.

10. If m is the minimum value of $f(x) = x^{5/3} + 5x^{2/3}$ on the interval $-5 \leq x \leq 1$ and M is the maximum value of $f(x) = x^{5/3} + 5x^{2/3}$ on the interval $-5 \leq x \leq 1$, then the value of $m + M$ is 6.

$$f'(x) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3} = \frac{2x + 10}{3\,x^{1/3}}$$



The critical points of f are

$$(x, y) = (-2, 3\,(-2)^{2/3})$$

and $(x, y) = (0, 0)$. The end points of the interval are $x = -5$ and $x = 1$, at which

$$f(-5) = 0 \text{ and } f(1) = 6.$$

Comparing the critical points and the endpoints,

$$m = 0 \text{ and } M = 6.$$