Math. Log. Quart. **50**, No. 3, 297 – 302 (2004) / **DOI** 10.1002/malq.200410101 / www.mlq-journal.org

Fragments of Martin's Maximum in generic extensions

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Received 20 November 2003, revised 16 March 2004, accepted 18 March 2004

Key words Martin's maximum, generic extension, forcing axioms, Banach-Mazur game.
MSC (2000) 03E30, 03E40

We show that large fragments of MM, e. g. the tree property and stationary reflection, are preserved by strongly $(\omega_1 + 1)$-game-closed forcings. PFA can be destroyed by a strongly $(\omega_1 + 1)$-game-closed forcing but not by an ω_2-closed.

1 Introduction

Recall longer versions of the Banach-Mazur game on a partial ordering $\mathcal{P}$:

Empty	p_0	p_2	$\cdots$	p_ξ	$p_{\xi+2}$	$\cdots$
Nonempty	p_1	p_3	$\cdots$	$p_{\xi+1}$	$p_{\xi+3}$	$\cdots$

where p_ξ, $\xi < \alpha$, is descending in $\mathcal{P}$ and Nonempty wins the game of length α if he can play α times.

A poset $\mathcal{P}$ is called *weakly α-game-closed* if Player Nonempty has a winning strategy in the Banach-Mazur game of length α, where Nonempty is allowed to play at limit stages. $\mathcal{P}$ is called *strongly α-game-closed* if Player Nonempty has a winning strategy in the same game, except where Empty is allowed to play at limit stages.

For an infinite cardinal κ, $\mathcal{P}$ is called *κ-closed* if any $\mathcal{P}$-descending chain of length $< \kappa$ has a lower bound in $\mathcal{P}$. In particular, an ω_1-closed poset is also called *σ-closed*. It is clear that κ^+-closed posets are strongly $(\kappa + 1)$-game-closed and strongly $(\kappa + 1)$-game-closed posets are weakly $(\kappa + 1)$-game-closed.

It was pointed out by Veličković in [10] (using technique from [3]):

Theorem 1.1 *Assume* MM.[1] *If $\mathcal{P}$ is weakly $(\omega_1 + 1)$-game-closed, then $V^{\mathcal{P}} \models$ "NS_{ω_1} is saturated".*

But weakly $(\omega_1 + 1)$-game-closed posets can add $\square_{\aleph_1}$ and therefore change the whole combinatorics of $\aleph_2$. We show in Sections 3 and 4 that a strongly $(\omega_1 + 1)$-game-closed extension will be considerably milder:

Theorem 1.2 *Assume* MM. *If $\mathcal{P}$ is strongly $(\omega_1 + 1)$-game-closed, then*

$$V^{\mathcal{P}} \models \textit{“stationary reflection”} + \textit{“there are no } \omega_2\textit{-Aronszajn-trees”}.$$

We can understand the impact of these forcings even better if we look at the following results of Sections 5 and 6:

Theorem 1.3 PFA *can be destroyed by a strongly $(\omega_1 + 1)$-game-closed forcing, but it is preserved by ω_2-closed forcings.*

Let us finish the introduction with some remarks and definitions. If $\mathcal{P}$ is a poset and $\mathbb{B}(\mathcal{P})$ is the corresponding regular open algebra, then $\mathcal{P}$ is strongly (weakly) α-game-closed if and only if $\mathbb{B}(\mathcal{P})$ is strongly (weakly) α-game-closed. In terms of forcing, we may therefore restrict ourselves to the cases where our game-closed posets are complete Boolean algebras. Unlike game closure properties, closure properties are not preserved under forcing

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[1] MM and PFA are defined below.

equivalence. Thus, we are using the term 'κ-closed' in a weaker sense: a κ-closed poset is a poset whose regular open algebra has a dense κ-closed (in the original sense) subset.

The *approachability property for* κ (AP_κ) holds if there exists a sequence $(C_\alpha : \alpha < \kappa^+)$ such that for any $\alpha < \kappa^+$,

(a) $C_\alpha \subseteq \kappa^+$, $\operatorname{otp} C_\alpha \le \kappa$,

and there is a club $C \subseteq \lim(\kappa^+)$ such that for every $\gamma \in C$,

(b) $C_\gamma \subseteq \gamma$ is club,

(c) the initial segments of C_γ are enumerated before γ, i. e. for each $\alpha < \gamma$ there exists $\beta < \gamma$ such that $C_\gamma \cap \alpha = C_\beta$.

It is easy to see from this definition that AP_κ is a consequence of $\square_\kappa$, indeed a consequence of all variations of square on this cardinality. It is also known that CH implies $\mathrm{AP}_{\aleph_1}$, and thus $\mathrm{AP}_{\aleph_1}$ is strictly weaker than $\square_{\aleph_1}$. The following remark is used very often throughout the paper.

Remark 1.4 The standard forcing to add a $\square_\kappa$-sequence with initial segments is weakly $(\kappa+1)$-game-closed. The analogous forcing to add an approachability sequence for κ with initial segments is strongly $(\kappa+1)$-game-closed. We leave this for the reader to check, the proofs can be found in [4, p. 255] and [11]. Let us denote these posets as $\mathbb{P}_{\square_\kappa}$ and $\mathbb{P}_{\mathrm{AP}_\kappa}$, respectively.

The following is known: If $\kappa > \omega$, then $\mathbb{P}_{\square_\kappa}$ is not strongly $(\kappa+1)$-closed, and if AP_κ fails, then $\mathbb{P}_{\mathrm{AP}_\kappa}$ is not κ^+-closed. Moreover, κ^+-closed posets never add an AP_κ-sequence unless AP_κ holds in the ground model, whereas $\mathbb{P}_{\mathrm{AP}_\kappa}$ does. [11] shows that under AP_κ, strongly $(\kappa+1)$-game-closed posets never add a $\square_\kappa$-sequence unless $\square_\kappa$ holds in the ground model, whereas $\mathbb{P}_{\square_\kappa}$ does.

Let Γ be a property of posets (like 'proper' or 'σ-closed'). $\mathrm{MA}(\Gamma)$ denotes the statement that whenever a poset $\mathcal{P}$ has property Γ and $\{D_\xi : \xi < \omega_1\}$ are dense subsets of $\mathcal{P}$, then there exists a filter G on $\mathcal{P}$ such that $D_\xi \cap G \ne \emptyset$ for all $\xi < \omega_1$. $\mathrm{MA}^+(\Gamma)$ denotes the statement that whenever $\mathcal{P}$ has property Γ, $\{D_\xi : \xi < \omega_1\}$ are dense subsets of $\mathcal{P}$, and $\dot{S}$ is a $\mathcal{P}$-name such that $\Vdash_{\mathcal{P}}$ "$\dot{S}$ is stationary in ω_1", then there exists a filter G on $\mathcal{P}$ such that $D_\xi \cap G \ne \emptyset$ for all $\xi < \omega_1$, and $\dot{S}[G] = \{\gamma < \omega_1 : (\exists q \in G)\,(q \Vdash_{\mathcal{P}} \check{\gamma} \in \dot{S})\}$ is stationary in ω_1. In particular, MA(proper) and MA(preserving the stationarity of subsets of ω_1) are also denoted as PFA and MM, respectively. It is known that MM implies MA^+(σ-closed) (see [8, p. 827]), but not PFA^+ (see [7]).

2 Amoeba forcing

The term Amoeba forcing has a history in set theory. We mean by this an iterated forcing construction $\mathbb{P} * \mathbb{Q}$ over a model of some $\mathrm{MA}(\Gamma)$ which achieves the following: if we apply the axiom $\mathrm{MA}(\Gamma)$ in the ground model and the resulting filter is $K \subseteq \mathbb{P} * \mathbb{Q}$, then there exists a lower bound $p \in \mathbb{P}$ for the first coordinate of K.

From now on assume that $\mathcal{P}$ is a strongly (ω_1+1)-game-closed forcing and let us fix a winning strategy σ for Nonempty. Instead of forcing with $\mathcal{P}$, we could also add a play of the Banach-Mazur game generically. Then this play induces a generic filter for $\mathcal{P}$. Define

$$\mathcal{R} = \{\langle p_\xi : \xi \le \gamma\rangle : \gamma < \omega_1 \text{ and } p_\xi,\ \xi \le \gamma, \text{ is a partial play according to } \sigma\}.$$

If $s = \langle p_\xi : \xi \le \gamma\rangle \in \mathcal{R}$ is such a partial play, we also denote the maximal condition p_γ by p_s. The ordering on $\mathcal{R}$ is usual extension.

Note that $\mathcal{R}$ contains $\mathcal{P}$ as a complete subalgebra by the projection mapping $i(s) = p_s$. Yet, it is a much stronger forcing: $\mathcal{R}$ will typically collapse the cardinality of $\mathcal{P}$ to $\aleph_1$.

Lemma 2.1 *$\mathcal{R}$ is σ-closed.*

Proof. Easy because we follow Nonempty's winning strategy. □

Lemma 2.2 *If $G \subseteq \mathcal{P}$ is generic, then $\mathcal{R}/\mathcal{P} = \{s \in \mathcal{R} : p_s \in G\}$ is σ-closed.*

Proof. If s_n, $n < \omega$, is a descending sequence in $\mathcal{R}/\mathcal{P}$, let γ be the length of the union $\bigcup_n s_n$. Then $q = \inf_n p_{s_n}$ is in G and $s = \bigcup_n s_n \cup \{(\gamma, q)\}$ is still a partial play according to σ. □

Lemma 2.2 points out the crucial difference between strongly and weakly game-closed forcings: the quotient $\mathcal{R}/\mathcal{P}$ will generally not be σ-closed if $\mathcal{P}$ is only weakly $(\omega_1 + 1)$-game-closed.

Lemma 2.3 *The forcing $\mathcal{P} * \mathcal{R}/\mathcal{P}$ is equivalent to $\mathcal{R}$.*

Proof. Note that the function $i : \mathcal{R} \longrightarrow \mathcal{P} * \mathcal{R}/\mathcal{P}$ sending s to (p_s, s) is a dense embedding. $\square$

3 Aronszajn-trees in generic extensions of PFA

The next theorem says that an ω_2-Aronszajn-tree can not be added with a strongly $(\omega_1 + 1)$-game-closed forcing notion, at least not a forcing constructed in ZFC. This contrasts the fact that we may always add an ω_2-Aronszajn-tree with a weakly $(\omega_1 + 1)$-game-closed forcing, even a $\square_{\aleph_1}$-sequence (cf. Remark 1.4).

The strategy in the following theorem is this: we start with a model of PFA, but instead of dealing with an ω_2-Aronszajn-tree, we are dealing with a $\mathcal{P}$-name $\dot{T}$ for such a tree, where $\mathcal{P}$ is strongly $(\omega_1 + 1)$-game-closed. Because of the strong closure properties of $\mathcal{P}$, we can quite often find $\mathcal{P}$-master conditions for substructures of size $\aleph_1$. This means we can decide a big enough structure after applying PFA in the ground model.

Theorem 3.1 *Assume* PFA. *If $\mathcal{P}$ is strongly $(\omega_1 + 1)$-game-closed, then ω_2 still has the tree property in $V^{\mathcal{P}}$.*

Proof. Assume that T is an ω_2-Aronszajn-tree in $V^{\mathcal{P}}$ and $\dot{T}$ is a $\mathcal{P}$-name for it. Let $\mathcal{R}$ be the poset from Section 2 and go to the extension $V^{\mathcal{R}}$: we find a generic play of the Banach-Mazur game $\langle \bar{p}_\xi : \xi < \omega_1 \rangle$. $\aleph_2$ is collapsed in this extension, so let $\bar{\delta}_\alpha$, $\alpha < \omega_1$, be a sequence converging to the ordinal ω_2. Then the cofinal subtree $T \restriction \{\bar{\delta}_\alpha : \alpha < \omega_1\}$ is a tree of height and size $\aleph_1$ in $V^{\mathcal{R}}$. Moreover, it has no cofinal branches by Silver's Lemma and the fact that we are forcing over $V^{\mathcal{P}}$ with a σ-closed final segment of $\mathcal{R}$ (Lemma 2.2). It is well-known that in this case we can specialize $T \restriction \{\bar{\delta}_\alpha : \alpha < \omega_1\}$ with a *ccc*-forcing $\mathcal{S}$ (proofs can be found in [4] or [7]). Now apply PFA to the proper poset $\mathcal{R} * \mathcal{S}$: Let $M \prec \mathrm{H}_\theta$ contain all countable ordinals and the relevant objects. Choose an M-generic filter for $\mathcal{R} * \mathcal{S}$ in the ground model. We get a play $\langle p_\xi : \xi < \omega_1 \rangle$ and a $\mathcal{P}$-name for a specializing function $\dot{f} : \dot{T} \restriction \{\delta_\alpha : \alpha < \omega_1\} \longrightarrow \omega$, where δ_α, $\alpha < \omega_1$, is converging to $\delta = M \cap \omega_2$. Now pick a lower bound q for the play $\langle p_\xi : \xi < \omega_1 \rangle$. Then q is a master condition for the structure M and $q \Vdash_{\mathcal{P}}$ "$\dot{f} : \dot{T} \restriction \{\delta_\alpha : \alpha < \omega_1\} \longrightarrow \omega$ is specializing". So this means that $q \Vdash_{\mathcal{P}}$ "$\dot{T}_{<\delta}$ has no cofinal branches", a contradiction. $\square$

Corollary 3.2 $\mathrm{AP}_{\aleph_1}$ *does not imply the existence of an ω_2-Aronszajn-tree.*

Proof. By Remark 1.4 and Theorem 3.1. $\square$

4 Preservation of MA$^+$(σ-closed)

A similar method gives a result about the reflection of stationary sets. Stationary reflection is generally preserved by strongly $(\omega_1 + 1)$-game-closed forcing notions, we show a bit more than that. Again, note that a weakly $(\omega_1 + 1)$-game-closed poset can add $\square_{\aleph_1}$ and therefore a non-reflecting stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$.

Theorem 4.1 *Assume* $\mathrm{MA}^+(\sigma\text{-closed})$. *If $\mathcal{P}$ is strongly $(\omega_1 + 1)$-game-closed, then $V^{\mathcal{P}} \models \mathrm{MA}^+(\sigma\text{-closed})$.*

Proof. Let $\mathcal{R}$ be as before, so $\mathcal{R}/\mathcal{P}$ is σ-closed. Now assume that $\mathcal{T}$ is a $\mathcal{P}$-name for a σ-closed poset, that $\dot{\tau}_\xi$, $\xi < \omega_1$, is a sequence of $\mathcal{P}$-names for dense subsets of $\mathcal{T}$, and let $\dot{S}$ be a $\mathcal{P} * \mathcal{T}$-name for a stationary subset of ω_1. In $V^{\mathcal{P}}$ look at the forcing notions $\mathcal{R}/\mathcal{P}$ and $\mathcal{T}$. By the Product Lemma we have

$$\mathcal{P} * (\mathcal{T} \times \mathcal{R}/\mathcal{P}) \simeq \mathcal{P} * (\mathcal{R}/\mathcal{P} \times \mathcal{T}) \simeq \mathcal{R} * \mathcal{T}.$$

Now note that $\mathcal{R} * \mathcal{T}$ is σ-closed and $\dot{S}$ is an $\mathcal{R} * \mathcal{T}$-name for a stationary subset of ω_1. Define the dense subsets of $\mathcal{R} * \mathcal{T}$: $D_\xi = \{(p, t, x) : p \Vdash_{\mathcal{P}} t \in \dot{\tau}_\xi\}$. Let $M \prec \mathrm{H}_\theta$ be a structure that contains all countable ordinals and the relevant objects. The structure M contains all dense sets D_ξ, $\xi < \omega_1$, in particular. Then apply $\mathrm{MA}^+(\sigma\text{-closed})$ in the ground model to the poset $\mathcal{R} * \mathcal{T}$: pick $K \subseteq \mathcal{R} * \mathcal{T}$ which is generic over M such that $\dot{S}[K]$ is stationary in ω_1. The filter K basically consists of a sequence $\langle p_\xi : \xi < \omega_1 \rangle$ and a $\mathcal{P}$-name for a filter $H \subseteq \mathcal{T}$. There is a lower bound q for $\langle p_\xi : \xi < \omega_1 \rangle$, thus $q \Vdash_{\mathcal{P}}$ "$H \subseteq \mathcal{T}$ is generic for $\dot{\tau}_\xi$, $\xi < \omega_1$)" and $q \Vdash_{\mathcal{P}}$ "$\dot{S}[H]$ is stationary in ω_1". This finishes the proof. $\square$

The last theorem can easily be generalized to $\mathrm{MA}^+(\sigma\text{-closed})$ for ω_1-many names for stationary sets. We did not choose to formulate the theorem in this generality.

Corollary 4.2 *For all $\kappa \geq \aleph_1$: $\mathrm{Con}(\mathrm{AP}_\kappa + \mathrm{MA}^+(\sigma\text{ -closed}) + \text{any reasonable cardinal arithmetic})$.*

Proof. By Remark 1.4 and Theorem 4.1. □

This last Corollary sharpens similar results that were obtained in [2]. Another application of the technique developed in Section 2 is a variation of the old consistency result of Friedman's statement. After consulting the relevant sections in [3], the reader will now easily find the proof for the following

Theorem 4.3 *Assume Martin's Maximum. If $\mathcal{P}$ is strongly (ω_1+1)-game-closed, then Friedman's statement holds in $V^{\mathcal{P}}$, i. e.*

$\Vdash_{\mathcal{P}}$ *"If $\kappa \geq \omega_2$ is regular and $A \subseteq \kappa \cap \mathrm{cof}(\omega)$ is stationary, then A contains a closed set of order-type ω_1".*

5 Approachability fails under PFA

A question arises from the last sections: what at all can we do with a strongly (ω_1+1)-game-closed forcing ? It is mentioned in the Introduction that we can add approachability sequences and it turns out that these will destroy PFA.

First another characterization of the approachability property which is used in the proof of Theorem 5.3:

Definition 5.1 A tree $T = (\omega_2, \preccurlyeq_T)$ is called *ω_2-directive* if

1. $\alpha \prec_T \beta$ implies $\alpha < \beta$ for every $\alpha, \beta < \omega_2$;
2. for every limit ordinal $\eta < \omega_2$ there is a branch b such that $\sup\, b = \eta$;
3. T has height ω_1.

Lemma 5.2 *The following are equivalent:*

1. $\mathrm{AP}_{\aleph_1}$.
2. *There is an ω_2-directive tree.*

Proof. See [11] for the proof. □

The next theorem says that in spite of the demonstrated weakness of $\mathrm{AP}_{\aleph_1}$, PFA still decides this statement to the negative. This proof was outlined to us by Matt Foreman and Stevo Todorčević independently (see acknowledgements).

Theorem 5.3 *Under* PFA*, the approachability property fails for $\aleph_1$.*

Proof. For every $\zeta < \omega_2$ fix a canonical function $\pi_\zeta : \omega_1 \longrightarrow \zeta$ which is onto ζ. To simplify the notation we write $\langle \zeta, \xi \rangle$ for $\pi_\zeta(\xi)$. Now assume $\mathrm{AP}_{\aleph_1}$ and let T be an ω_2-directive tree by Lemma 5.2. In the following let $\mathfrak{B}_T = \{b_\xi : \xi < \omega_2\}$ be an enumeration of all branches through T of order-type ω_1. Define $\mathbb{P}_0 = \mathrm{Add}(\omega) * \mathrm{Col}(\omega_1, \omega_2)$, where $\mathrm{Add}(\omega)$ is the countable poset to add a Cohen real. Pick a $\mathbb{P}_0$-name $\dot{h}$ for a function $h : \omega_1 \longrightarrow \omega_2$ that is cofinal in ω_2. In the model $V^{\mathbb{P}_0}$, T is a tree of height ω_1 and size $\aleph_1$. It is important to note that T has $\aleph_1$-many branches in the extension because $\mathbb{P}_0$ does not add any new cofinal branches through T:

Claim 5.3.1 *Assume that T is a tree of uncountable height and $\mathbb{Q}$ is σ-closed in $V^{\mathrm{Add}(\omega)}$. Then every cofinal branch through T in $V^{\mathrm{Add}(\omega)*\mathbb{Q}}$ is already in V.*

Claim 5.3.1 was exploited for the first time in [6] and pointed out later by various people. Continuing our proof, we know from Baumgartner's argument in [1] that now there is a 1-1 mapping $g : \mathfrak{B}_T \longrightarrow T$ that assigns to every branch a point in that branch. Define the subtree $S = \{x \in T : (\forall b \in \mathfrak{B}_T)\,(x \in b \rightarrow x \preccurlyeq_T g(b))\}$.

Claim 5.3.2 *S has no uncountable branches.*

Proof. Assume b is an uncountable branch through T. Then any $x \in b$ is not in S whenever $g(b) \prec_T x$. □

In the ground model there are $\mathbb{P}_0$-names $\dot{g}$ and $\dot{S}$ for the function g and the subtree S, respectively.

Now define $\mathbb{P} = \mathbb{P}_0 * \mathrm{Spec}(S)$, where $\mathrm{Spec}(S)$ is specializing the tree S, i. e. this forcing introduces a function $f : S \longrightarrow \omega$ such that $f(x) \neq f(x')$ for all comparable $x \neq x'$. As before, let $\dot{f}$ be a $\mathbb{P}$-name for this function.

Claim 5.3.3 $\mathbb{P}$ *is proper.*

Proof. Note that $\mathbb{P}$ is of type *ccc* $*$ (σ-closed) $*$ *ccc*. □

We want to apply PFA now. It turns out that the dense sets have to be chosen very carefully for this. Let us pick a filter $G \subseteq \mathbb{P}$ which is generic for the following dense subsets of $\mathbb{P}$, where $\zeta,\ \xi < \omega_1$:

$$\begin{aligned}
D_\xi &= \{p \in \mathbb{P} : \text{there exists } \beta < \omega_2 \text{ such that } p \Vdash \dot{h}(\xi) = \beta\},\\
E_{\zeta\xi} &= \{p \in \mathbb{P} : \text{there exists } x \in T \text{ such that } p \Vdash \dot{g}(b_{\langle \dot{h}(\zeta), \xi\rangle}) = x\},\\
F_{\zeta\xi} &= \{p \in \mathbb{P} : \text{there exists } n \in \mathbb{N} \text{ such that } p \Vdash \dot{f}(\langle \dot{h}(\zeta), \xi\rangle) = n \text{ or}\\
&\qquad \text{there are } \alpha,\ \beta < \omega_1 \text{ such that } p \Vdash \langle \dot{h}(\zeta), \xi\rangle \in b_{\langle \dot{h}(\alpha), \beta\rangle} \mathbin{\&} \dot{g}(b_{\langle \dot{h}(\alpha), \beta\rangle}) \prec_T \langle \dot{h}(\zeta), \xi\rangle\},\\
H_{\zeta\xi} &= \{p \in \mathbb{P} : \text{there are } \alpha,\ \beta < \omega_1 \text{ such that } p \Vdash b_{\langle \dot{h}(\zeta), \xi\rangle} \subseteq \langle \dot{h}(\alpha), \beta\rangle\}.
\end{aligned}$$

Once we have the filter G we can define approximations of $h,\ g,\ S$ and f in the ground model. Let $\bar{h} : \omega_1 \longrightarrow \delta$ and $\bar{g} : \{b_\xi\}_{\xi<\delta} \longrightarrow T$ be defined by

$$\bar{h}(\xi) = \gamma \ \text{ iff }\ (\exists p \in G)\, p \Vdash \dot{h}(\xi) = \gamma, \qquad \bar{g}(b_\gamma) = x \ \text{ iff }\ (\exists p \in G)\, p \Vdash \dot{g}(b_\gamma) = x,$$

where $\delta = \sup_{\xi<\omega_1} \bar{h}(\xi)$ is an ordinal of cofinality ω_1. Now define

$$\bar{S} = \{x \in T : (\forall \gamma < \delta)\, (x \in b_\gamma \rightarrow x \preccurlyeq_T \bar{g}(b_\gamma))\}$$

and $\bar{f} : \bar{S} \longrightarrow \omega$ by $\bar{f}(x) = n$ iff there exist $p \in G$ such that $p \Vdash \dot{f}(x) = n$. This is the reason we included the dense sets $F_{\zeta\xi}$ in the list: if G does not decide the $\dot{f}$-value of x, then G must contain a condition that forces x to be outside of $\bar{S}$. That way, $\bar{f}$ is defined on all of $\bar{S}$.

Claim 5.3.4 $\bar{f}$ *has the property that* $\bar{f}(x) \neq \bar{f}(x')$ *for all* $x \neq x' \in \bar{S}$ *that are comparable.*

Proof. Let $x,\ x' \in \bar{S}$ be comparable. Assume that $p \Vdash \dot{f}(x) = n$ and $p' \Vdash \dot{f}(x') = n'$, where $p,\ p'$ are in G. But G is a filter, so there is $s \in G$ such that $s \leq_{\mathbb{P}} p,\ p'$. Then $s \Vdash \dot{f}(x) = n \mathbin{\&} \dot{f}(x') = n'$, so $n \neq n'$ must hold and $\bar{f}(x) = n \neq n' = \bar{f}(x')$. □

Claim 5.3.5 *The function* $\bar{g} : \{b_\xi\}_{\xi<\delta} \longrightarrow T$ *is* 1–1.

Proof. Similar to the previous Claim. □

Claim 5.3.6 *All branches in* $\{b_\xi\}_{\xi<\delta}$ *are bounded below* δ*, i. e. if* $\xi < \delta$*, then there is* $\alpha < \delta$ *such that* $b_\xi \subseteq \alpha$.

Proof. This is by choice of the dense sets $H_{\zeta\xi}$, $\zeta,\ \xi < \omega_1$. □

Now fix an ω_1-branch b through T that converges to δ. There are no uncountable branches through $\bar{S}$ by Claim 5.3.4, so let $\eta < \omega_1$ be the minimal ordinal such that $b \restriction \eta \notin \bar{S}$. Define a regressive function $d : \omega_1 \setminus \eta \longrightarrow \omega_1$ by letting $d(\alpha) = \mathrm{ht}(\bar{g}(b_{\xi(\alpha)}))$, where $\xi(\alpha) < \delta$ is such that $b \restriction \alpha \in b_{\xi(\alpha)}$ and $\bar{g}(b_{\xi(\alpha)}) \prec_T b \restriction \alpha$. Note that there is some such ordinal by the definition of $\bar{S}$. We Press Down and get a stationary $E \subseteq \omega_1$ with the property that $d(\alpha) = \gamma_0$ for all $\alpha \in E$. $\bar{g}$ is 1–1 by Claim 5.3.5, so it is impossible that $\xi(\alpha) \neq \xi(\beta)$ for $\alpha,\ \beta \in E$. So let $\xi(\alpha) = \xi$ be fixed for all $\alpha \in E$. But since $b \restriction \alpha \in b_\xi$ for cofinally many α's, the branch b must be b_ξ. So b is a branch in $\{b_\xi\}_{\xi<\delta}$ that converges to δ. But this contradicts Claim 5.3.6. □

6 Preservation of PFA

Finally, we would like to point out that an ω_2-closed forcing can not do anything to PFA. This should be compared with Theorem 5.3: that theorem together with Remark 1.4 said that a strongly $(\omega_1{+}1)$-game-closed forcing might destroy PFA. The observations in this section strengthen over a result of Larson in [5], where it is proved that PFA and MM are preserved by ω_2-directed-closed posets.

Theorem 6.1 *Assume* PFA. *If* $\mathbb{P}$ *is* ω_2*-closed, then* $V^{\mathbb{P}} \models \mathrm{PFA}$.

Proof. Assume that $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ is proper", where $\mathbb{Q}$ is a $\mathbb{P}$-name for a partial ordering. Let $\dot{\tau}_\xi$, $\xi < \omega_1$, be a sequence of $\mathbb{P}$-names for dense subsets of $\mathbb{Q}$. In the generic extension by $\mathbb{P} * \mathbb{Q}$, let $G^* \subseteq \mathbb{P}$ be generic over V and define the poset $\mathbb{R}$ by $\mathbb{R} = \{\langle r_\alpha : \alpha \leq \gamma\rangle : r_\alpha \geq_{\mathbb{P}} r_\beta$ for $\alpha \leq \beta \leq \gamma$, $r_\alpha \in G^*$ for $\alpha \leq \gamma$, and $\gamma < \omega_1\}$, ordered by usual extension. Basically, $\mathbb{R}$ shoots a cofinal ω_1-sequence through the filter G^*. Namely, it follows from genericity that every condition in G^* is forced to have a refinement in the $\mathbb{R}$-generic sequence. We fix a $\mathbb{P} * \mathbb{Q} * \mathbb{R}$-name $\langle \dot{p}_\alpha : \alpha < \omega_1\rangle$ for the $\mathbb{R}$-generic object.

Claim 6.1.1 $\mathbb{R}$ *is forced to be* σ*-closed.*

Proof. This follows easily from two facts: Every countable sequence through G^* in $V^{\mathbb{P}*\mathbb{Q}}$ can be covered by a countable sequence in $V^{\mathbb{P}}$ since $\mathbb{Q}$ is proper. But G^* is closed under countable sequences in $V^{\mathbb{P}}$. □

So $\mathbb{P} * \mathbb{Q} * \mathbb{R}$ is proper. As in the proof of Theorem 4.1, we let $D_\xi = \{(p, q, r) \in \mathbb{P} * \mathbb{Q} * \mathbb{R} : p \Vdash_{\mathbb{P}} q \in \dot{\tau}_\xi\}$. Now apply MM in the ground model to the poset $\mathbb{P} * \mathbb{Q} * \mathbb{R}$: Let $M \prec \mathrm{H}_\theta$ be elementary such that M contains the countable ordinals and all objects so far. Then find a filter $K \subseteq \mathbb{P} * \mathbb{Q} * \mathbb{R}$ which is generic over M. We let $G \subseteq \mathbb{P}$ be the projection of K to the first coordinate. The projection to the second coordinate of K is a $\mathbb{P}$-name for a filter $H \subseteq \mathbb{Q}$. Define the descending sequence $\langle p_\alpha : \alpha < \omega_1\rangle$ by letting $p_\alpha = p$ iff there is $k \in K$ such that $k \Vdash \dot{p}_\alpha = p$. Note that every condition in G is refined in $\langle p_\alpha : \alpha < \omega_1\rangle$. But that sequence converges in $\mathbb{P}$, say $p' \leq_{\mathbb{P}} p_\alpha$ for $\alpha < \omega_1$. Then $p' \Vdash_{\mathbb{P}}$ "$H \subseteq \mathbb{Q}$ is generic for $\dot{\tau}_\xi$, $\xi < \omega_1$". □

The above argument can be carried out for PFA^+ as well.

Question 6.2 Is MM preserved by ω_2-closed forcings ? Note that the proof of Theorem 6.1 breaks down if for example $\mathbb{P}$ is a Cohen subset of ω_2 and $\mathbb{Q}$ is Namba forcing. In this case, $\mathbb{R}$ must collapse $\aleph_1$.

Acknowledgements Both authors would like to express their gratitude towards Matt Foreman and Stevo Todorčević for the knowledgeable support they provided during the work on this paper. Part of this research was done while the second author was visiting University of California, Irvine, with support from the Japanese Ministry of Education, Culture and Science. He would like to express his gratitude to the mathematical community at Irvine. The second author was partially supported by Grant-in-Aid for Scientific Research (No. 15540115), the Japanese Ministry of Education, Culture and Science.

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