Forcing indestructibility of set-theoretic axioms

Bernhard König
Université Paris 7

Starting point is an easy and folkloristic theorem:

Theorem 1. MM *is preserved by forcings that are* $<\omega_2$**-directed-closed**, *i.e. if* $V \models$ MM *and* $\mathbb{P}$ *is* $<\omega_2$*-directed-closed then*

$$V^{\mathbb{P}} \models \text{MM}.$$

An equally easy theorem is:

Theorem 2. BMM *is preserved by forcings that are* $<\omega_2$**-distributive**, *i.e. if* $V \models$ BMM *and* $\mathbb{P}$ *is* $<\omega_2$*-distributive then*

$$V^{\mathbb{P}} \models \text{BMM}.$$

Clearly, the stronger the closure properties of $\mathbb{P}$, the larger a fragment of MM can we hope to preserve.

We consider the following properties of forcings, increasing in logical strength:

(1) $< \omega_2$-distributive
(2) weakly $< \omega_2$-game-closed
(3) strongly $< \omega_2$-game-closed
(4) $< \omega_2$-closed
(5) $< \omega_2$-directed-closed

Our program is the following: for each property (1)-(5) we associate a fragment of MM that gets preserved by the forcing with the corresponding property.

The outgrowth of this program is a whole bunch of new (and interesting) consistency results. Here are examples of what can be "done" with the respective forcings:

There is a ... forcing	that adds ....
(2) weakly game-closed	$\square_\kappa$ for any κ
(3) strongly game-closed	AP_κ for any κ
(4) ω_2-closed	regressive κ-Kurepatree
(5) ω_2-directed-closed	κ-Kurepatree

The corresponding preservation results are the following:

Theorem 3 (Folklore).
MM *is preserved by forcings that are* ω_2-**directed-closed**.

Theorem 4 (Folklore).
BMM *is preserved by forcings that are* ω_2-**distributive**.

Theorem 5 (Velickovic).
If $V \models \mathrm{MM}$ *and* $\mathbb{P}$ *is* **weakly game-closed** *then*

$$V^{\mathbb{P}} \models \textit{ saturation of } NS_{\omega_1}.$$

Theorem 6 (König,Yoshinobu).
If $V \models \mathrm{MM}$ *and* $\mathbb{P}$ *is* $< \omega_2$-**closed** *then*

$$V^{\mathbb{P}} \models \mathrm{MM}(\Gamma_{\mathrm{cov}}).$$

Where $\mathbb{P} \in \Gamma_{\mathrm{cov}}$ if every countable set of ordinals in $V^{\mathbb{P}}$ can be covered by a countable set in V.

Theorem 7 (König).
If $V \models \mathrm{MM}$ *and* $\mathbb{P}$ *is* **strongly game-closed** *then*

$$V^{\mathbb{P}} \models \mathrm{MM}(\Gamma_\Sigma).$$

Where $\mathbb{Q} \in \Gamma_\Sigma$ if it is of the form $\mathbb{Q}_0 * \mathbb{Q}_1$, $\mathbb{Q}_0$ adds no countable sequences and forces $\mathbb{Q}_1$ to be of size $\leq \aleph_1$.

So we have the table:

If $V \models \mathrm{MM}$ and $\mathbb{P}$ is ...	then ... is true in $V^{\mathbb{P}}$.
(1) ω_2-distributive	BMM
(2) weakly game-closed	saturation of NS_{ω_1}
(3) strongly game-closed	$\mathrm{BMM} + \mathrm{MM}(\Gamma_\Sigma)$
(4) ω_2-closed	$\mathrm{BMM} + \mathrm{MM}(\Gamma_{\mathrm{cov}})$
(5) ω_2-directed-closed	MM

It can easily be checked:

Remark 8.

(i) $\mathrm{MM}(\Gamma_\Sigma)$ *implies Moore's* MRP, *the strong reflection principle* SRP *and Todorcevic's P-ideal dichotomy.*

(ii) $\mathrm{MM}(\Gamma_{\mathrm{cov}})$ *implies* PFA *and basically all known applications of* MM.

We reap the benefits of our program: a variety of new consistency results are the corollaries to the results above.

Just a flavor:

Corollary 9. PFA + SRP *is consistent with regressive Kurepatrees.*

Corollary 10. *Neither* MRP, SRP *nor the P-ideal dichotomy imply failure of* $\mathrm{AP}_{\aleph_1}$ [*or* $\mathrm{AP}_{\aleph_\omega}$] *even though full* MM *does.*

Corollary 11. $\mathrm{AP}_{\aleph_1}$ *does not imply the existence of an* $\aleph_2$*-Aronszajntree.*
[*Since these trees are killed by* $\mathrm{MM}(\Gamma_\Sigma)$.]

And finally:

Question 12 (Oberwolfach lunch).
Does MM *imply that* $\aleph_\omega$ *is Jonsson?*

We generically add the right "blend" of Kurepatrees and we get:

Answer 13.
MM *does* **not** *imply that* $\aleph_\omega$ *is Jonsson.*