WINNING STRATEGIES AND TACTICS IN CLUB GAMES

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Abstract. We present results concerning winning strategies and tactics in club games on $\mathcal{P}_{\omega_1}\lambda$. We show that there is generally no winning tactic for the player trying to get inside the club. The bound-countable game turns out to be rather fruitful and adds to some previous results about the construction of elementary substructures, their localization in certain intervals, and the realization of arbitrarily large gaps. We show that Player II has a winning strategy in the bound-countable game, thus establishing a new ZFC result. The applications given are new proofs for two cardinal diamonds.

1. INTRODUCTION

Club games are well known from the literature. We consider different versions of these types of games in this paper. In general, Player I sets up a constraint that Player II has to deal with during the course of the game. In club games, this constraint is usually to try and close off all sets played so far under a given function f which generates a club $\mathcal{C}_f$ in $\mathcal{P}_{\omega_1}\lambda$. Many variations of these games can easily be seen to have winning strategies for Player II, but it's less trivial to see whether Player II has a tactic. We feel that this kind of problem has never been encountered in the literature, which is what should make parts of the paper interesting. Another type of club game is considered, the *bound-countable game*. We show that Player II has a winning strategy for this game in ZFC, improving over earlier results and Namba-lemmas in this direction.

The bound-countable game has some interesting applications: it allows us to construct substructures which realize arbitrarily large gaps. This means that we can localize their elements in certain intervals. Problems like these are related to results formerly established in [13]

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and [4] but our techniques enable us to add some new applications, mostly concerning two-cardinal diamonds.

We do not intend to repeat all the standard definitions about winning strategies and tactics. The interested reader is referred to [7], we believe that our notation is fairly standard. Let us just mention really quickly that a *strategy* for one of the players is a function that maps any possible initial state of the game to the set of responses at that point and therefore tells the corresponding player what to play. A *winning strategy* is a strategy that will always end up with a win for the corresponding player. A *tactic* on the other hand is a strategy that only depends on the very last move of the opponent, instead of the whole history of the game. Thus, the player with a strategy has full knowledge about the history of the game, whereas the player with a tactic has partial knowledge about the history of the game, namely the last move only. A tactic is sometimes also called a *positional strategy.* There are some interesting questions concerning the differences between strategies and tactics, this paper highlights and answers some of them.

The clubs considered are clubs in $\mathcal{P}_{\omega_1}\lambda$ and it is generally assumed that λ is at least ω_2. We know that clubs in $\mathcal{P}_{\omega_1}\lambda$ are generated by functions $f : [\lambda]^{<\omega} \longrightarrow \lambda$ and we assume throughout that any such f includes Skolem functions for H_λ. In other words, members of

$$\mathcal{C}_f = \{X \in \mathcal{P}_{\omega_1}\lambda : X \text{ is closed under } f\}$$

are of the form $X \prec H_\lambda$. If we don't make the assumption that f closes off under the basic Skolem functions, the entire club game might become trivial. All our games depend on an f of this form and if we say that Player II has a winning strategy in one of the club games below, we mean that he has a winning strategy for any such given function $f : [\lambda]^{<\omega} \longrightarrow \lambda$.

1 Definition. The *singleton-singleton game* is

$$\begin{array}{c|ccccccccccc} \text{I} & \alpha_0 & & \alpha_2 & & \alpha_4 & & \alpha_6 & & \dots & \\ \hline \text{II} & & \alpha_1 & & \alpha_3 & & \alpha_5 & & \alpha_7 & & \dots \end{array}$$

where $\alpha_i < \lambda$ for all $i < \omega$ and II wins if $\{\alpha_i\}_{i<\omega}$ is closed under f.

2 Definition. The *countable-countable game* is

$$\begin{array}{c|ccccccccccc} \text{I} & X_0 & & X_2 & & X_4 & & X_6 & & \dots & \\ \hline \text{II} & & X_1 & & X_3 & & X_5 & & X_7 & & \dots \end{array}$$

where $X_i \in \mathcal{P}_{\omega_1}\lambda$ for all $i < \omega$ and II wins if $\bigcup_{i<\omega} X_i$ is closed under f.

Note that the X_i's are not required to be increasing in the countable-countable game. If the X_i's were required to be increasing then Player II would have a tactic, as can easily be checked. The following is easy to see.

3 Remark. *Player II has a winning strategy in both the singleton-singleton and the countable-countable game on $\mathcal{P}_{\omega_1}\lambda$ for any λ.*

A slightly different type of club game:

4 Definition. Assume λ is regular.[1] The *bound-countable game* is

$$\begin{array}{c|cccccccccc}
\text{I} & X & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \dots & \\
\hline
\text{II} & & X_0 & & X_1 & & X_2 & & X_3 & & \dots
\end{array}$$

where $X \subseteq X_0$ and for all $i < \omega$,

(1) $\emptyset \neq X_i \in \mathcal{P}_{\omega_1}\lambda$,
(2) $X_i << \alpha_{i+1} << X_{i+1} << \lambda$.

Player II wins if $\bigcup_{i<\omega} X_i$ is closed under f.

Let us mention an important result from [11] which says that many club-guessing sequences can be constructed in ZFC alone.

5 Definition. Let $\mu < \nu$ both be regular. Denote the set

$$\{\gamma < \nu : \mathrm{cf}(\gamma) = \mu\}$$

by S^μ_ν. A *club guessing sequence on S^μ_ν* is of the form C_γ ($\gamma \in S^\mu_\nu$) such that C_γ is an unbounded subset of γ of order type μ and if $D \subseteq \nu$ is club then $\{\gamma \in S^\mu_\nu : C_\gamma \subseteq D\}$ is stationary in ν.

Shelah [11] constructs such a sequence and establishes the following:

6 Theorem. *Let μ, ν be regular such that $\mu^+ < \nu$. Then there is a club guessing sequence on S^μ_ν.*

2. THERE ARE STRATEGIES BUT NO TACTICS

First we prove some results which show that Player will typically have no tactic in these types of club games.

7 Theorem. *Player II has no tactic in the singleton-singleton game on $\mathcal{P}_{\omega_1}\lambda$.*

This is an easy consequence of the next theorem.

8 Theorem. *Player II has no tactic in the countable-countable game on any $\mathcal{P}_{\omega_1}\lambda$.*

Proof. Let τ be a tactic for Player II.

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[1] The bound-countable game makes less sense if $\mathrm{cf}(\lambda) = \omega$ since X_0 could be cofinal in λ.

8.1 Claim. *For any* $x, y \in \mathcal{P}_{\omega_1}\lambda$ *we must have:*

$$\tau(x) \subseteq \tau(y) \text{ or } \tau(y) \subseteq \tau(x).$$

Proof. Otherwise Player I could play x at all even stages and y at all odd stages. The payoff set of this game would be $\tau(x) \cup \tau(y)$ but note that such a set can never be closed under Skolem functions if there is some $\alpha \in \tau(x) \setminus \tau(y)$ and $\beta \in \tau(y) \setminus \tau(x)$ since the union would have to contain the pair $\langle \alpha, \beta \rangle$. □

Given the claim, the set

$$\mathcal{K} = \{\tau(x) : x \in \mathcal{P}_{\omega_1}\lambda\}$$

of τ-responses must be a $\subseteq$-chain. Then clearly, $\mathcal{K}$ can be at most countable which is a contradiction. □

The proof of Theorem 8 was originally different and required the assumption that λ is not Jonsson. I would like to thank Yasuo Yoshinobu for pointing out to me that this assumption is not necessary. Turning our attention to the bound-countable game from now on, we have an interesting ZFC theorem.

9 Theorem. *Player II has a winning strategy in the bound-countable game on* $\mathcal{P}_{\omega_1}\lambda$ *for any regular* λ.

Proof. Let $f : [\lambda]^{<\omega} \longrightarrow \lambda$ be the game-relevant function. First note that the bound-countable game is determined, so it's enough to show that Player I does not have a winning strategy. To this end, assume that σ is a winning strategy for Player I in which he plays X in his first move. Construct an $\in$-chain of models

$$M_n \prec H_\theta \ (n < \omega)$$

for some large enough θ such that $X, \sigma, f \in M_0$, and such that each model M_n $(n < \omega)$ is *internally club* witnessed by a continuous $\subseteq$-chain

$$M_n = \bigcup \langle M_n^\xi : \xi < \omega_1 \rangle,$$

where each M_n^ξ is countable and in M_n. Let $M_\omega = \bigcup_{n<\omega} M_n$ and $\delta_i = \sup(M_i \cap \lambda)$ for all $i \leq \omega$. Now choose a set $z(0)$ such that

1. $X \subseteq z(0) \in \mathcal{P}_{\omega_1}\delta_\omega$
2. $z(0)$ is cofinal in δ_ω.

Define $z_n(0) = z(0) \cap \delta_n$. Note that $z_n(0)$ might not be in M_n. We will construct recursively for all $n, i < \omega$ the objects

- $z(i) \in \mathcal{P}_{\omega_1}\delta_\omega$
- $z_n(i) \in \mathcal{P}_{\omega_1}\delta_n$
- $\xi_n(i) < \omega_1$.

If $z(i)$, $z_n(i)$, $\xi_n(i)$ are constructed for all $n < \omega$, then define:

$$z(i+1) = \lambda \cap \bigcup_{n<\omega} M_n^{\xi_n(i+1)} \tag{2.1}$$

where $\xi_n(i+1) < \omega_1$ is such that

$$\mathrm{cl}_f(z_n(i)) \subseteq M_n^{\xi_n(i+1)}. \tag{2.2}$$

Then set

$$z_n(i+1) = z(i+1) \cap \delta_n. \tag{2.3}$$

At the end of the day, let $z(\omega) = \bigcup_{i<\omega} z(i)$ and $z_n(\omega) = z(\omega) \cap \delta_n$. We have the following claims:

9.1 Claim. *$z(\omega)$ is f-closed.*

Proof of Claim 9.1. Let $\alpha_0, \dots, \alpha_{j-1} \in z(\omega)$. Then there is $n < \omega$ and $i < \omega$ such that

$$\alpha_0, \dots, \alpha_{j-1} \in z(i) \cap \delta_n = z_n(i).$$

But this implies that

$$f(\alpha_0, \dots, \alpha_{j-1}) \in \mathrm{cl}_f(z_n(i)) \subseteq z(i+1).$$

Hence, $f(\alpha_0, \dots, \alpha_{j-1}) \in z(\omega)$. □

9.2 Claim. *$z_n(\omega) = \lambda \cap \bigcup_{i<\omega} M_n^{\xi_n(i)}$ for all $n < \omega$.*

Proof of Claim 9.2. "$\supseteq$" is clear and for the other direction assume that $\alpha \in z(\omega) \cap \delta_n$. Then $\alpha \in z(i)$ for some $i < \omega$. Hence,

$$\alpha \in z_n(i) = z(i) \cap \delta_n.$$

Therefore, $\alpha \in M_n^{\xi_n(i+1)}$ by definition of $z(i+1)$. □

9.3 Claim. *$z_n(\omega) \in M_n$ for all $n < \omega$.*

Proof of Claim 9.3. Follows from Claim 9.2 since the model M_n is internally club. □

Now we construct a play of the game in which Player II plays in his first move the set $X_0 = z_0(\omega)$ and in his further moves the sets

$$X_n = z_n(\omega) \setminus \delta_{n-1} \; (1 \leq n < \omega).$$

We still have to show that this is a legal play: since $\sigma \in M_0$ we conclude that α_{n+1}, Player I's response to $X_n \in M_n$, must be in M_n as well. Therefore, $\alpha_{n+1} < \delta_n$ and Player II can play $X_{n+1} = z_n(\omega) \setminus \delta_n$ in his next move. So we constructed a legal play in which Player II plays X_n ($n < \omega$) at his nth move, Player I plays according to σ, and

$$\bigcup_{n<\omega} X_n = \bigcup_{n<\omega} z_n(\omega) = z(\omega)$$

is f-closed. Thus II wins, a contradiction. $\square$

Theorem 9 says that there are stationarily many countable structures $N \prec H_\lambda$ that can be iteratively end-extended. This improves over results previously achieved in [13] and [4]. The new aspect of Theorem 9 is that we know the initial segments of that particular structure during the finite stages of its construction. It turns out that we actually have an application of this phenomenon in Section 4 where the knowledge of the initial segments of the structure is of crucial importance already during its construction. Theorem 9 can also be viewed as a Namba-lemma.

10 Theorem. *Let $f : [\lambda]^{<\omega} \longrightarrow \lambda$ where λ is regular. Then every Namba tree $T \subseteq \lambda^{<\omega}$ has a labelling $x_s \in \mathcal{P}_{\omega_1}\lambda$ ($s \in T$) such that*

(a) *$s <_T t$ implies that x_t end-extends x_s,*
(b) *if $s^\frown\alpha \in T$ then $\min(x_{s^\frown\alpha} \setminus x_s) \geq \alpha$, and*
(c) *if $b = \{s_n\}_{n<\omega}$ is a branch through T, then $x_b = \bigcup_{n<\omega} x_{s_n}$ is closed under f.*

Can Player II have a tactic in the bound-countable game? At first sight, it may seem that Player II might have a tactic in the bound-countable game if he has indiscernibles at his disposal. However,

11 Theorem. *Player II has no tactic in the bound-countable game on $\mathcal{P}_{\omega_1}\lambda$ for any regular λ.*

Proof. Assume that τ is a tactic for Player II in the bound-countable game. Then define

$$\begin{aligned} \delta(\alpha) &= \min(\tau(\alpha)) \\ \varepsilon(\alpha) &= \min\{\varepsilon \geq \delta(\alpha) : \varepsilon \notin \tau(\alpha)\} \\ \xi(\alpha) &= \mathrm{otp}(\varepsilon(\alpha) \setminus \delta(\alpha)). \end{aligned}$$

Note that $\xi(\alpha)$ is a countable ordinal. So find an infinite set $H \subseteq \lambda$ such that $\xi(\alpha) = \xi_0$ for all $\alpha \in H$. Now we describe a play in which Player II plays according to τ but loses still, this would suffice. Let Player I play in his first move a set X such that $\xi_0 + 1 \subseteq X$. In his further moves, Player I plays $\alpha_i \in H$ for all $i < \omega$. Say that the responses of Player II are X_i ($i < \omega$). We have that $\xi_0 \in X$ and

$$\varepsilon(\alpha_i) = \delta(\alpha_i) + \xi_0. \tag{2.4}$$

But note that

$$\delta(\alpha_i), \xi_0 \in \bigcup_{i<\omega} X_i. \tag{2.5}$$

This means that Player II loses: since $\varepsilon(\alpha_i) \notin \bigcup_{i<\omega} X_i$, (2.4) and (2.5) imply that $\bigcup_{i<\omega} X_i$ is not closed under Skolem functions. □

3. PARTIAL ORDERINGS

It is interesting to note that there are historical examples in which the difference between a strategy and a tactic played a crucial role. We only mention two examples, the first one being the existence of a normal ideal $\mathcal{I}$ on ω_2 such that Player Nonempty has a winning strategy in the Banach-Mazur game played on $\mathcal{I}^+$. This was established by Galvin, Jech, and Magidor in [5]. Around the same time, Laver showed in unpublished work that this can actually be strengthened: using the same model as in the result just mentioned, Laver shows that the ideal $\mathcal{I}$ has the stronger property that $\mathcal{I}^+$ has a σ-closed dense subset which in turn implies that Player Nonempty has a tactic in the Banach-Mazur game on $\mathcal{I}^+$.

Another related classical problem is the existence of a complete Boolean algebra for which Player II has a winning strategy in the Banach-Mazur game but there is no σ-closed dense subset. We believe that this question was first asked by Jech in [6], but variants of it can probably be traced back even to [9]. We would like to remind the reader of a partial result in this direction due to Jech and Shelah [8] which claims that it is consistent that there be such a complete Boolean algebra. Even though not mentioned there, it can be checked without too much effort that Player II does not even have a tactic for that algebra. So Jech and Shelah indeed prove:

12 Theorem. *It is consistent that there is a partial ordering $\mathbb{P}$ for which Player II has a winning strategy in the Banach-Mazur game but no tactic.*

It is still open whether there is a ZFC-example of such a complete Boolean algebra. Another interesting result in this direction can be found in [1] which answers the above question for games on topological spaces but not for complete Boolean algebras.

4. TWO CARDINAL DIAMOND PRINCIPLES

In this section, we give an application of Section 2 which concerns two-cardinal diamonds. These have been considered previously in the literature, see for example [2].

13 Definition. Let κ and λ be cardinals. Then $\diamondsuit_{\kappa,\lambda}$ is the statement that there is a sequence $(v_z : z \in \mathcal{P}_\kappa \lambda)$ such that for every $W \subseteq \lambda$, the set $\{z \in \mathcal{P}_\kappa \lambda : v_z = W \cap z\}$ is stationary. We also say that *diamond holds on* $\mathcal{P}_\kappa \lambda$ in this case.

The following is proved in [3] and [12].

14 Theorem. $\diamondsuit_{\kappa,\lambda}$ *holds whenever* $2^{<\kappa} < \lambda$.

We comment on a result originally due to [10] and give an alternative proof that has independent interest, as can be seen from the additional result below. Another advantage of our proof is that it provides a uniform construction for all regular λ, while [10] and [12] construct $\diamondsuit_{\omega_1,\lambda}$ by splitting up into cases. See also the discussion of Shelah's result in [12], where the author tries to elaborate on some arguments from [10].

15 Theorem. $\diamondsuit_{\omega_1,\lambda}$ *holds for all cardinals* $\lambda > \omega_1$.

Proof. We also need to split up into cases, but have a uniform construction at least for all regular λ.

λ **regular:** We fix a coding device for the rest of the proof: if $r \in 2^\omega$ and $n \in \omega$, then we define the *section* $(r)_n$ by letting

$$(r)_n(k) = r(2^n(2k+1)).$$

We also fix a bijection $\varphi : [\omega]^{\aleph_0} \longrightarrow [\omega_1]^{\aleph_0}$ and define the operation

$$^* : [\omega]^{\aleph_0} \longrightarrow [\omega_1]^{\aleph_0}$$

by letting $r^* = \bigcup_{n<\omega} \varphi((r)_n)$. When reading these definitions, note that we freely indentify a subset $r \subseteq \omega$ with its *characteristic function* $\chi_r : \omega \longrightarrow 2$ defined by: $\chi_r(n) = 1$ iff $n \in r$.

Let C_δ $(\delta \in S^\omega_\lambda)$ be a club-guessing sequence as in Theorem 6. We let δ_n be the nth element of C_δ. If $z \in [\lambda]^{\aleph_0}$, we define $v_z \subseteq z$ by letting $v_z = \pi_z^{-1}{}''\mathrm{pat}(z)^*$, where

$$\mathrm{pat}(z) = \{n < \omega : z \cap [\delta_n, \delta_{n+1}) \neq \emptyset\}$$

is the *pattern of* z *on* λ and π_z the transitive collapse of z. To show that $(v_z : z \in [\lambda]^{\aleph_0})$ is a $\diamondsuit_{\omega_1,\lambda}$-sequence, fix $f : [\lambda]^{<\omega} \longrightarrow \lambda$ and $W \subseteq \lambda$. We have to find a $z \in [\lambda]^{\aleph_0}$ such that

(1) z is closed under f, and
(2) $v_z = W \cap z$.

Build a continuous sequence N_ξ $(\xi < \lambda)$ of elementary substructures of some large enough structure, containing everything in sight, and such that $|N_\xi| < \lambda$ and $N_\xi \cap \lambda \in \lambda$ for all $\xi < \lambda$. Now define the club

$$C = \{\xi < \lambda : \xi = N_\xi \cap \lambda\}$$

and find $\delta \in \lambda \cap \mathrm{cof}(\omega)$ such that $C_\delta \subseteq C$. Let $N_\lambda = \bigcup_{\xi<\lambda} N_\xi$. Now let σ be a winning strategy for Player II in the bound-countable game on $\mathcal{P}_{\omega_1}\lambda$. For future arguments it is important to notice that for any $\xi < \lambda$:

$$\text{if } X, \alpha_0, \dots, \alpha_{n-1} \in N_\xi \text{ then } \sigma(X, \alpha_0, \dots, \alpha_{n-1}) \in N_\xi. \tag{4.1}$$

The idea of the proof is roughly as follows: we construct a set z through countably many initial segments z_n $(n < \omega)$ inside the structures N_{δ_n} $(n < \omega)$ and while this decides more and more information about initial segments of $W \cap z$, we have to make sure that the pattern of z on λ will later correctly code the information about these initial segments. So the goal is to build the set z such that

(3) $\mathrm{pat}(z) = r$ for some $r \in [\omega]^{\aleph_0}$ and
(4) $W \cap z = \pi_z^{-1}{}'' r^*$.

Conditions (3) and (4) would suffice: by definition of v_z we achieve (1) and (2). Proceed by induction: starting with the first σ-response $z_0 = \sigma(\emptyset)$, first define the section $(r)_0$, then the next approximation z_1, then the next section $(r)_1$ and so on. So assume that $z_i, (r)_i$ are defined for all $i < n$ and look at the set $w_n = W \cap z_n$. Modulo the collapse π_{z_n}, this set w_n can be viewed as a countable set a_n of ordinals below ω_1, formally $a_n = \pi_{z_n}{}'' w_n$. We let $(r)_n = \varphi^{-1}(a_n)$. That in mind, we have to define z_{n+1} and there is two cases here:

Case 1. there is some $m \geq n$ such that $r(m) = 1$ and $r(i) = 0$ for all $n \geq i > m$. In particular, $r(i)$ is defined for all these i's.

In this case we pick z_{n+1} such that $z_{n+1} \cap [\delta_n, \delta_{n+1}) = \emptyset$ for all $n \geq i > m$ and $z_{n+1} \cap [\delta_m, \delta_{m+1}) \neq \emptyset$. This choice is possible by (4.1), by the rules of the bound-countable game, and by elementarity of the structures N_{δ_n} $(n < \omega)$.

Case 2. there is some $m \geq n$ such that $r(m) = 1$ but not all $r(i)$ $(n \geq i > m)$ are yet defined.

Then we pick z_{n+1} such that $\sup(z_{n+1}) < \delta_n$ which is possible by elementarity of the structure N_{δ_n}.

The construction of Cases 1 and 2 makes sure that we achieve (3). On the other hand, $(r)_i$ codes the section w_i inside of x_{s_i}, so by definition of the operation * we have that $r^* = \bigcup_{n<\omega} a_n$ which implies (4).

λ **singular:** By Theorem 14, we may assume that $2^{\aleph_0} \geq \lambda^{\aleph_0}$. We mimic the construction for $\diamondsuit_{\omega_1,\omega_2}$, i.e. we code $W \cap z$ $(z \in \mathcal{P}_{\omega_1}\lambda)$ into the pattern of z on ω_2. This uses a club guessing sequence on $S^{\omega}_{\omega_2}$, except that now we use a surjection

$$\varphi : [\omega]^{\aleph_0} \longrightarrow [\lambda]^{\aleph_0}$$

and define $v_z = \mathrm{pat}(z)^*$. The rest is similar to the previous construction. $\square$

Since our construction is uniform on all regular cardinals, we also get the following ZFC theorem.

16 Theorem. *Let* $\langle \{\delta_n\}_{n<\omega} : \delta \in S^\omega_\lambda \rangle$ *be club-guessing for a regular cardinal* λ *and let the function* $\mathrm{pat} : \mathcal{P}_{\omega_1}\lambda \longrightarrow [\omega]^{\aleph_0}$ *be defined by*

$$\mathrm{pat}(z) = \{n < \omega : z \cap [\delta_n, \delta_{n+1}) \neq \emptyset\}.$$

Then for every $r \in [\omega]^{\aleph_0}$ *there is a* $\diamondsuit_{\omega_1,\lambda}$*-sequence* $\langle v_z : z \in \mathcal{E}_{\subseteq r}\rangle$ *on*

$$\mathcal{E}_{\subseteq r} = \{x \in \mathcal{P}_{\omega_1}\lambda : \mathrm{pat}(x) \subseteq r\}.$$

Let us give another application of the techniques above. The following end-extension property is well-known:

17 Definition. Assume that μ is regular. Then $\mathrm{CC}^*(\mu)$ is the statement that every countable structure $N \prec H_\theta$ can be *μ-end-extended*, i.e. there are arbitrarily large $\delta < \mu$ such that

$$\mathrm{Sk}_{H_\theta}(N) \cap \delta = \mathrm{Sk}_{H_\theta}(N \cup \{\delta\}) \cap \delta.$$

For regular μ, this statement is known to be true after Levy-collapsing a measurable cardinal to μ. The statement $\mathrm{CC}^*(\omega_2)$ is what is usually denoted by CC^* or *strong Chang's conjecture.*

We give another application of our previous techniques that is related to results in [12].

18 Theorem. *Assume* $\mathrm{CC}^*(\mu)$ *for some regular* $\mu \geq \omega_3$*. Then* $\diamondsuit_{\omega_2,\lambda}$ *holds for all cardinals* $\lambda \geq \mu$*.*

The proof will be sketched. It is similar to the proof of Theorem 15 except that we use the end-extension properties of $\mathrm{CC}^*(\mu)$ instead of Theorem 9. We only point out the differences to the previous proof.

Proof of Theorem 18. If $2^{\aleph_1} < \lambda$ then we are done by Theorem 14, so we may assume $2^{\aleph_1} \geq \lambda^{\aleph_1}$. Fix a club-guessing sequence on

$$\{\alpha < \mu : \mathrm{cf}(\alpha) = \omega_1\}$$

given by Theorem 6. The definition of the v_z's is similar to before except that we define the *pattern of z on λ* by

$$\mathrm{pat}(z) = \{i < \omega_1 : z \cap [\delta_i, \delta_{i+1}) \neq \emptyset\}.$$

Assume $W \subseteq \lambda$ and that f generates a club on $\mathcal{P}_{\omega_2}\lambda$. Then construct an ω_1-chain $\langle N_i : i < \omega_1\rangle$ of countable substructures of H_λ which are μ-end-extending one another. Code the information about the sets $W \cap N_\xi$ ($\xi < \omega_1$) into the pattern of the chain $\langle N_\xi : \xi < \omega_1\rangle$ on μ. This would use a surjection φ from $[\omega_1]^{\omega_1}$ onto $[\lambda]^{\omega_1}$. The rest is similar to the previous construction. $\square$

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