A game in the reals

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Definition 1. If $X \subseteq \mathbb{R}$, we define the game $\mathbb{G}_X$ as follows:

I	s_0	s_1	s_2	s_3	$\dots$
II	i_0	i_1	i_2	i_3	$\dots$

where s_n is a finite binary sequence and $i_n \in \{0, 1\}$.

We say that player I wins the game if the concatenated sequence $(s_n, i_n : n < \omega)$ is in the set X.

Theorem 2 (Davis).

(1) *X is perfect if and only if I has a winning strategy in $\mathbb{G}_X$.*

(2) *X is countable if and only if II has a winning strategy in $\mathbb{G}_X$.*

Question 3. *Can we generalize Davis' theorem?*

I. A game on trees

Instead of the tree of finite binary sequences, we play a similar game in the tree of binary sequences of arbitrary length δ.

We replace the notion of countable with *"Every limit is a finite-support limit".*

Definition 4.

$$\mathbb{Q}_{<\delta} = \{f : \delta \longrightarrow 2 | f(\alpha) \neq 0 \text{ for only finitely many } \alpha\text{'s}\}$$

We modify the game:

Definition 5. Let T be an arbitrary binary tree of height δ. We define the game $\mathbb{G}_T$ as follows:

I	s_0		s_1		s_2		s_3		$\ldots$	
II		s_0		s_1		s_2		s_3		$\ldots$

where $s_n \in 2^{<\delta}$ and $i_n \in \{0, 1\}$.

We say that player I wins the game if the concatenated sequence $(s_n, i_n : n < \omega)$ is an element of the tree T.

Then we get the following generalization of Davis' theorem:

Theorem 6. *The following are equivalent for a tree of height* δ*:*

(1) $T \cong \mathbb{Q}_{<\delta}$

(2) *player II has a winning strategy in the game* $\mathbb{G}_T$

II. An application

In $V^{\mathrm{Coll}(\omega_1,<\kappa)}$ (κ supercompact), the following principle holds true:

GRP: Let $\mathbb{G}$ be an arbitrary game. If player II wins the game $\mathbb{G}$ played on any structure of size $\aleph_1$, then player II wins the game in the real world.

where "played on a structure" means that the players' moves are restricted to that particular structure.

Using theorem 6, it can easily be seen:

Proposition 7. GRP *implies the following reflection principle* $\mathbb{Q}$R*:*

if $\mathrm{cf}(\delta) \geq \omega_2$ *and* T *is a tree such that* $T_{<\gamma} \cong \mathbb{Q}_{<\gamma}$ *for all* $\gamma < \delta$ *then* $T_{<\delta} \cong \mathbb{Q}_{<\delta}$

Corollary 8. *If* κ *is supercompact, then*

$$V^{\mathrm{Coll}(\omega_1, <\kappa)} \models \mathbb{Q}\mathrm{R}$$

The principle $\mathbb{Q}$R implies the reflection of stationary subsets of $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$.