

## Guessing clubs in the generalized club filter

by

**Bernhard König** (Toronto), **Paul Larson** (Oxford, OH)  
and **Yasuo Yoshinobu** (Nagoya)

**Abstract.** We present principles for guessing clubs in the generalized club filter on  $\mathcal{P}_\kappa\lambda$ . These principles are shown to be weaker than classical diamond principles but often serve as sufficient substitutes. One application is a new construction of a  $\lambda^+$ -Suslin-tree using assumptions different from previous constructions. The other application partly solves open problems regarding the cofinality of reflection points for stationary subsets of  $[\lambda]^{\aleph_0}$ .

**1. Introduction.** Club guessing principles have been studied intensely in the literature, a major source being [10]. But in all of these references, the guessing sequences anticipate clubs of ordinals. The purpose of this note is to introduce principles that guess clubs in the generalized club filter on  $\mathcal{P}_\kappa\lambda$ . Throughout the whole paper, the notion of a club always refers to the club filter that is generated by the sets

$$\mathcal{C}_f = \{x \in \mathcal{P}_\kappa\lambda : x \text{ is closed under } f\},$$

where  $f : <^\omega\lambda \rightarrow \lambda$ . Some references refer to this as “strongly club” as opposed to “Jech clubs” which are unbounded sets that are closed under chains of length less than  $\kappa$ . We generally prefer to write  $[\lambda]^{<\kappa}$  for  $\mathcal{P}_\kappa\lambda$ . We will also make the implicit assumption that  $\kappa$  and  $\lambda$  are regular cardinals and usually  $\kappa \leq \lambda$ .

Section 2 of this article contains the definition of and basic facts about  $\lambda^*$ , the newly introduced principle. We show in Section 3 that the guessing of clubs in the generalized club filter is a fairly weak assumption if the guessing attempts are on ordinals of small cofinality. For example,  $2^\lambda = \lambda^+$  would suffice to guarantee a variety of guessing principles for clubs in  $[\lambda^+]^{<\kappa}$ . In Section 4 we give an independence result using iterated forcing to demonstrate that even full GCH does not imply  $\lambda^*$  if the guessing attempts are

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made on ordinals of maximal cofinality. The last two sections deal with applications: in Section 5, a  $\lambda^+$ -Suslin-tree is constructed from GCH and a club guessing principle on ordinals of maximal cofinality. This is related to the old problem if GCH always constructs a Suslin tree on successors of regular cardinals. Finally, in Section 6 we show that guessing on ordinals of cofinality  $\omega$  can be used to thin out stationary subsets of  $[\omega_n]^{N_0}$  and thereby remove all possible reflection points with countable cofinality but preserving stationarity of the original set. Problems regarding the cofinality of reflection points for stationary subsets of  $[\lambda]^{N_0}$  have frequently been asked in the literature.

As additional guidelines for general set theory, we recommend the sources [6] and [8]. For more information about issues related to proper forcing and iterations, we suggest [11].

**2. The principle.** The following definition seems to be in the spirit of guessing clubs in the generalized club filter on  $[\lambda]^{<\kappa}$ .

1. DEFINITION. Let  $E \subseteq \lambda$  be stationary. Then  $\lambda^*(\kappa, E)$  is the statement that there is a sequence  $\langle \mathcal{F}_\delta : \delta \in E \rangle$  such that

- (1)  $\mathcal{F}_\delta$  is club in  $[\delta]^{<\kappa}$  for all  $\delta$  in  $E$ ,
- (2) for all clubs  $\mathcal{D} \subseteq [\lambda]^{<\kappa}$  there is a club  $C \subseteq \lambda$  such that for all  $\delta \in C \cap E$  we have  $\mathcal{F}_\delta \leq^* \mathcal{D}$ .

In (2),  $\mathcal{A} \leq^* \mathcal{B}$  means that there is  $x \in \mathcal{A}$  of size less than  $\kappa$  such that  $x \subseteq y \in \mathcal{B}$  implies  $y \in \mathcal{B}$  for all  $y$ .

We also say that  $\langle \mathcal{F}_\delta : \delta \in E \rangle$  is *tail club guessing*. Defining  $\mathcal{A} \leq^* \mathcal{B}$  like this seems to be the right notion for saying that “a tail of  $\mathcal{A}$  is included in  $\mathcal{B}$ ” in the context of subsets of  $[\lambda]^{<\kappa}$ . The cardinal  $\kappa$  is supposed to be clear from the context whenever we use this notation. Note that our new statement can be viewed as a  $\diamond^*$  spin-off <sup>(1)</sup>. We will show that  $\lambda^*(\kappa, E)$  is strictly weaker than  $\diamond^*(E)$ . The following facts help determining the status of  $\lambda^*(\kappa, E)$ :

2. LEMMA. *Let  $E \subseteq \lambda$  be stationary.*

- (1)  $\diamond^*(E)$  implies  $\lambda^*(\kappa, E)$  for all  $\kappa \leq \lambda$ .
- (2)  $\lambda^*(\kappa, E)$  is preserved by  $\gamma^+$ -cc forcings for any  $\gamma < \kappa$ .

*Proof.* For (1), let  $\diamond^*(E)$  guess all functions  $f : {}^{<\omega}\lambda \rightarrow \lambda$  via a sequence  $\langle \mathcal{S}_\delta : \delta \in E \rangle$ . Then each  $\mathcal{S}_\delta$  consists of fewer than  $\lambda$ -many functions

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<sup>(1)</sup> If  $E \subseteq \lambda$  then  $\diamond^*(E)$  means that there is a sequence  $\langle \mathcal{S}_\delta : \delta \in E \rangle$  where  $|\mathcal{S}_\delta| \leq |\delta|$  and such that for every  $S \subseteq \lambda$  there is a club  $C \subseteq \lambda$  such that for all  $\delta \in C \cap E$  we have  $S \cap \delta \in \mathcal{S}_\delta$ . Standard arguments show that  $\diamond^*(\lambda)$  holds in the constructible universe if and only if  $\lambda$  is not ineffable in  $L$  [1, p. 328].

$f_\xi^\delta : {}^{<\omega}\delta \rightarrow \delta$  ( $\xi < |\delta|$ ) that are guessing each  $f : {}^{<\omega}\lambda \rightarrow \lambda$  club many times. Now for each  $\delta \in E$  find a club  $\mathcal{F}_\delta$  in  $[\delta]^{<\kappa}$  such that  $y$  is closed under  $f_\xi^\delta$  whenever  $\xi \in y \in \mathcal{F}_\delta$ . One easily checks that this suffices.

(2) follows easily from the following fact:

2.1. CLAIM. *If  $\mathcal{D} \subseteq [\lambda]^{<\kappa}$  is a club in some  $\gamma^+$ -cc extension, then there is a club  $\mathcal{D}_0 \leq^* \mathcal{D}$  in the ground model.*

*Proof of Claim 2.1.* Let  $\dot{\mathcal{D}}$  be a name for a club in  $[\lambda]^{<\kappa}$  in the  $\gamma^+$ -cc extension. Then

$$\mathcal{D}_0 = \{x \in [\lambda]^{<\kappa} : \Vdash x \in \dot{\mathcal{D}}\}$$

is a club in  $V$ . Straightforward arguments will show that  $\mathcal{D}_0 \leq^* \mathcal{D}$  is witnessed by  $\gamma$ . ■

This completes the proof of Lemma 2. ■

Notice finally that both principles  $\lambda^*(\kappa, E)$  and  $\diamond^*(E)$  increase in logical strength as  $E$  gets bigger.

**3. Small cofinality.** Let us denote the set  $\{\gamma \in [\lambda, \lambda^+) : \omega \leq \text{cf}(\gamma) < \kappa\}$  by  $S_{\lambda^+}^{<\kappa}$ . The next theorem shows that  $\lambda^*(\kappa, S_{\lambda^+}^{<\kappa})$  is pretty weak in logical strength. When compared to Lemma 2(1), the assumptions needed here are far weaker than the previous  $\diamond^*(S_{\lambda^+}^{<\kappa})$ .

3. THEOREM. *Let  $\kappa \leq \lambda$ . The following are equivalent <sup>(2)</sup>:*

- (i)  $\lambda^*(\kappa, S_{\lambda^+}^{<\kappa})$ .
- (ii) *There is a club  $\mathcal{F} \subseteq [\lambda^+]^{<\kappa}$  such that for every club  $\mathcal{D} \subseteq [\lambda^+]^{<\kappa}$  there is a club  $C \subseteq \lambda^+$  such that for all  $\delta \in C \cap S_{\lambda^+}^{<\kappa}$  we have  $\mathcal{F} \cap [\delta]^{<\kappa} \leq^* \mathcal{D}$ .*
- (iii) *There is a collection of  $\lambda^+$ -many clubs in  $[\lambda]^{<\kappa}$  which is cofinal in the  $\geq^*$ -ordering.*

Considering (ii), it is an interesting fact that the witness for the principle  $\lambda^*(\kappa, S_{\lambda^+}^{<\kappa})$  can actually be taken to be a single club  $\mathcal{F}$  in  $[\lambda^+]^{<\kappa}$ . We still chose to formulate  $\lambda^*(\kappa, S_{\lambda^+}^{<\kappa})$  in the way given above because it is more in the style of classical guessing principles.

*Proof of Theorem 3.* It is straightforward to check that (ii) implies (i), so we show (iii)  $\Rightarrow$  (ii). To this end, suppose that  $\mathcal{C}_\alpha$  ( $\alpha < \lambda^+$ ) is a  $\geq^*$ -cofinal collection of clubs on  $[\lambda]^{<\kappa}$ . For each  $\eta \in [\lambda, \lambda^+)$  fix a bijection  $g_\eta : \lambda \rightarrow \eta$  and let

$$g_{\eta^*} : [\lambda]^{<\kappa} \rightarrow [\eta]^{<\kappa}$$

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<sup>(2)</sup> The global assumption that  $\lambda$  is regular is actually not necessary for this particular theorem.

be the induced bijection. If  $\lambda \leq \eta < \lambda^+$  and  $\alpha < \lambda^+$ , then we set  $\mathcal{C}_\alpha^\eta = g_{\eta^*} \mathcal{C}_\alpha$ . One checks that  $\mathcal{C}_\alpha^\eta$  ( $\alpha < \lambda^+$ ) is  $\geq^*$ -cofinal in the clubs on  $[\eta]^{<\kappa}$  for each  $\eta \in [\lambda, \lambda^+)$ . Now fix a bijection  $g : [\lambda, \lambda^+) \times \lambda^+ \rightarrow \lambda^+$  and let

$$F_{g(\eta, \alpha)} = \{x \in [\lambda^+]^{<\kappa} : x \cap \eta \in \mathcal{C}_\alpha^\eta\}$$

for each  $(\eta, \alpha) \in [\lambda, \lambda^+) \times \lambda^+$ . Note that  $F_\gamma$  is club in  $[\lambda^+]^{<\kappa}$  for each  $\gamma < \lambda^+$ . Define

$$\mathcal{F} = \bigtriangleup_{\gamma < \lambda^+} F_\gamma = \{x \in [\lambda^+]^{<\kappa} : \forall \gamma \in x (x \cap \gamma \in F_\gamma)\}.$$

Clearly,  $\mathcal{F}$  is a club subset of  $[\lambda^+]^{<\kappa}$ . We show that this  $\mathcal{F}$  works: let  $\mathcal{D}$  be any club subset of  $[\lambda^+]^{<\kappa}$  and let  $f : {}^{<\omega}\lambda^+ \rightarrow \lambda^+$  generate  $\mathcal{D}$ . Then set

$$D = \{\eta \in [\lambda, \lambda^+) : \eta \text{ is closed under } f\}.$$

Note that  $D$  is a club subset of  $\lambda^+$ . For each  $\eta \in D$  there exists an  $h(\eta)$  such that  $\mathcal{C}_{h(\eta)}^\eta \leq^* \mathcal{D} \cap [\eta]^{<\kappa}$ . Let  $D'$  be the set of ordinals which are closed under both  $g$  and  $h$  and set  $D'' = S_{\lambda^+}^{<\kappa} \cap D' \cap \lim D$ . Note that  $D''$  is a relative club subset of  $S_{\lambda^+}^{<\kappa}$ . For each  $\delta \in D''$  there is an increasing sequence  $\delta_i$  ( $i < \text{cf}(\delta)$ ) of elements of  $D$  converging to  $\delta$ . For each  $i < \text{cf}(\delta)$  pick an  $x_i \in [\delta_i]^{<\kappa}$  which witnesses  $\mathcal{C}_{h(\delta_i)}^{\delta_i} \leq^* \mathcal{D} \cap [\delta_i]^{<\kappa}$ , i.e.  $x_i \subseteq y \in \mathcal{C}_{h(\delta_i)}^{\delta_i}$  implies  $y \in \mathcal{D}$ . Now let

$$x = \bigcup_{i < \text{cf}(\delta)} (x_i \cup \{g(\delta_i, h(\delta_i))\}),$$

and note that  $x \in [\delta]^{<\kappa}$ . Whenever  $x \subseteq y \in \mathcal{F} \cap [\delta]^{<\kappa}$  then we infer for each  $i < \text{cf}(\delta)$  that  $y \in F_{g(\delta_i, h(\delta_i))}$  and thus  $y \cap \delta_i \in \mathcal{C}_{h(\delta_i)}^{\delta_i}$ . But since  $x_i \subseteq y \cap \delta_i$  we deduce that  $y \cap \delta_i$  is in  $\mathcal{D}$  and therefore closed under  $f$ . Then clearly,  $y$  is closed under  $f$ . This shows that  $\mathcal{F} \cap [\delta]^{<\kappa} \leq^* \mathcal{D}$  witnessed by  $x$ .

Regarding (i) $\Rightarrow$ (iii), let  $\langle \mathcal{F}_\delta : \delta \in E \rangle$  be a  $\lambda^*(\kappa, S_{\lambda^+}^{<\kappa})$ -sequence and, for each  $\delta \in S_{\lambda^+}^{<\kappa}$ , fix an increasing ladder sequence  $\delta_i$  ( $i < \text{cf}(\delta)$ ) such that  $\lambda \leq \delta_0$ . We may assume without restriction that all  $x \in \mathcal{F}_\delta$  are closed under the bijections  $g_{\delta_i} : \lambda \rightarrow \delta_i$  ( $\delta_i \in x$ ) used above. Now define a sequence of clubs in  $[\lambda]^{<\kappa}$  by letting

$$\mathcal{C}_\delta = \{x \cap \lambda : x \in \mathcal{F}_\delta\}$$

for all  $\delta \in S_{\lambda^+}^{<\kappa}$ .

3.1. CLAIM.  $\mathcal{C}_\delta$  ( $\delta \in S_{\lambda^+}^{<\kappa}$ ) is  $\geq^*$ -cofinal in the clubs on  $[\lambda]^{<\kappa}$ .

*Proof of Claim 3.1.* If  $\mathcal{C} \subseteq [\lambda]^{<\kappa}$  is club, define

$$(3.1) \quad \mathcal{D} = \{y \in [\lambda^+]^{<\kappa} : y \cap \lambda \in \mathcal{C}\}.$$

Then find  $\delta \in S_{\lambda^+}^{<\kappa}$  such that  $\mathcal{F}_\delta \leq^* \mathcal{D}$  witnessed by some  $b \in [\delta]^{<\kappa}$ , i.e.

$$(3.2) \quad \forall x \ b \subseteq x \in \mathcal{F}_\delta \text{ implies } x \in \mathcal{D}.$$

Now set

$$(3.3) \quad a = \{\delta_i : i < \text{cf}(\delta)\} \cup \bigcup_{i < \text{cf}(\delta)} g_{\delta_i}^{-1} \text{''}(b \cap \delta_i).$$

We claim that  $\mathcal{C}_\delta \leq^* \mathcal{C}$  is witnessed by  $a$ . So assume there is an  $x \in \mathcal{F}_\delta$  such that  $a \subseteq x \cap \lambda$ . Then  $x$  is closed under the relevant bijections  $g_{\delta_i}$  for all  $i < \text{cf}(\delta)$ , hence  $b \subseteq x$ . By (3.2),  $x$  is in  $\mathcal{D}$  and therefore  $x \cap \lambda \in \mathcal{C}$ . This suffices. ■

By Claim 3.1, we have found a collection of size  $\lambda^+$  of clubs in  $[\lambda]^{<\kappa}$  that is cofinal in the  $\geq^*$ -ordering. This finishes the proof of Theorem 3. ■

We remark that the simple cardinal arithmetic  $2^\lambda = \lambda^+$  implies that there are  $\lambda^+$ -many clubs in  $[\lambda]^{<\kappa}$ , and the collection of all these generates the club filter and is therefore cofinal in the  $\geq^*$ -ordering. It will be shown in the next section of this article that tail club guessing principles defined on ordinals of higher cofinality are considerably stronger than that.

As to the possible failure of  $\lambda^*(\kappa, S_{\lambda^+}^{<\kappa})$ , if  $\lambda$  is regular one can add  $\lambda^{++}$ -many Cohen-subsets of  $\lambda$  to create a model in which the equivalent statements (i)–(iii) of Lemma 3 are false. We leave the details to the interested reader.

**4. Maximal cofinality.** We assume GCH throughout this section and remember that  $\kappa$  and  $\lambda$  are always assumed to be regular. Similar to previous notation, we denote the set  $\{\gamma < \lambda^+ : \text{cf}(\gamma) = \lambda\}$  by  $S_{\lambda^+}^\lambda$ . We want to investigate the status of  $\lambda^*(\kappa, S_{\lambda^+}^\lambda)$ : it will be shown that  $\lambda^*(\kappa, S_{\lambda^+}^\lambda)$  does not follow from GCH which means that it is much stronger than  $\lambda^*(\kappa, S_{\lambda^+}^{<\kappa})$  and cannot be characterized analogously to Theorem 3.

Let  $\langle \mathcal{F}_\delta : \delta \in S_{\lambda^+}^\lambda \rangle$  be a  $\lambda^*(\kappa, S_{\lambda^+}^\lambda)$ -sequence. Then define a forcing  $\mathcal{Q}_\mathcal{F}$  in the following way: conditions are functions  $f : {}^{<\omega}\gamma \rightarrow \gamma$ , where  $\gamma < \lambda^+$  is such that for all  $\beta \in S_{\lambda^+}^\lambda \cap (\gamma + 1)$  we have  $\mathcal{F}_\beta \not\leq^* \mathcal{C}_f$ , where  $\mathcal{C}_f$  is the club generated by  $f$ . The ordering on  $\mathcal{Q}_\mathcal{F}$  is reverse inclusion. Notice that a straightforward argument shows that every condition in  $\mathcal{Q}_\mathcal{F}$  can be extended to have arbitrarily large domain.

4. LEMMA.  $\mathcal{Q}_\mathcal{F}$  is  $\lambda$ -closed and  $\lambda^+$ -distributive.

*Proof.* The  $\lambda$ -closure should be clear, and to show  $\lambda^+$ -distributivity, let  $\dot{\tau}$  be a name for a  $\lambda$ -sequence of ordinals,  $g$  an arbitrary condition in  $\mathcal{Q}_\mathcal{F}$ , and  $M$  an elementary substructure of  $H_\theta$  for some sufficiently large regular  $\theta$  such that  $|M| = \lambda$  with  $\mathcal{F}, g, \dot{\tau} \in M$ ,  ${}^{<\lambda}M \subseteq M$ , and  $\delta = M \cap \lambda^+ \in S_{\lambda^+}^\lambda$ . Fix a sequence  $\langle \delta_i : i < \lambda \rangle$  such that  $\delta_i \nearrow \delta$  and let  $x_i$  ( $i < \lambda$ ) be an enumeration of  $[\delta]^{<\kappa}$ . Now build a descending sequence of conditions  $\langle f_i : i < \lambda \rangle$  of

$M \cap \mathcal{Q}_{\mathcal{F}}$  with  $f_0 \leq g$  such that for all  $i < \lambda$ ,

- (1)  ${}^{<\omega}\delta_i \subseteq \text{dom}(f_i)$ ,
- (2)  $f_i$  decides the value of  $\dot{\tau}$  at  $i$ ,
- (3) there is  $x_i \subseteq y \in \mathcal{F}_\delta$  such that  $y$  is not closed under  $f_i$ .

This construction can be carried out, in particular since  $\mathcal{Q}_{\mathcal{F}}$  is  $\lambda$ -closed. Now let  $p = \bigcup_{i < \lambda} f_i$ . Note that  $\text{dom}(p) = {}^{<\omega}\delta$  by (1). But then  $p$  is a condition below  $g$  by (3) and decides  $\dot{\tau}$  by (2). ■

Given Lemma 4 and the remark before this lemma, it follows that a generic filter  $G \subseteq \mathcal{Q}_{\mathcal{F}}$  adds a club  $\mathcal{D}_G \subseteq [\lambda^+]^{<\kappa}$  whose existence destroys the tail club guessing properties of  $\langle \mathcal{F}_\delta : \delta \in S_{\lambda^+}^\lambda \rangle$ . The rest of the section depends heavily on iteration lemmas from [13], so we would like to remind the reader of some definitions from that paper.

5. DEFINITION. If  $S \subseteq S_{\lambda^+}^\lambda$  is stationary then a substructure  $M \prec \mathbb{H}_\theta$  of size  $\lambda$  will be called *S-good* whenever  $M \cap \lambda^+ \in S$  and  $M$  is closed under sequences of length less than  $\lambda$ . A  $\lambda$ -closed forcing notion  $\mathcal{Q}$  is called *strongly S-complete* if for all but non-stationarily many *S-good* structures  $M$ , every  $(M, \mathcal{Q})$ -generic sequence of conditions  $\langle q_\xi : \xi < \lambda \rangle$  has a lower bound in  $\mathcal{Q}$ .

We chose not to reproduce Shelah's notions in the most general form. Our presentation here is basically a special case of the machinery in [13]. The same holds for future definitions.

6. LEMMA. *Strongly S-complete forcings are  $\lambda^+$ -distributive and preserve stationary subsets of  $S$ .* ■

7. REMARK. Let  $E \subseteq S_{\lambda^+}^\lambda$  be stationary. The poset  $\mathcal{Q}_{\mathcal{F}}(E)$  is as above but with the weakened requirement that only for all  $\beta \in E \cap (\gamma + 1)$  we have  $\mathcal{F}_\beta \not\leq^* \mathcal{C}_f$ . Then  $\mathcal{Q}_{\mathcal{F}}(E)$  is again  $\lambda$ -closed and  $\lambda^+$ -distributive. Moreover, if we let  $S = S_{\lambda^+}^\lambda \setminus E$  then  $\mathcal{Q}_{\mathcal{F}}(E)$  is strongly *S-complete*.

*Proof.* The proof for  $\lambda$ -closure and  $\lambda^+$ -distributivity is as before in Lemma 4. So we only have to show that  $\mathcal{Q}_{\mathcal{F}}(E)$  is strongly *S-complete*. To this end, let  $N \prec \mathbb{H}_\theta$  be an *S-good* elementary substructure of size  $\lambda$  such that  $\delta = N \cap \lambda^+ \in S$  and let  $\langle p_i : i < \lambda \rangle$  be a  $\mathcal{Q}_{\mathcal{F}}(E)$ -generic sequence over  $N$ . We define  $p^* = \bigcup_{i < \lambda} p_i$ . Then  $p^*$  is clearly a condition in  $\mathcal{Q}_{\mathcal{F}}(E)$  since  $S$  and  $E$  are disjoint and therefore  $\delta \notin E$ . ■

8. DEFINITION. Let  $\theta$  be a sufficiently large regular cardinal. A continuous increasing sequence  $(N_i : i \leq \lambda)$  is called *S-suitable* if for all  $i \leq \lambda$ :

- (1)  $N_i \prec \mathbb{H}_\theta$  is of size  $\lambda$ ,
- (2)  $(N_\xi : \xi \leq i) \in N_{i+1}$  for  $i < \lambda$ ,

and there is a club  $X \subseteq \lambda$  such that for all  $i \in X$ ,

- (3)  $N_{i+1}$  is *S-good*.

9. DEFINITION. A forcing notion  $\mathcal{Q}$  is *really  $S$ -complete* if it is

- (1)  $\lambda$ -closed,
- (2) strongly  $S$ -complete, and
- (3) whenever  $\bar{N} = (N_i : i \leq \lambda)$  is  $S$ -suitable witnessed by the club  $X \subseteq \lambda$  and  $r \in N_0 \cap \mathcal{Q}$  then INC(omplete) does not have a winning strategy in the following game  $\mathcal{G}(\bar{N}, X, r)$  of length  $\lambda$ :

COM	$\zeta_0, p_0$	...	$\zeta_i, p_i$	...
INC	$\bar{q}_0$	...	$\bar{q}_i$	...

where for all  $i < \lambda$ ,

- (a)  $\zeta_i \in X$  and  $\zeta_i < \zeta_j$  for all  $i < j$ ,
- (b)  $p_i \in N_{\zeta_{i+1}} \cap \mathcal{Q}$ ,
- (c)  $\bar{q}_i = (q_i(\xi) : \xi < \lambda) \subseteq N_{\zeta_{i+1}} \cap \mathcal{Q}$  is  $\mathcal{Q}$ -generic over  $N_{\zeta_{i+1}}$ ,
- (d)  $\bar{q}_i \in N_{\zeta_{i+2}}$ ,
- (e)  $r \geq p_i \geq q_i(0)$ ,
- (f)  $q_i(\xi) \geq p_j$  for all  $\xi < \lambda$  and  $i < j$ .

The player COM(plete) wins the play of the game iff the sequence  $(p_i : i < \lambda)$  has a lower bound in  $\mathcal{Q}$ .

10. REMARK.

- (i) Recalling the definitions above, we see that the sequences  $\bar{q}_i$  played by INC will, modulo club, always have a lower bound in  $\mathcal{Q}$ . This is because  $\mathcal{Q}$  is strongly  $S$ -complete, the sequence  $\bar{q}_i$  is  $\mathcal{Q}$ -generic over  $N_{\zeta_{i+1}}$ , and  $N_{\zeta_{i+1}}$  is  $S$ -good.
- (ii) Standard arguments show that really  $S$ -complete forcings preserve all stationary subsets of  $\lambda^+$ .

The following theorem is one of the crucial iteration lemmas of [13] that will be used in the proof of Theorem 13.

11. THEOREM. *Let  $\langle \mathbb{P}_i, \tilde{\mathcal{Q}}_i : i < \gamma \rangle$  be a  $\lambda$ -support iteration such that for each  $i < \gamma$ ,*

$$\Vdash_{\mathbb{P}_i} \text{“}\tilde{\mathcal{Q}}_i \text{ is really } S\text{-complete.} \text{”}$$

*Then the forcing notion  $\mathbb{P}_\gamma$  is really  $S$ -complete. ■*

12. LEMMA. *Suppose  $E \subseteq S_{\lambda^+}^\lambda$  and  $S = S_{\lambda^+}^\lambda \setminus E$  are stationary. Then  $\mathcal{Q}_{\mathcal{F}}(E)$  is really  $S$ -complete.*

*Proof.* We already showed in Remark 7 that  $\mathcal{Q}_{\mathcal{F}}(E)$  is  $\lambda$ -closed and strongly  $S$ -complete. So we are left with showing (3) of Definition 9. To this end, let  $(N_i : i \leq \lambda)$  be an  $S$ -suitable sequence witnessed by the club  $X \subseteq \lambda$  and let  $r \in N_0 \cap \mathcal{Q}$ . We actually describe a winning strategy for player COM in the game  $\mathcal{G}(\bar{N}, X, r)$ .

We have no problem unless  $\delta = N_\lambda \cap \lambda^+ \in E$ . The following argument is similar to the proof of Lemma 4: player COM creates his strategy by fixing an enumeration  $x_i$  ( $i < \lambda$ ) of  $[\delta]^{<\kappa}$ . At stage  $i < \lambda$ , he picks  $\zeta_i \in X$  above all  $\zeta_\xi$ 's played so far with the additional requirements that there is  $y \in \mathcal{F}_\delta$  such that  $x_i \subseteq y$  and

$$(4.1) \quad y \not\subseteq \bigcup_{\xi < i} N_{\zeta_{\xi+1}}.$$

Then COM extends all conditions played so far to a  $p_i$  such that

$$(4.2) \quad p_i \Vdash \text{“}y \text{ is not closed under the generic } G\text{”}.$$

The requirement (4.1) guarantees that this can be carried out. But now  $(p_i : i < \lambda)$  has a lower bound in  $\mathcal{Q}$  since we stipulated (4.2) for unboundedly many  $y$ 's. ■

13. THEOREM. GCH does not imply  $\lambda^*(\kappa, S_{\lambda^+}^\lambda)$  for any  $\aleph_0 < \kappa \leq \lambda$ .

*Proof.* Start with a model of GCH and fix two stationary sets  $E, S \subseteq S_{\lambda^+}^\lambda$  such that  $E \cap S = \emptyset$ . Now define a  $\lambda$ -support iteration of length  $\lambda^{++}$ . In each step  $i < \lambda^{++}$ , we force with  $\tilde{\mathbb{Q}}_{\mathcal{F}^i}(E)$  to deal with a guessing sequence of the form  $\mathcal{F}^i = \langle \mathcal{F}_\delta^i : \delta \in S_{\lambda^+}^\lambda \rangle$  that is given to us by a book-keeping device. Note that this will destroy the guessing properties of  $\langle \mathcal{F}_\delta^i : \delta \in S_{\lambda^+}^\lambda \rangle$ . Remember also that each  $\tilde{\mathbb{Q}}_{\mathcal{F}^i}(E)$  is forced to be really  $S$ -complete by Lemma 12. We have thus defined a  $\lambda$ -support iteration of the form

$$\langle \mathbb{P}_i, \tilde{\mathbb{Q}}_{\mathcal{F}^i}(E) : i < \lambda^{++} \rangle,$$

so by Theorem 11 we conclude that  $\mathcal{P} = \mathcal{P}_{\lambda^{++}}$  is really  $S$ -complete. If we consult Lemma 6 and Remark 10(ii), this is enough to make sure that  $\mathcal{P}$  is  $\lambda^+$ -distributive and that all stationary subsets of  $\lambda^+$  are preserved. Finally,  $\mathcal{P}$  has the  $\lambda^{++}$ -chain condition by the standard line of reasoning using the fact that every iterand has size  $\leq \lambda^+$ . The  $\lambda^{++}$ -chain condition ensures, as in standard arguments, that every sequence  $\langle \mathcal{F}_\delta : \delta \in S_{\lambda^+}^\lambda \rangle$  appears at some initial step of the iteration, so that every potential sequence is finally taken care of. With the properties just mentioned, it is also easy to see that  $\mathcal{P}$  preserves GCH. This finishes the proof. ■

The authors do not know if GCH is consistent with the failure of  $\lambda^*(\kappa, S)$  for every stationary  $S \subseteq S_{\lambda^+}^\lambda$  since the methods presented here cannot settle this question. It is conjectured though that the older (and more involved) Shelah techniques of [14] can be applied to show that the above is actually consistent.

**5. Suslin trees.** Let us turn to applications of the club guessing principles presented above. We give an interesting application of  $\lambda^*$ -sequences defined on the set  $S_{\lambda^+}^\lambda$ . The following theorem of Shelah from [10, p. 126]

is needed, where  $\text{nacc}(A)$  is the set of all non-accumulation points of  $A$ , i.e. the set  $A \setminus \lim(A)$ .

14. THEOREM. *Assume that  $\lambda$  is regular uncountable. There exists a sequence  $(C_\alpha : \alpha \in S)$  such that*

- (1)  $S \subseteq S_{\lambda^+}^\lambda$  is stationary,
- (2)  $C_\alpha \subseteq \alpha$  is a club of order-type  $\lambda$ ,
- (3)  $\text{nacc}(C_\alpha) \subseteq S_{\lambda^+}^\lambda$ ,
- (4) if  $D \subseteq \lambda^+$  is club then there is  $\alpha \in S$  such that

$$\sup(D \cap \text{nacc}(C_\alpha)) = \alpha. \blacksquare$$

The application we present is the construction of a  $\lambda^+$ -Suslin tree from GCH and  $\lambda^*(\lambda, S_{\lambda^+}^\lambda)$ . Note that the consistency of GCH + “no  $\omega_2$ -Suslin trees” is still an open question. The following construction originally raised hopes that GCH actually does imply the existence of an  $\omega_2$ -Suslin tree. But in light of Theorem 13, this old question is now more open than ever. It should be mentioned that Jensen [7] was the first to construct a  $\lambda^+$ -Suslin tree in a similar fashion but using stronger square and guessing principles in the constructible universe.

15. THEOREM. *If  $\lambda$  is regular uncountable then*

$$2^{<\lambda} = \lambda + 2^\lambda = \lambda^+ + \lambda^*(\lambda, S_{\lambda^+}^\lambda)$$

*implies the existence of a  $\lambda$ -closed  $\lambda^+$ -Suslin tree.*

*Proof.* Fix enumerations  $\mathcal{P}(\alpha) = \{W_\gamma^\alpha\}_{\gamma < \lambda^+}$  for all  $\alpha < \lambda^+$  and then define  $W_X^\alpha = \{W_\gamma^\alpha\}_{\gamma \in X}$ . Let  $(C_\alpha : \alpha \in S)$  be as in Theorem 14, where we enumerate  $C_\alpha = \{\alpha_\xi\}_{\xi < \lambda}$  in increasing order. Furthermore, take  $\langle \mathcal{F}_\delta : \delta \in S_{\lambda^+}^\lambda \rangle$  to be  $\lambda^*(\lambda, S_{\lambda^+}^\lambda)$ -guessing. Remember that  $\mathcal{F}_\delta$  is club in  $[\delta]^{<\lambda}$ , so  $\mathcal{F}_\delta$  has a  $\subseteq$ -cofinal subfamily  $\mathcal{G}_\delta = \{G_\delta^\xi : \xi < \lambda\}$  such that

- $G_\delta^\xi \in [\delta]^{<\lambda}$  for all  $\xi < \lambda$ ,
- the sequence  $(G_\delta^\xi : \xi < \lambda)$  is continuously  $\subseteq$ -increasing,
- $\alpha_{\xi+1} \in G_\alpha^{\xi+1}$ .

Now construct a binary  $\lambda^+$ -tree  $T$  by induction on the levels so that the following holds for every  $\alpha < \lambda^+$ :

$$(5.1) \quad \text{For every } x \in T_{<\alpha} \text{ there is } y \in T_\alpha \text{ such that } x <_T y,$$

and simultaneously carry along an enumeration of  $T$  in the usual way. To start, let  $T_0 = \lambda$ . Once  $T_{\leq \alpha}$  is constructed, let every  $x \in T_\alpha$  have exactly two successors at level  $\alpha + 1$ . If  $\text{cf}(\alpha) < \lambda$  then  $T_\alpha$  extends all branches through  $T_{<\alpha}$ , and if  $\alpha \in S_{\lambda^+}^\lambda \setminus S$  then choose any normal extension  $T_\alpha$  of size  $\lambda$ . If  $\alpha \in S$  then for any  $x \in T_{<\alpha}$  we will construct a branch  $b_x$  through  $x$  and cofinal in  $T_{<\alpha}$ : first pick  $\xi_0 < \lambda$  such that  $\alpha_{\xi_0}$  is larger

than the height of  $x$ , and pick  $x_{\xi_0} \in T_{\alpha_{\xi_0}}$  above  $x$ . Now by induction on  $\xi \in [\xi_0, \lambda)$  we will construct  $x_\xi \in T_{\alpha_\xi}$  as follows: if  $x_\xi$  is constructed, then pick  $x_{\xi+1} \in T_{\alpha_{\xi+1}}$  above  $x_\xi$  so that the branch determined by  $x_{\xi+1}$  intersects with all  $A \in W_{G_\alpha^{\xi+1}}^{\alpha_{\xi+1}}$  that are maximal antichains in  $T_{<\alpha_{\xi+1}}$ . Note that this is possible because  $\alpha_{\xi+1}$  has cofinality  $\lambda$  and we are diagonalizing through fewer than  $\lambda$ -many antichains. If  $\xi < \lambda$  is a limit, then let  $x_\xi$  be the limit of  $\{x_\eta\}_{\xi_0 \leq \eta < \xi}$  in  $T_{<\alpha}$ . At the end of the day, let  $b_x$  be the  $\alpha$ -branch determined by the sequence  $\langle x_\xi \rangle_{\xi_0 \leq \xi < \lambda}$ .

Now let  $T_\alpha = \{b_x : x \in T_{<\alpha}\}$  and  $T = \bigcup_{\alpha < \lambda^+} T_\alpha$ . Note that by an easy inductive argument, (5.1) was achieved at all levels. The following claim will finish the proof.

15.1. CLAIM. *T has no antichains of size  $\lambda^+$ .*

*Proof of Claim 15.1.* Assume that  $\mathcal{A}$  is a maximal antichain of size  $\lambda^+$  and let the function  $f : \lambda^+ \rightarrow \lambda^+$  be defined by  $f(\alpha) = \gamma$  iff  $\mathcal{A} \cap \alpha = W_\gamma^\alpha$ . Then define the club

$$D = \{\delta < \lambda^+ : \mathcal{A} \cap \delta \text{ is a maximal antichain in } T_{<\delta} \text{ and } T_{<\delta} = \delta\}.$$

Now we can use both guessing sequences simultaneously to find  $\delta \in S$  such that

$$(5.2) \quad \sup(D \cap \text{nacc}(C_\delta)) = \delta,$$

$$(5.3) \quad \text{each set in a tail of } \mathcal{G}_\delta \text{ is closed under } f.$$

If  $b \in T_\delta$  then by construction there is  $x \in T_{<\delta}$  such that  $b = b_x$ . In view of (5.2) and (5.3), there is  $\xi < \lambda$  such that

- $\text{ht}(x) < \delta_{\xi+1}$ ,
- $\delta_{\xi+1} \in D$ ,
- $G_\delta^{\xi+1}$  is closed under  $f$ .

This means, again by construction, that  $b_x \cap \mathcal{A} \cap \delta_{\xi+1} \neq \emptyset$ , since

$$\mathcal{A} \cap \delta_{\xi+1} \in W_{G_\delta^{\xi+1}}^{\delta_{\xi+1}}$$

as  $\delta_{\xi+1} \in G_\delta^{\xi+1}$  and  $G_\delta^{\xi+1}$  is closed under  $f$ . So  $b_x \cap \mathcal{A} \neq \emptyset$ , which shows that  $\mathcal{A}$  is sealed off at  $\delta$ . ■

This completes the proof of Theorem 15. ■

**6. Stationary reflection.** We would like to shift the attention to the following question which was asked in [2], [3], [4], and other places in the literature. This question has been formulated in various ways, but the basic problem reads:

16. QUESTION. *Are the following equivalent for a regular  $\lambda$ ?*

- *Every stationary  $\mathcal{E} \subseteq [\lambda]^{\aleph_0}$  reflects to a set  $X$  of size  $\aleph_1$  containing all countable ordinals <sup>(3)</sup>.*
- *Every stationary  $\mathcal{E} \subseteq [\lambda]^{\aleph_0}$  reflects to a set  $X$  of size  $\aleph_1$  containing all countable ordinals with the additional property that  $\text{cf}(\text{otp } X) = \omega_1$ .*

Using the principle  $\lambda^*(\aleph_1, S_\lambda^\omega)$ , we can now shed new light on this question even though the most general case seems to be still open. Remember that  $S_\lambda^\omega$  denotes the collection of all  $\omega$ -cofinal ordinals below the cardinal  $\lambda$ .

17. DEFINITION. If  $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$  and  $x \in [\lambda]^{\aleph_0}$  then we define

$$\mathcal{B}(x) = \{y \in \mathcal{B} : x \subseteq y\},$$

the set of all supersets of  $x$  in  $\mathcal{B}$ . Furthermore, the union of all supersets of  $x$  in  $\mathcal{B}$ , i.e.

$$\bigcup \mathcal{B}(x) = \bigcup \{y \in \mathcal{B} : x \subseteq y\},$$

is said to be the  $\mathcal{B}$ -coverage of  $x$ .

We need the following sequence of lemmas.

18. LEMMA. *If  $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$  then we can partition  $\mathcal{B}$  into two sets  $\mathcal{B}^{(0)}$  and  $\mathcal{B}^{(1)}$  such that*

- (1)  $\mathcal{B}^{(0)}$  has no  $\subseteq$ -increasing chains of length  $\omega_1$ ,
- (2) every  $x \in \mathcal{B}^{(1)}$  has uncountable  $\mathcal{B}^{(1)}$ -coverage.

*Proof.* Given  $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$ , we iteratively remove all sets with countable coverage, i.e. define

$$\begin{aligned} \mathcal{B}_0 &= \mathcal{B}, \\ \mathcal{B}_{\xi+1} &= \{x \in \mathcal{B}_\xi : x \text{ has uncountable } \mathcal{B}_\xi\text{-coverage}\}, \\ \mathcal{B}_\xi &= \bigcap_{\zeta < \xi} \mathcal{B}_\zeta \quad \text{if } \xi \text{ is limit.} \end{aligned}$$

There must be an ordinal  $\infty$  such that  $\mathcal{B}_\infty = \mathcal{B}_{\infty+1}$ . Then set

$$\mathcal{B}^{(1)} = \mathcal{B}_\infty \quad \text{and} \quad \mathcal{B}^{(0)} = \mathcal{B} \setminus \mathcal{B}^{(1)}.$$

Clearly, every member of  $\mathcal{B}^{(1)}$  has uncountable  $\mathcal{B}^{(1)}$ -coverage because  $\mathcal{B}_\infty = \mathcal{B}_{\infty+1}$ . On the other hand it is straightforward to check that, by construction of the  $\mathcal{B}_\xi$ 's,  $\mathcal{B}^{(0)}$  does not contain any  $\subseteq$ -increasing chains of length  $\omega_1$ . ■

19. LEMMA. *Suppose  $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$  for some regular  $\lambda$  and  $x_\alpha$  ( $\alpha < \gamma$ ) is a (possibly incomplete) list of members of  $\mathcal{B}$ . For each  $\alpha < \gamma$ , assume*

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<sup>(3)</sup> Following [3], we say for short that  $\mathcal{E}$  reflects to  $X$ .

that  $\mathcal{A}_\alpha$  is a  $\subseteq$ -cofinal subset of  $\mathcal{B}(x_\alpha)$  that does not contain any continuous, increasing  $\subsetneq$ -chains of length  $\omega + 1$ . Define a sequence  $\mathcal{A}'_\alpha$  ( $\alpha < \gamma$ ) inductively:

$$\mathcal{A}'_\alpha = \{y \in \mathcal{A}_\alpha : \forall \xi < \alpha \forall x \in \mathcal{A}'_\xi \ y \not\subseteq x \text{ and } x \not\subseteq y\}.$$

Then  $\mathcal{A}' = \bigcup_{\alpha < \gamma} \mathcal{A}'_\alpha$  is cofinal in  $\bigcup_{\alpha < \gamma} \mathcal{B}(x_\alpha)$  and contains no continuous, increasing  $\subsetneq$ -chain of length  $\omega + 1$ .

*Proof.* The proof is by induction on  $\gamma$ . To check cofinality, consider an  $x \in \mathcal{B}(x_\alpha)$  for some  $\alpha < \gamma$ . Since  $\mathcal{A}_\alpha$  is  $\subseteq$ -cofinal in  $\mathcal{B}(x_\alpha)$ , we may assume that  $x \in \mathcal{A}_\alpha$ . Without loss of generality,  $x \notin \mathcal{A}'_\alpha$  and  $x$  is not contained in any member of  $\bigcup_{\xi < \alpha} \mathcal{A}'_\xi$ . So there must be an  $a \in \mathcal{A}'_\xi$  for some  $\xi < \alpha$  such that  $a \subset x$ . But note that by induction hypothesis,  $\bigcup_{\zeta < \alpha} \mathcal{A}'_\zeta$  is  $\subseteq$ -cofinal in  $\bigcup_{\zeta < \alpha} \mathcal{A}_\zeta$ . Note also that  $\mathcal{A}_\xi \cap \mathcal{B}(a)$  is  $\subseteq$ -cofinal in  $\mathcal{B}(a)$ . Thus, there must be  $y \in \bigcup_{\zeta < \alpha} \mathcal{A}'_\zeta$  such that  $x \subseteq y$ .

The lack of continuous, increasing  $\subsetneq$ -chains of length  $\omega + 1$  in  $\mathcal{A}'$  follows by construction, using the fact that no individual  $\mathcal{A}_\alpha$  contains such a sequence. ■

20. LEMMA. Assume CH and  $1 \leq n < \omega$ . Let  $\mathcal{B} \subseteq [\omega_n]^{\aleph_0}$  be such that every member of  $\mathcal{B}$  has uncountable  $\mathcal{B}$ -coverage. Then there is  $\mathcal{A} \subseteq \mathcal{B}$  which is  $\subseteq$ -cofinal in  $\mathcal{B}$  but contains no continuous, increasing  $\subsetneq$ -sequence of length  $\omega + 1$ .

*Proof.* We argue by induction on  $n$ . The following claim is crucial.

20.1. CLAIM. Assume that every  $y \in \mathcal{B}(x)$  has  $\mathcal{B}$ -coverage of cardinality  $\omega_n$ . Then there is  $\mathcal{A}(x)$  which is  $\subseteq$ -cofinal in  $\mathcal{B}(x)$  such that  $\mathcal{A}(x)$  does not contain any continuous, increasing  $\subsetneq$ -chains of length  $\omega + 1$ .

*Proof of Claim 20.1.* Enumerate  $\mathcal{B}(x) = \{x_\alpha : \alpha < \omega_n\}$ ; this uses CH. For each  $\alpha < \omega_n$ , choose  $y_\alpha \in \mathcal{B}(x_\alpha)$  such that

$$y_\alpha \not\subseteq \bigcup_{\xi < \alpha} y_\xi,$$

or in other words, every  $y_\alpha$  contains a brand new ordinal. Then the set  $\mathcal{A}(x) = \{y_\alpha : \alpha < \omega_n\}$  contains no continuous, increasing  $\subsetneq$ -chains of length  $\omega + 1$ . ■

So we have two cases in which we can thin out a set  $\mathcal{B}(x)$  to a  $\subseteq$ -cofinal  $\mathcal{A}(x)$  which contains no continuous, increasing  $\subsetneq$ -chains of length  $\omega + 1$ .

CASE 1: If every superset of  $x$  in  $\mathcal{B}$  has  $\mathcal{B}$ -coverage of cardinality  $\omega_n$ , this is by Claim 20.1.

CASE 2: If  $x$  has  $\mathcal{B}$ -coverage of cardinality less than  $\omega_n$ , this is by induction hypothesis.

It is straightforward to check that the set of  $x \in \mathcal{B}$  satisfying Case 1 or Case 2 is  $\subseteq$ -cofinal in  $\mathcal{B}$ . So we have a situation that allows us to apply Lemma 19. This finishes the proof of Lemma 20. ■

21. LEMMA. *Assume CH and  $1 \leq n < \omega$ . Then for every  $\mathcal{B} \subseteq [\omega_n]^{\aleph_0}$  there is an  $\mathcal{A} \subseteq \mathcal{B}$  such that  $\mathcal{A}$  is  $\subseteq$ -cofinal in  $\mathcal{B}$  and  $\mathcal{A}$  does not reflect to any set of size  $\aleph_1$ .*

*Proof.* First partition  $\mathcal{B}$  into two pieces as in Lemma 18. Then apply Lemma 20 to the piece  $\mathcal{B}^{(1)}$  to get an  $\mathcal{A}^{(1)} \subseteq \mathcal{B}^{(1)}$ . Now  $\mathcal{B}^{(0)} \cup \mathcal{A}^{(1)}$  will do the job. ■

22. THEOREM. *Assume CH and  $\lambda^*(\aleph_1, S_{\aleph_n}^\omega)$  for some  $2 \leq n < \omega$ . Then every stationary  $\mathcal{B} \subseteq [\omega_n]^{\aleph_0}$  can be refined to a stationary  $\mathcal{A} \subseteq \mathcal{B}$  with the property that  $\{x \in \mathcal{A} : \sup(x) = \gamma\}$  does not reflect to a set of size  $\aleph_1$  for all  $\gamma \in S_{\aleph_n}^\omega$ .*

*Proof.* Let  $\mathcal{F} \subseteq [\omega_n]^{\aleph_0}$  be as stated in Theorem 3(ii). We may assume that every element of  $\mathcal{F}$  has limit order type and therefore a supremum in  $S_{\aleph_n}^\omega$ . For each  $\eta \in S_{\aleph_n}^\omega$ , let  $\mathcal{F}^\eta = \{x \in \mathcal{F} : \sup(x) = \eta\}$ . If  $\mathcal{B} \subseteq [\omega_n]^{\aleph_0}$  is stationary, we may also assume that  $\mathcal{B} \subseteq \mathcal{F}$ . For every  $\eta \in S_{\aleph_n}^\omega$ , apply Lemma 21 to find a  $\subseteq$ -cofinal  $\mathcal{A}^\eta \subseteq \mathcal{B} \cap \mathcal{F}^\eta$  such that  $\mathcal{A}^\eta$  does not reflect to any set of size  $\aleph_1$ . Let

$$\mathcal{A} = \bigcup_{\eta \in S_{\aleph_n}^\omega} \mathcal{A}^\eta.$$

It suffices to show that  $\mathcal{A}$  is stationary, so pick a club  $\mathcal{D} \subseteq [\omega_n]^{\aleph_0}$ . We may assume that  $\mathcal{D} \subseteq \mathcal{F}$ . Let  $\mathcal{D}^*$  be the set of  $x \in [\omega_n]^{\aleph_0}$  containing increasing ordinals  $\langle \mu_i : i < \omega \rangle$  such that

- $\sup_{i < \omega} \mu_i = \sup x$ ,
- $\mathcal{D} \cap [\mu_i]^{\aleph_0}$  is club in  $[\mu_i]^{\aleph_0}$  for each  $i < \omega$ ,
- every element of  $\mathcal{F} \cap [\mu_i]^{\aleph_0}$  containing  $x \cap \mu_i$  is in  $\mathcal{D}$ .

It is easy to see that  $\mathcal{D}^*$  is club. Now pick  $x \in \mathcal{B} \cap \mathcal{D}^*$  and let  $\sup x = \eta$ . Then  $x \in \mathcal{B} \cap \mathcal{F}^\eta$ , and thus there is  $y \in \mathcal{A}^\eta$  containing  $x$ . By the definition of  $\mathcal{D}^*$  we have  $y \cap \mu_i \in \mathcal{D}$  for each  $i < \omega$ , and therefore  $y \in \mathcal{D} \cap \mathcal{A}$ . ■

23. COROLLARY. *Assume CH and  $2^{\aleph_{n-1}} = \aleph_n$  for some  $2 \leq n < \omega$ . If every stationary subset of  $[\omega_n]^{\aleph_0}$  reflects, then every stationary subset of  $[\omega_n]^{\aleph_0}$  reflects to a set of ordinals of uncountable order-type.*

*Proof.* Directly from Theorems 3 and 22. ■

If  $n = 2$  then we can get by without CH.

24. COROLLARY. *Assume  $2^{\aleph_1} = \aleph_2$ . If every stationary subset of  $[\omega_2]^{\aleph_0}$  reflects, then every stationary subset of  $[\omega_2]^{\aleph_0}$  reflects to a set of ordinals of uncountable order-type.*

*Proof.* Note that if  $\mathcal{E} \subseteq [\omega_2]^{\aleph_0}$  is stationary, we may assume that

$$E = \{\delta \in S_{\aleph_2}^{\aleph_0} : \mathcal{E} \cap [\delta]^{\aleph_0} \text{ is stationary in } [\delta]^{\aleph_0}\}$$

is stationary. Otherwise, throw away all elements of  $\mathcal{E}$  whose supremum is in  $E$  to reach the conclusion of the corollary.

So given that  $E$  is stationary, we can do without Lemmas 18–21: if  $\mathcal{E} \cap [\delta]^{\aleph_0}$  is stationary for some  $\delta < \omega_2$ , it is straightforward to find an unbounded subset thereof which contains no continuous, increasing  $\subsetneq$ -sequence of length  $\omega + 1$ . Now repeat the proof of Theorem 22. ■

So the outcome of this section is that the Generalized Continuum Hypothesis is enough to provide a partial answer to Question 16 but we were not able to push the method beyond  $\aleph_\omega$ . It is possible that similar arguments can be carried out for higher cardinals but there are some problems. For instance, Lemma 20 is false if we replace the cardinal  $\omega_n$  with  $\omega_\omega$ : assuming that  $\aleph_\omega$  is strong limit, Shelah [12] constructs a club subset of  $[\omega_\omega]^{\aleph_0}$  with the property that every unbounded subset of it is stationary.

To wrap things up, let us mention a result communicated to us by Donder which says that, in the constructible universe, a much better refinement than Theorem 22 is possible. This is the following fact which has also been established independently by Sakai [9].

25. THEOREM. *In  $L$ , for all uncountable regular  $\lambda$ , every stationary  $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$  can be refined to a stationary  $\mathcal{A} \subseteq \mathcal{B}$  such that the sup-function on  $\mathcal{A}$  is 1-1.*

The interested reader will find that the proof of Theorem 25 is pretty close to the usual condensation argument that  $\diamond_\lambda$  holds in  $L$ . Note that the conclusion of Theorem 25 is stronger than the conclusion of Theorem 22 but it also requires the stronger assumption of  $V = L$ . In [9], Matsubara and Sakai use a model of Gitik [5] to show that there can be an inaccessible cardinal  $\lambda$  and a stationary  $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$  such that  $\mathcal{B}$  cannot be thinned out to a stationary subset on which the sup-function is 1-1. But it seems to be open whether the same can happen if  $\lambda$  is a successor cardinal. In other words, the answer to the following question is not known:

26. QUESTION. *Let  $\lambda > \omega_1$  be a successor cardinal. Is it possible to prove in ZFC that every stationary  $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$  can be thinned out to a stationary  $\mathcal{A} \subseteq \mathcal{B}$  on which the sup-function is 1-1?*

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Department of Mathematics  
University of Toronto  
Toronto, Ontario, Canada M5S 2E4  
E-mail: bkoenig@math.toronto.edu

Department of Mathematics and Statistics  
Miami University  
Oxford, OH 45056, U.S.A.  
E-mail: larsonpb@muohio.edu

Graduate School of Information Science  
Nagoya University  
Furocho, Chikusa-ku  
Nagoya 464-8601, Japan  
E-mail: yosinobu@math.nagoya-u.ac.jp

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