A NON-SPECIAL TREE

BERNHARD KÖNIG

Abstract. A Cohen real will always add an $\mathbb{R}$-embeddable Aronszajn-tree that is not $\mathbb{Q}$-embeddable.

1. Characterizations in terms of substructures

We consider trees of height κ^+ for a regular κ. According to this height, elementary substructures are assumed to be of cardinality κ and they contain all ordinals $< \kappa$. Furthermore, they are assumed to contain the trees in question.

1.1 Definition. We say that an elementary substructure $N \prec \mathrm{H}_\theta$ is T-*Suslin*, if every converging and cofinal branch $K \subseteq N \cap T$ is (N, T)-*generic*, i.e. $K \cap A \neq \emptyset$ for all maximal antichains $A \in N$.

$$\mathit{SUSLIN}_T = \{N \prec \mathrm{H}_\theta : N \text{ is } T\text{-Suslin}\}$$

1.2 Lemma. *The following are equivalent for any tree* T*:*

(1) T *is Suslin.*
(2) SUSLIN_T *is club in* $[\mathrm{H}_\theta]^\kappa$*.*
(3) SUSLIN_T *is stationary in* $[\mathrm{H}_\theta]^\kappa$*.*

Proof. (1) $\Longrightarrow$ (2): if $T \in N \prec \mathrm{H}_\theta$, $K \subseteq N \cap T$ cofinal and $A \in N$ a maximal antichain, then A is bounded in the universe, so there is a bound $\varepsilon < \delta = N \cap \kappa^+$ on the heights of elements of $A \cap N$. Pick $x \in K$ above ε. Then x will have a predecessor in A, so K hits A.

(3) $\Longrightarrow$ (1): let A be any maximal antichain in the tree T. Pick a T-Suslin N containing A as an element and let $\delta = N \cap \kappa^+$. Note that $N \cap T = T_{<\delta}$ is an initial segment of T. By Suslinity of N, we know that every converging cofinal $K \subseteq N \cap T$ hits A, so A is sealed off at the point δ. $\square$

1.3 Definition. $N \prec \mathrm{H}_\theta$ is *nearly* T-*dense*, if there is a converging and cofinal $K \subseteq N \cap T$ and an antichain $A \in N$ that *captures* the limit of K, i.e. $\lim(K) \in A$. $N \prec \mathrm{H}_\theta$ is T-*dense*, if for every converging and cofinal branch $K \subseteq N \cap T$, there is an antichain $A \in N$ that captures the limit of K. If an elementary submodel is not T-dense, we call it T-*wide*. If it's not nearly T-dense, we call it *strongly* T-*wide*.

1.4 Definition.

$$\begin{aligned}
\mathit{DENSE}_T &= \{N \prec \mathrm{H}_\theta : N \text{ is } T\text{-dense}\},\\
\mathit{DENSE}_T^+ &= \{N \prec \mathrm{H}_\theta : N \text{ is nearly } T\text{-dense}\},\\
\mathit{WIDE}_T &= \{N \prec \mathrm{H}_\theta : N \text{ is } T\text{-wide}\} = [\mathrm{H}_\theta]^\kappa \setminus \mathit{DENSE}_T,\\
\mathit{WIDE}_T^- &= \{N \prec \mathrm{H}_\theta : M \text{ is strongly } T\text{-wide}\} = [\mathrm{H}_\theta]^\kappa \setminus \mathit{DENSE}_T^+.
\end{aligned}$$

1.5 Note. *We have the following inclusions:*

$$\mathit{SUSLIN}_T \subseteq \mathit{WIDE}_T^- \subseteq \mathit{WIDE}_T,$$
$$\mathit{DENSE}_T \subseteq \mathit{DENSE}_T^+.$$

□

1.6 Lemma. *If $N \prec \mathrm{H}_\theta$ and $K \subseteq N \cap T$ is converging cofinal, then K cannot be captured by both a chain and an antichain in N.*

Proof. Assume that the antichain A and the chain L capture K. Then $N \models A \cap L \neq \emptyset$. But this means that L hits the antichain A at two different points. A contradiction. □

1.7 Lemma. *If the chain $L \in N \prec \mathrm{H}_\theta$ captures the converging cofinal chain $K \subseteq N \cap T$, then K witnesses non-Suslinity of N, i.e. there is a maximal antichain $A \in N$ such that $K \cap A = \emptyset$.*

Proof. A is simply the trivial branch-antichain, that can be defined and maximalized in N. □

1.8 Lemma. *If an antichain A is in the elementary submodel $N \prec \mathrm{H}_\theta$, then for $\delta = N \cap \kappa^+$:*

(a) *A is cofinal iff $A \setminus T_{<\delta} \neq \emptyset$.*
(b) *If A is club, then $A \cap T_\delta \neq \emptyset$.*
(c) *If $A \cap T_\delta \neq \emptyset$, then A is stationary.*

Proof. We only need to show (c): for this, it suffices to prove that N thinks of A as a stationary antichain. For any club $C \in N$, $\delta \in C$ will hold, so $C \cap (\mathrm{ht}''A) \neq \emptyset$. By elementarity, $N \models C \cap (\mathrm{ht}''A) \neq \emptyset$. So A is indeed stationary in N. □

1.9 Lemma. *The following are equivalent for any tree T:*

(1) *T has a stationary antichain.*
(2) *DENSE_T^+ is stationary in $[\mathrm{H}_\theta]^\kappa$.*

Proof. (1) $\Longrightarrow$ (2): assume that A is a stationary antichain in the tree T. In every club $\mathcal{D} \subseteq [\mathrm{H}_\theta]^\kappa$, we can find an elementary $N \prec \mathrm{H}_\theta$ such that $A \in N$ and $\delta = N \cap \kappa^+ \in \mathrm{ht}''A$. Let $x \in A \cap T_\delta$. Now, $\mathrm{pred}(x) \subseteq N \cap T$ is converging cofinal and $A \in N$ captures the limit point x. So N is nearly dense.

(2) $\Longrightarrow$ (1): if DENSE_T^+ is stationary, we can choose a nearly T-dense N such that $T \in N$. Then there is a converging cofinal $K \subseteq N \cap T$ and $A \in N$ with $\lim(K) \in A$. By Lemma 1.8(c), we know that A is a stationary antichain in the tree T. □

1.10 Lemma. *The following are equivalent for any tree T:*

1. *T is a special Aronszajn-tree.*
2. *DENSE_T is club in $[\mathrm{H}_\theta]^\kappa$.*

Proof. The characterization of speciality that we use is the following: there is a regressive $f : T \longrightarrow T$ whose preimages are antichains.

(1) $\Longrightarrow$ (2): assume that T is a special Aronszajn-tree via the specializing function $f : T \longrightarrow T$. If $T, f \in N \prec \mathrm{H}_\theta$ and $K \subseteq N \cap T$ converging cofinal, we let $x = \lim(K)$. Since f is regressive, we have that $y = f(x) \in N$. We can hence define $A = f^{-1}(y) \in N$. Note that A is an antichain, since f is specializing. But A captures x, the limit of K. Therefore, N is T-dense.

(2) $\Longrightarrow$ (1): choose a continuous ϵ-chain $\langle M_\xi : \xi < \kappa^+\rangle$ of T-dense structures. Let $\delta_\xi = M_\xi \cap \kappa^+$ for all $\xi < \kappa^+$. For $t \in T_{\delta_\xi}$, define

$$f(t) = \text{some antichain } A \in M_\xi \text{ such that } t \in A.$$

Note that this is possible by density of M_ξ and that f is a regressive mapping defined on the club $\{\delta_\xi : \xi < \kappa^+\}$. Finally, we check that if $f(t) = f(t') = A$, then t and t' are both in A and hence incomparable. From this we know that T is special. □

1.11 Definition. An Aronszajn-tree T is called *stat-special*, if there is a stationary $E \subseteq \kappa^+$ and a regressive $f : T \restriction E \longrightarrow T$ whose preimages are antichains.

The same argument as in the proof of Lemma 1.10 yields:

1.12 Lemma. *The following are equivalent for any tree T:*

1. *T is a stat-special Aronszajn-tree.*
2. *DENSE_T is stationary in $[\mathrm{H}_\theta]^\kappa$.*

□

1.13 Lemma. *If T is a coherent tree, then*

$$\mathit{DENSE}_T = \mathit{DENSE}_T^+.$$

Proof. Assume that T is coherent and let $N \prec \mathrm{H}_\theta$ be nearly T-dense, where $\delta = N \cap \kappa^+$. By definition, there is a converging cofinal chain $K \subseteq N \cap T$ that is captured by the antichain $A \in N$. Now, for any converging cofinal $K' \subseteq N \cap T$, let $\gamma < \delta$ be such that

$$K = \pi(K' \restriction \gamma)^\frown K' \restriction [\gamma, \delta)$$

for some permutation $\pi : T_\gamma \longrightarrow T_\gamma$. Then

$$A' = \{x \in T : \pi(x \restriction \gamma)^\frown x \restriction [\gamma, \mathrm{ht}(x)) \in A\}$$

is an antichain in N that captures the chain K'. This shows that N is T-dense. ☐

1.14 Corollary. *If T is a coherent Aronszajn-tree, then T is stat-special if and only if T has a stationary antichain.* ☐

1.15 Corollary. *If T is a coherent Aronszajn-tree with a club-antichain, then T is special.* ☐

In the next section, we modify the construction of a Suslin-tree from a Cohen-real in [1] to get an $\mathbb{R}$-embeddable non-special ω_1-tree from the same assumption.

2. A COHEN-REAL CONSTRUCTION

From now on assume that $c : \omega \longrightarrow [\omega]^{<\omega}$ is a Cohen-generic real and $e_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ a *coherent* sequence of 1-1 functions, i.e. $\{\xi < \alpha : e_\alpha(\xi) \neq e_\beta(\xi)\}$ is finite for all $\alpha < \beta < \omega_1$. Let C_α $(\alpha < \omega_1)$ be such that

(1) $C_\alpha \subseteq \alpha$ is unbounded,
(2) $C_{\alpha+1} = \{\alpha\}$,
(3) $\mathrm{otp}(C_\alpha) \leq \omega$.

Then we enlarge this sequence to a sequence D_α $(\alpha < \omega_1)$ in the following way:

$$D_\alpha = \{\xi < \alpha : C_\alpha(n) \leq \xi < C_\alpha(n+1), e_\alpha(\xi) \in c(n)\},$$

where $C_\alpha(0) = 0$ and $C_\alpha(n)$ is the nth element of C_α for $0 < n < \omega$. Now let $\varrho_1 = \varrho_1(D_\alpha : \alpha < \omega_1)$ and $T = T(\varrho_1)$.

2.1 Definition. $C_\alpha \restriction n = \{C_\alpha(m) : m < n\}$.

2.2 Proposition. *T has no stationary antichains.*

Proof. Assume that A is a stationary subset of T. Without restriction we may let $A \subseteq \{\varrho_1(\cdot, \alpha) : \alpha < \omega_1\}$, so that we actually indentify A with a stationary set of ordinals. Note that if A is stationary in $V^{\mathbb{C}}$, then there is a stationary $A_0 \in V$ such that $A_0 \subseteq A$. So we may assume without restriction that A is in the ground model. It suffices to show that the set of conditions forcing that A is not an antichain is dense in $\mathbb{C}$. We fix a condition $p : n_0 \longrightarrow [\omega]^{<\omega} \in \mathbb{C}$ and our aim is to extend it to a condition $q \leq p$ that connects two points in the set A. Let k be a bound for all the places fixed by p, e.g. we can let

$$k = \max\{l \in p(m) : m < n_0\}.$$

It is possible to shrink A and assume that all sets of the form

$$F_k(\alpha) = \{\xi < \alpha : e_\alpha(\xi) \le k\} \quad (\alpha \in A)$$

are isomorphic in their structural properties (i.e. they are equal and $e_\alpha(\xi) = e_\beta(\xi)$ for all $\xi \in F_k(\alpha), F_k(\beta)$, $\alpha < \beta \in A$). We construct a continuous ϵ-chain of submodels $\langle N_\xi : \xi < \omega_1\rangle$ where we let $A \in N_0$ and we set $\delta_\xi = N_\xi \cap \omega_1$. Note that $\xi = \delta_\xi$ holds almost everywhere. Now we can find $N \prec \mathrm{H}_\theta$, $\xi < \omega_1$ such that $N \cap \omega_1 = \delta_\xi = \delta \in A$ and

$$\langle N_\xi : \xi < \omega_1\rangle, \langle C_\alpha : \alpha < \omega_1\rangle, \langle D_\alpha : \alpha < \omega_1\rangle, A \in N.$$

Pick $n > n_0$ and $\zeta < \xi$ such that $C_\delta \cap \delta_\zeta = C_\delta \upharpoonright n$. N_ζ thinks that there is a countable ordinal $\gamma > C_\delta(n)$ in the stationary set A such that $C_\gamma \upharpoonright n = C_\delta \upharpoonright n$, since δ is such an ordinal in the universe. Recapitulating this, we know that γ has the following property:

$$C_\gamma \upharpoonright n = C_\delta \upharpoonright n = C_\delta \cap \gamma.$$

Now we can try and make the following true by extending p to a condition $q : n \longrightarrow [\omega]^{<\omega}$:

(i) $\gamma \in D_\delta$,
(ii) $D_\delta \cap \gamma \sqsubseteq D_\gamma$.

It is easy to establish (i) by letting $e_\delta(\gamma) \in q(n-1)$. By the coherence properties of the function $e : [\omega_1]^2 \longrightarrow \omega$, we know that the set of differences $\Delta = \{C_\gamma(n-1) \le \xi < C_\gamma(n) : e_\gamma(\xi) \ne e_\delta(\xi)\}$ is finite. Let $E_\gamma = \{e_\gamma(\xi) : \xi \in \Delta\}$, $E_\delta = \{e_\delta(\xi) : \xi \in \Delta\}$ and define $E = E_\gamma \cup E_\delta$. Note that E is bounded below ω. Now let $\sim$ be the smallest equivalence relation on E that contains the pairs $(e_\gamma(\xi), e_\delta(\xi))$ $(\xi \in \Delta)$. If we choose $q(n-1)$ to be the whole equivalence class of $e_\delta(\gamma)$, we know that (i) will hold and since this equivalence class takes care of the differences between e_γ and e_δ, it follows that

$$D_\gamma \cap C_\gamma(n) = D_\delta \cap \gamma.$$

So (i) and (ii) are true. But these imply that $\varrho_1(\cdot, \gamma) \subseteq \varrho_1(\cdot, \delta)$. This finishes the proof. □

REFERENCES

[1] Stevo Todorčević. Partitioning pairs of countable ordinals. *Acta Mathematica*, 159:261–294, 1987.