Preservation of Forcing-Axioms

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I. Game-closed posets

Definition 1. We say that a poset $\mathbb{P}$ is *weakly* ω_1*-game-closed* if player II wins the following game:

$$\begin{array}{c|cccccccccccc} \text{I} & p_0 & & p_2 & & \cdots & & & p_{\omega+1} & & p_{\omega+3} & & \cdots \\ \hline \text{II} & & p_1 & & p_3 & & \cdots & p_\omega & & p_{\omega+2} & & \cdots & \end{array}$$

i.e. player II plays at limit stages. Here, $p_\alpha \geq_{\mathbb{P}} p_\beta$ if $\alpha < \beta < \omega_1$ and II wins the play if there is $p \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_\alpha$ for all $\alpha < \omega_1$.

$\mathbb{P}$ is *strongly ω_1-game-closed* if player II wins the modified game:

I	p_0		p_2		$\dots$		p_ω		$p_{\omega+2}$		$\dots$	
II		p_1		p_3		$\dots$		$p_{\omega+1}$		$p_{\omega+3}$		$\dots$

i.e. player I plays at limit stages. Same rules otherwise.

It's easy to see that:

$\leq \omega_1$-closure$\Longrightarrow$ strong ω_1-game-closure$\Longrightarrow$ weak ω_1-game-closure

Examples: The standard forcing to add a $\square_{\aleph_1}$-sequence is weakly ω_1-game-closed.

$\mathrm{AP}_{\aleph_1}$ can be added with a strongly ω_1-game-closed forcing.

$\mathrm{AP}_{\aleph_1}$ is the following approachability property: there exists a sequence $(C_\alpha : \alpha < \omega_2)$ such that for any $\alpha < \omega_2$:

(a) $C_\alpha \subseteq \omega_2$, otp $C_\alpha \leq \omega_1$,

and there is a club $C \subseteq \lim(\omega_2)$ such that for every $\gamma \in C$:

(b) $C_\gamma \subseteq \gamma$ is club,

(c) the initial segments of C_γ are enumerated before γ,

i.e. $\forall \alpha < \gamma \ \exists \beta < \gamma \ C_\gamma \cap \alpha = C_\beta$.

The following is known:

Theorem 2 (Foreman,Magidor,Shelah and Velickovic). *Assume* MM*. If* $\mathbb{P}$ *is weakly* ω_1*-game-closed then*

$$V^{\mathbb{P}} \models \mathrm{NS}_{\omega_1} \textit{ is saturated.}$$

But a weakly ω_1-game-closed forcing can add $\square_{\aleph_1}$ and this will completely mess up the combinatorics of ω_2: $\square_{\aleph_1}$ gives us an ω_2-Aronszajn-tree and a nonreflecting subset of $\omega_2 \cap \mathrm{cof}(\omega)$.

Question 3. *What about strong ω_1-game-closure?*

Theorem 4. *Assume* MM. *If* $\mathbb{P}$ *is strongly ω_1-game-closed then*

$$V^{\mathbb{P}} \models \text{"\textit{stationary reflection}"} + \text{"\textit{no }}\omega_2\text{\textit{-Aronszajn-trees}"}$$

Actually: if $\mathbb{P}$ is strongly ω_1-game-closed

Theorem 5.

$$V \models \mathrm{MA}^{+}(\sigma\text{-}\textit{closed}) \Longrightarrow V^{\mathbb{P}} \models \mathrm{MA}^{+}(\sigma\text{-}\textit{closed})$$

$$V \models \mathrm{PFA} \Longrightarrow V^{\mathbb{P}} \models \text{"\textit{no }}\omega_2\text{\textit{-Aronszajn-trees}"}$$

Corollary 6.

(1) $\mathrm{AP}_{\aleph_1} \not\Longrightarrow$ *"there is an ω_2-Aronszajn-tree"*

(2) $\mathrm{Con}(\mathrm{MA}^+(\sigma\text{-}\mathit{closed}) + \mathrm{AP}_{\aleph_1})$

Corollary 7.

There is no strongly ω_1-game-closed forcing which adds an ω_2-Aronszajn-tree nor a non-reflecting subset. (Although there is a weakly ω_1-game-closed forcing.)

So the following question seems natural.

Question 8. *Can a strongly* ω_1*-game-closed forcing destroy* MM *or* PFA*?*

Is MM *or* PFA *consistent with* $\mathrm{AP}_{\aleph_1}$*?*

Answer:

Theorem 9.

$\mathrm{AP}_{\aleph_1}$ *fails under* PFA.

Remember that $\mathrm{AP}_{\aleph_1}$ can be added by a strongly ω_1-game-closed forcing notion.

Question 10. *Is there a* ($\leq \omega_1$)*-closed forcing which forces* $\mathrm{AP}_{\aleph_1}$*?*

The following can easily be checked:

Note 11. *Both* MM *and* PFA *are preserved by* $\leq \omega_1$*-directed-closed forcings.*

We refine this:

Theorem 12. *Assume* PFA*.*

If $\mathbb{P}$ *is* $\leq \omega_1$*-closed then* $V^{\mathbb{P}} \models \mathrm{PFA}$*.*

(And therefore $V^{\mathbb{P}} \models \neg\mathrm{AP}_{\aleph_1}$*.)*

Conclusion:

The mildest forcing to add

- $\Box_{\aleph_1}$ is weakly ω_1-game-closed
- $\mathrm{AP}_{\aleph_1}$ is strongly ω_1-game-closed.