Archive for Mathematical Logic manuscript No.
(will be inserted by the editor)

Bernhard König

Generic Compactness Reformulated

Received: date / Revised version: date – © Springer-Verlag 2003

Abstract. We point out a connection between reflection principles and generic large cardinals. One principle of pure reflection is introduced that is as strong as generic supercompactness of ω_2 by σ-closed forcing. This new concept implies CH and extends the reflection principles for stationary sets in a canonical way.

Introduction

Set theorists are familiar with the following well-known reflection of stationary sets called RP:

> For every set $\mathcal{E}$ stationary in $[\omega_2]^{\aleph_0}$ there is a continuous $\in$-chain $\langle M_\xi : \xi < \omega_1 \rangle$ of elementary submodels of some H_λ such that $\{\xi < \omega_1 : M_\xi \cap \omega_2 \in \mathcal{E}\}$ is stationary in ω_1.

This is one of the strongest principles to reflect stationary subsets of ω_2 and known to leave the power of the continuum undecided, which might be $\aleph_1$ or $\aleph_2$ under RP (see [1], [2], [4] and [17]). An equivalent version of RP is used in the literature sometimes: 'for every stationary $\mathcal{E} \subseteq [\omega_2]^{\aleph_0}$ there is an ω_1-cofinal ordinal $\delta < \omega_2$ such that $\mathcal{E} \cap [\delta]^{\aleph_0}$ is stationary in $[\delta]^{\aleph_0}$'.

Let us reformulate this principle in yet another way. We recall the *club-game* $\mathbb{G}^\delta(\mathcal{E})$ for an arbitrary set $\mathcal{E} \subseteq [\omega_2]^{\aleph_0}$: players I and II take turns in playing ordinals

I	α_0		α_2		α_4		$\dots$	
II		α_1		α_3		α_5		$\dots$

where all α_i's are below δ and I wins the game if $\{\alpha_i : i < \omega\} \in \mathcal{E}$.

This game is well known. We give a short proof of the following since there doesn't seem to be a reference for it:

Bernhard König: Department of Mathematics, 276 Multipurpose Science & Technology Bldg., University of California Irvine, Irvine CA 92697-3875, USA
e-mail: `bkoenig@math.uci.edu`

Key words or phrases: generic large cardinals, games, stationary reflection

Mathematics Subject Classification (2000): 03E50,03E55

Proposition 1. *Let δ be uncountable.*

(a) I has a winning strategy in $\mathbb{G}^\delta(\mathcal{E})$ iff $\mathcal{E} \cap [\delta]^{\aleph_0}$ is club,
(b) II has a winning strategy in $\mathbb{G}^\delta(\mathcal{E})$ iff $\mathcal{E} \cap [\delta]^{\aleph_0}$ is non-stationary.

Proof. (a) and (b) can be seen equivalent by switching the players. We will show (a): if I wins, we choose a winning strategy σ and N such that $\sigma \in N \prec \mathrm{H}_\lambda$. Now play against this strategy and let player II list all elements of $N \cap \delta$, no matter what I responds. Since player I wins, $N \cap \delta \in \mathcal{E}$. We just showed that $\{N \cap \delta : \sigma \in N \prec \mathrm{H}_\lambda\} \subseteq \mathcal{E}$, so $\mathcal{E} \cap [\delta]^{\aleph_0}$ is club.

Assuming $\mathcal{E} \cap [\delta]^{\aleph_0}$ is club, we define a winning strategy for player I inductively: if II plays the ordinal α_n in the nth move, player I can find $A_n \in \mathcal{E}$ that contains all α_i $(i \leq n)$ and such that $A_{n-1} \subseteq A_n$. Using a pairing function, he creates his strategy by making sure that he lists all elements of $\bigcup_{i<\omega} A_n$. This last set is in $\mathcal{E}$, so I wins the play. □

Now it is straightforward to check that RP is equivalent to the following:

> If there is an ω_1-club $C \subseteq \omega_2$ such that II has a winning strategy in the club-game $\mathbb{G}^\delta(\mathcal{E})$ for every $\delta \in C$, then II has a winning strategy in the club-game $\mathbb{G}^{\omega_2}(\mathcal{E})$.

Let us cut this short: RP says that club-games of the form $\mathbb{G}(\mathcal{E})$ are *reflected* at the cardinal ω_2. The following question is natural.

Question. *Is it possible that every game be reflected at ω_2?*

We will discover later in this article that a principle of full game reflection is tightly connected with the generic compactness of ω_2.

1. Preliminaries

We ventured deep into the topic right from the start, simply to catch the reader's attention. Let us define the basic notions that we used already. Our notations ${}^\delta\gamma$ for the set of all functions from δ into γ and ${}^{<\delta}\gamma$ for the set of all functions from ordinals smaller than δ into γ are common. We use the symbol $[A]^\lambda$ for the set of all subsets of A with cardinality λ, $[A]^{<\lambda}$ is defined analogously. $\mathfrak{P}(A)$ denotes the power set of A.

We call a set $C \subseteq \kappa$ *club* if it is closed and unbounded in κ. We define the notion of club for subsets of $[X]^\kappa$: $\mathcal{C}$ is *closed and unbounded in* $[X]^\kappa$ if

(i) for all $a \in [X]^\kappa$ there is a $b \in \mathcal{C}$ such that $a \subseteq b$,
(ii) whenever $\langle a_\xi : \xi < \kappa \rangle \subseteq \mathcal{C}$ is an increasing sequence, then $\bigcup_{\xi<\kappa} a_\xi \in \mathcal{C}$.

The above definition is common. Note that if $\kappa > \omega$, our notion of club no longer coincides with the property of being closed under an algebra of functions.

We confuse the notions of being club and containing a club. Furthermore, we need a more general version of clubs: a set of ordinals is λ-*closed* if it is closed under sequences of order-type λ. A λ-closed and unbounded set is called λ-*club*. The same generalization carries over to clubs in $[X]^\kappa$.

The string $\mathrm{lh}(s)$ is a notation for the length of a sequence s. We write $M \prec N$ to say that M is an elementary submodel of N. An increasing chain of models $\langle M_\xi : \xi < \omega_1 \rangle$ is an ϵ-*chain* if $M_\xi \in M_\eta$ for all $\xi < \eta < \omega_1$. The term $Sk(X)$ denotes the Skolem closure of the set X when it should be clear from the context which superstructure we are working in, usually some H_λ the collection of all sets hereditarily of size smaller than λ. Pretty often it will go without saying that H_λ is sufficiently large. Furthermore, we denote ideals by letters like $\mathcal{I}$ or $\mathcal{J}$, where $\mathcal{I}^+$ is the collection of all positive sets with respect to the ideal $\mathcal{I}$. NS_λ is the ideal of non-stationary subsets of λ, the subscript will be dropped if it is clear from the context. An ideal on κ is κ-*complete* if it closed under unions of size less than κ.

ZFC^- is ZFC minus Power set and we use an abbreviation in the context of elementary embeddings: $j : M \longrightarrow N$ means that j is a non-trivial elementary embedding from M into N such that M and N are transitive. The *critical point* of such an embedding, i.e. the first ordinal moved by j, is denoted by $\mathrm{cp}(j)$.

When it comes to forcing, we use the classical notation where $p \leq q$ means that p is a stronger condition than q. Names are denoted with dots on top (e.g. $\dot\tau$) but our notation shall not be too strict in this.

If λ is regular, we define the posets $Col(\lambda, \kappa)$ and $Coll(\lambda, < \kappa)$ to be the $(< \lambda)$-closed Levy Collapses of κ to λ and of everything less than κ to λ respectively:

$$Col(\lambda, \kappa) = \{p : \alpha \longrightarrow \kappa \,|\, \alpha < \lambda\}, \tag{1.1}$$

$$Coll(\lambda, < \kappa) = \{p \,|\, \mathrm{dom}(p) \subseteq \kappa \times \lambda,\ p(\delta, \alpha) < \delta \text{ and } p \text{ has cardinality less than } \lambda\}. \tag{1.2}$$

In both of these cases, the ordering is reverse inclusion.

Rado's conjecture is known as the following statement: a family of intervals of a linearly ordered set is the union of countably many disjoint subfamilies if and only if every subfamily of size $\aleph_1$ has this property.

Finally, we need the Extension Lemma which was first proved by Silver:

Lemma 1 (Extension Lemma). *Let $j : M \longrightarrow N$ and assume that G is $\mathbb{P}$-generic over M, H is $j(\mathbb{P})$-generic over N for a poset $\mathbb{P}$. If $j''G \subseteq H$ then there is a unique extension $j^* : M[G] \longrightarrow N[H]$ of j such that $j^*(G) = H$.*

Proof. For each $\mathbb{P}$-name $\dot\tau$, simply let $j^*(\dot\tau[G]) = j(\dot\tau)[H]$. $\square$

2. Reflecting games

Let's pick up on what was said in the beginning. We want to formulate our principle of game reflection using the ideas of the introduction. This is based upon a very general notion of a game.

In what is going to follow, θ stands for an arbitrary regular cardinal.

Definition 2. If $\mathcal{A} \subseteq {}^{<\omega_1}\theta$, the game $\mathbb{G}(\mathcal{A})$ has length ω_1 and is played as follows:

I	α_0	α_1	$\dots$	α_ξ	$\alpha_{\xi+1}$	$\dots$
II	β_0	β_1	$\dots$	β_ξ	$\beta_{\xi+1}$	$\dots$

both players I and II play ordinals below θ and

$$\text{II wins iff } \langle \alpha_\xi, \beta_\xi : \xi < \omega_1 \rangle \in [\mathcal{A}],$$

where $[\mathcal{A}] = \{ f \in {}^{\omega_1}\theta : f \restriction \xi \in \mathcal{A} \text{ for all } \xi < \omega_1 \}$.

For $B \subseteq \mathrm{H}_\lambda$, define the game $\mathbb{G}^B(\mathcal{A})$ by letting the winning conditions be the same as in $\mathbb{G}(\mathcal{A})$ but imposing the restriction on both players to play ordinals in $B \cap \theta$.

Definition 3. The *Game Reflection Principle* or GRP is the following statement:

> Let $\mathcal{A} \subseteq {}^{<\omega_1}\omega_2$. If there is an ω_1-club $C \subseteq \omega_2$ such that II has a winning strategy in $\mathbb{G}^\alpha(\mathcal{A})$ for every $\alpha \in C$, then II has a winning strategy in $\mathbb{G}(\mathcal{A})$.

The *global Game Reflection Principle* or GRP^+ is the following statement:

> Let θ be regular and $\mathcal{A} \subseteq {}^{<\omega_1}\theta$. If there is an ω_1-club $\mathcal{C} \subseteq [\theta]^{\aleph_1}$ such that II has a winning strategy in $\mathbb{G}^B(\mathcal{A})$ for every $B \in \mathcal{C}$, then II has a winning strategy in $\mathbb{G}(\mathcal{A})$.

The statement of GRP^+ for $\theta = \omega_2$ is just GRP.

Definition 4. A substructure $M \prec \mathrm{H}_\lambda$ of size $\aleph_1$ is called *ϵ-approachable* if it is the limit of an ϵ-chain of countable elementary substructures, i.e. there is an ϵ-chain $\langle M_\xi : \xi < \omega_1 \rangle$ with $M = \bigcup_{\xi<\omega_1} M_\xi$.

We denote the set of all ϵ-approachable substructures of H_λ of size $\aleph_1$ by EA.

Note.

- EA is ω_1-club in $[\mathrm{H}_\lambda]^{\aleph_1}$.
- $\mathrm{EA} \restriction \theta = \{ M \cap \theta : M \in \mathrm{EA} \}$ is ω_1-club in $[\theta]^{\aleph_1}$.
- $\mathrm{EA} \restriction \omega_2 = \{ M \cap \omega_2 : M \in \mathrm{EA} \}$ is ω_1-club in ω_2.

We use this Note crucially: it suffices to show that II wins $\mathbb{G}^M(\mathcal{A})$ for all $M \in \mathrm{EA}$ to apply the Game Reflection Principle.

It should be mentioned that every internally approachable M in the sense of [4] is ϵ-approachable.[1] The other direction might not be true in general as the referee pointed out, but we will show that these two notions, internally approachable and ϵ-approachable, are the same under CH:

[1] M of size $\aleph_1$ is called *internally approachable* if M is the limit of a sequence $\langle N_\xi : \xi < \omega_1 \rangle$ such that $\langle N_\xi : \xi < \eta \rangle \in M$ for every $\eta < \omega_1$. If M is like this, build an ϵ-chain by letting $M_{\alpha+1} = Sk(M_\alpha, \langle N_\xi : \xi \le \alpha \rangle)$.

Lemma 5. *Let $M \prec \mathrm{H}_\lambda$ be of size $\aleph_1$. The following are equivalent under* CH*:*

(1) M is ϵ-approachable,
(2) ${}^\omega M \subseteq M$.

Proof. (1) $\Longrightarrow$ (2): let M be the union of an ϵ-chain $\langle M_\xi : \xi < \omega_1 \rangle$. Now if $A \in [M]^{\aleph_0}$, there is $\zeta < \omega_1$ such that $A \subseteq M_\zeta$. But we are given that $\mathfrak{P}(M_\zeta) \in M_{\zeta+1} \subseteq M$ and $\mathfrak{P}(M_\zeta)$ has cardinality $\aleph_1$ by CH. Note that M contains all countable ordinals, so $A \in \mathfrak{P}(M_\zeta) \subseteq M$ and we are done.

(2) $\Longrightarrow$ (1): let $\{x_\alpha\}_{\alpha<\omega_1}$ enumerate M. Build an increasing ϵ-chain by letting

$$M_{\xi+1} = Sk(M_\xi, x_\xi).$$

The union of this chain will end up being exactly M since M is closed under countable sequences. □

Remark.

(a) Notice that both formulations of the Game Reflection Principle are completely false when we replace the notion of an ordinary strategy with the notion of a positional strategy.
(b) We are going to show in Theorem 8 that GRP implies the Continuum Hypothesis. We can hence assume by Lemma 5 that all ϵ-approachable substructures we consider are closed under countable sequences.

Let us note that there is a canonical ideal associated with GRP:

Definition 6. The *Game Reflection Ideal* $\mathcal{J}_{\mathrm{GR}}$ is defined as follows: for any $X \subseteq \omega_2$, let $X \in \mathcal{J}_{\mathrm{GR}}$ if

> there is $\mathcal{A} \subseteq {}^{<\omega_1}\omega_2$ such that II has no winning strategy for $\mathbb{G}(\mathcal{A})$, but II has a winning strategy for $\mathbb{G}^\alpha(\mathcal{A})$ whenever $\alpha \in X$.

The ideal $\mathcal{J}_{\mathrm{GR}}$ is non-trivial if and only if GRP holds.

Proposition 7. *$\mathcal{J}_{\mathrm{GR}}$ is normal.*

Proof. Assume that X_ν ($\nu < \omega_2$) is a sequence in $\mathcal{J}_{\mathrm{GR}}$ witnessed by the games $\mathcal{A}_\nu$ ($\nu < \omega_2$). Build a game $\mathcal{B} \subseteq {}^{<\omega_1}\omega_2$ by letting player II choose the index ν for the game $\mathcal{A}_\nu$ that he wants to play and then resume with player I's first move. II will then win the game $\mathbb{G}(\mathcal{B})$ iff he can win the upcoming $\mathcal{A}_\nu$-game. Note that II does not have a winning strategy for $\mathcal{B}$ since he has none for the games $\mathcal{A}_\nu$ ($\nu < \omega_2$). But now let $\alpha \in \nabla_{\nu<\omega_2} X_\nu$. In this case, there is $\nu^* < \alpha$ such that $\alpha \in X_{\nu^*}$. We claim that player II has a winning strategy in the game $\mathbb{G}^\alpha(\mathcal{B})$ if he initiates it by selecting ν^*: since $\mathcal{A}_{\nu^*}$ witnesses that X_{ν^*} is a member of the ideal and $\alpha \in X_{\nu^*}$, we conclude that there is a winning strategy for II in the game $\mathbb{G}^\alpha(\mathcal{A}_{\nu^*})$. Therefore, II wins $\mathbb{G}^\alpha(\mathcal{B})$, and $\mathcal{B}$ witnesses that $\nabla_{\nu<\omega_2} X_\nu \in \mathcal{J}_{\mathrm{GR}}$. □

We will now fulfill the promise given earlier in this section. Theorem 8 shows that the Π^1_1-reflection of GRP implies CH. This contrasts the well-known fact that the tree property for ω_2, which is just as well a Π^1_1-reflection, implies $\neg$CH.

Theorem 8. *The Game Reflection Principle implies the Continuum Hypothesis.*

Proof. Assume that $2^{\aleph_0} \geq \aleph_2$. With the Axiom of Choice we can construct a Bernstein set, i.e. a set $B \subseteq \mathbb{R}$ of size continuum that does not contain a perfect subset (see e.g. [9, p.48]). So in particular we get a set $A \subseteq \mathbb{R}$ of size $\aleph_2$ that does not contain a perfect subset. Enumerate $A = \{r_\beta : \beta < \omega_2\}$ and define the game $\mathbb{G}_{\mathrm{CH}}$ as follows: a typical play of this game is

I	α_0	α_1	α_2	α_3	$\dots$
II	i_0	i_1	i_2	i_3	$\dots$

where $\alpha_n < \omega_2$ and $i_n \in \{0,1\}$ $(n < \omega)$. We say that II wins the game if $\langle i_n : n < \omega \rangle = r_\beta \in A$ and $\beta > \sup_{n<\omega} \alpha_n$. Note that player II has a winning strategy in the game $\mathbb{G}^M_{\mathrm{CH}}$ for every internally approachable $M \prec \mathrm{H}_\lambda$ since he can simply play the real $r_{(M\cap\omega_2)}$. By the Game Reflection Principle, II has a winning strategy σ for the game $\mathbb{G}_{\mathrm{CH}}$. From this we deduce a contradiction: let us identify our winning strategy with a function $\sigma : {}^{<\omega}\omega_2 \longrightarrow {}^{<\omega}2$ such that $\mathrm{lh}(\boldsymbol{\zeta}) = \mathrm{lh}(\sigma(\boldsymbol{\zeta}))$. For every $s \in {}^{<\omega}2$, choose a sequence of ordinals $\boldsymbol{\zeta}_s$ such that

(i) $\boldsymbol{\zeta}_r \subseteq \boldsymbol{\zeta}_s$ whenever $r \subseteq s$ are members of ${}^{<\omega}2$,
(ii) $\mathrm{lh}(r) = \mathrm{lh}(s) \longrightarrow \mathrm{lh}(\boldsymbol{\zeta}_r) = \mathrm{lh}(\boldsymbol{\zeta}_s)$,
(iii) $\sigma(\boldsymbol{\zeta}_{s^\frown 0}) \neq \sigma(\boldsymbol{\zeta}_{s^\frown 1})$ holds for every $s \in {}^{<\omega}2$.

Note that the length of $\boldsymbol{\zeta}_s$ typically differs from the length of s. Such a sequence exists because σ is a winning strategy for player II. If there were no such splitting, player I could predict the final outcome of II's choices and beat this very real. But now we claim:

Claim. $S = \{\sigma(\boldsymbol{\zeta}_s) \upharpoonright n : s \in {}^{<\omega}2\,,\ n \leq \mathrm{lh}(\boldsymbol{\zeta}_s)\}$ is a perfect tree.

Proof. By condition (iii) in the construction of the sequences $\boldsymbol{\zeta}_s$, we know that for every element of S, there are two incomparable ones above. This is enough to prove our Claim. $\dashv$

But $[S]$ is a subset of A, since for every branch through S there is a play associated to it and moreover it is played according to II's winning strategy σ. So A contains a perfect subset by our claim. This finishes the proof, because $A \supseteq X$ was chosen to avoid at least one point in every perfect set. $\square$

The consistency of the Game Reflection Principle will be established in Corollary 18. We could go for this right now, but prefer to give an equivalent formulation first in Section 3 and make the proofs more transparent.

Our next result points further down the road: no winning strategies are added in ω-closed forcing extensions. See the importance of it in the proofs of Theorems 14 and 17.

Lemma 9. *Let $\mathcal{A} \subseteq {}^{<\omega_1}\theta$ for some regular θ and $\mathcal{A} \in V$. If $\mathbb{P}$ is ω-closed and $\dot{\sigma}$ a $\mathbb{P}$-name for a winning strategy in the game $\mathbb{G}(\mathcal{A})$, then there is a winning strategy τ for $\mathbb{G}(\mathcal{A})$ in V.*

Proof. For simplicity we assume that $\dot{\sigma}$ is a name for a winning strategy of player II. We construct a winning strategy τ for player II in the ground model, using ω-closedness of $\mathbb{P}$.

Define τ as follows: first choose a condition $p_\emptyset \Vdash \dot{\sigma}$ is winning. Now for all countable sequences of ordinals $s = \langle \alpha_\xi : \xi < \gamma \rangle$, find conditions $p_s \in \mathbb{P}$ such that $p_s \Vdash \dot{\sigma}(\alpha_\xi : \xi < \gamma) = \beta_s$ for some $\beta_s \in V$ and

$$s \subseteq s' \longrightarrow p_s \geq p_{s'}.$$

Finally, let $\tau(\alpha_\xi : \xi < \gamma) = \beta_s$. We show that τ is winning for player II: for if there is a play

I	α_0	α_1	$\dots$	α_ξ	$\alpha_{\xi+1}$	$\dots$
II	β_0	β_1	$\dots$	β_ξ	$\beta_{\xi+1}$	$\dots$

according to τ, we claim that II wins this play. Assume otherwise, then there is $\gamma < \omega_1$ and a sequence $s = \langle \alpha_\xi : \xi < \gamma \rangle$ such that $\langle \alpha_\xi, \beta_\xi : \xi < \gamma \rangle \notin \mathcal{A}$. Remember that we constructed our tree of conditions in a way such that $p_s \Vdash$ 'II wins the play α_ξ, β_ξ $(\xi < \gamma)$', since p_s extends $p_\emptyset$. We choose any $\mathbb{P}$-generic filter H in V that contains the condition p_s. This will make the following true:

$$V[H] \models \langle \alpha_\xi, \beta_\xi : \xi < \gamma \rangle \in \mathcal{A}. \tag{2.1}$$

But $\mathcal{A}$ is in the ground model, so we conclude

$$V \models \langle \alpha_\xi, \beta_\xi : \xi < \gamma \rangle \in \mathcal{A}. \tag{2.2}$$

We have reached a contradiction. This proves that τ is a winning strategy for II in V. $\square$

In the near future, we are sometimes going to play with arbitrary objects instead of ordinals. This is no restriction though, because only the cardinality of the underlying set matters: just fix any enumeration, make sure that it appears in all referred to structures and define the payoff-set relative to this enumeration. Note that we have to pay attention to this only in the case of the weaker GRP, where the underlying set is supposed to have cardinality at most $\aleph_2$.

3. Generic large cardinals

Now we shall prove a main result: we can characterize the new principle of Game Reflection in terms of generic embeddings. The project of axiomatizing mathematics with the help of generic large cardinals has recently been pursued by Cummings and Foreman (see [3] and [5]). These mathematical universes, that we are going to live in while proving Theorems 11,14 and 17, have not been as carefully axiomatized as the respective forcing and reflection axioms that go along with $\neg$CH. Examples of known axioms in this well-studied 'world of $\neg$CH' are SRP, SPFA or Woodin's (*) (see [18]).

Definition 10. We redefine the hierarchy of large cardinals modulo forcing extensions. The following properties can be true for smaller cardinals as well. Let κ be a cardinal and Γ a class of posets.

κ is *generically weak compact by* Γ if whenever the transitive structure $M \models \mathrm{ZFC}^-$ is of size κ with $\kappa \in M$, then there is $\mathbb{P} \in \Gamma$ such that the generic extension $V^{\mathbb{P}}$ supports $j : M \longrightarrow N$ with $\mathrm{cp}(j) = \kappa$.

κ is *generically supercompact by* Γ if for every regular λ there is $\mathbb{P} \in \Gamma$ such that $V^{\mathbb{P}}$ supports $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

It is usually the case that generic large cardinals have the same consistency strength as their classical counterparts. A famous exception to this rule has been the notion of generically almost huge though (see [11] and [4]). But it might not entail such logical strength if a cardinal κ is generically weak compact, supercompact, etc. by the class of all posets. From now on, we will restrict Γ to the class of all ω-closed posets. This turns out to have considerable impact on the combinatorics of the cardinal κ (see Section 4).

We have the well-known fact:

Theorem 11. *Let* $\mathbb{P} = Coll(\omega_1, < \kappa)$.

(a) If κ *is weakly compact then*

$$V^{\mathbb{P}} \models \omega_2 \textit{ is generically weak compact by } \omega\textit{-closed forcing.}$$

(b) If κ *is supercompact then*

$$V^{\mathbb{P}} \models \omega_2 \textit{ is generically supercompact by } \omega\textit{-closed forcing.}$$

Proof. **(a):** Assume that G is $\mathbb{P}$-generic. If in $V^{\mathbb{P}}$, $M^* \models \mathrm{ZFC}^-$ is of size κ and contains κ as an element, choose $M \models \mathrm{ZFC}^-$ of size κ in the ground model such that $M^* \subseteq M[G]$. This can be accomplished by taking the Skolem Hull of a big enough set of names. We may assume without restriction that $M^* = M[G]$. Now since κ is weakly compact, there is $j : M \longrightarrow N$ with $\mathrm{cp}(j) = \kappa$. Note that

$$j(\mathbb{P}) = \mathbb{P}_{j(\kappa)} = Coll(\omega_1, < j(\kappa)),$$

so we can identify $j(\mathbb{P}) \cong \mathbb{P} * \mathbb{S}$, where $\mathbb{S} = \mathit{Coll}(\omega_1, [\kappa, j(\kappa)))$. Let H be $\mathbb{S}$-generic over $V[G]$. Using Silver's argument (see Lemma 1), we can extend j to

$$j^* : M[G] \longrightarrow N[G * H].$$

This j^* exists in $V^{j(\mathbb{P})} = V^{\mathbb{P}*\mathbb{S}}$, but $\mathbb{S}$ is ω-closed, so we are done.

(b): This is an easy variation of (a). For any λ fix $j : V \longrightarrow M$ in the ground model such that $\mathrm{cp}(j) = \kappa$ and $j''\lambda \in M$. Just like before we can identify $j(\mathbb{P})$ with $\mathbb{P} * \mathbb{S}$, where $\mathbb{S} = \mathit{Coll}(\omega_1, [\kappa, j(\kappa)))$. If G is $\mathbb{P}$-generic over V and H is $\mathbb{S}$-generic over $V[G]$, Silver's Lemma will apply again to provide us with

$$j^* : V[G] \longrightarrow M[G * H].$$

Of course, $j''\lambda \in M[G * H]$ and $\mathbb{S}$ is once more ω-closed. This finishes the proof. □

The following Lemma is of considerable import and will be used quite often later:

Lemma 12. *Let* $\mathrm{cf}(\kappa) > \omega$. *Given that* $M \prec \mathrm{H}_\lambda$, $\delta = M \cap \kappa$ *is an ordinal below* κ *and* $\delta \in E$ *where* $E \in M$. *Then* E *is stationary in* κ.

Proof. By elementarity it is enough to show that E hits every club in M. So let $C \in M$ be club in κ. Then $C \cap \delta$ is unbounded in δ, thus we have $\delta \in C$. This means $\delta \in C \cap E \neq \emptyset$. □

Definition 13. Let $\mathcal{F} \subseteq {}^X\lambda$, for λ an ordinal and X an arbitrary set. Define the *filter-game* $\mathfrak{G}(\mathcal{F})$:

I	f_0	f_1	$\dots$	f_ξ	$f_{\xi+1}$	$\dots$
II	α_0	α_1	$\dots$	α_ξ	$\alpha_{\xi+1}$	$\dots$

where $f_\xi \in \mathcal{F}$, $\alpha_\xi < \lambda$ $(\xi < \omega_1)$. II wins if the set

$$\bigcap_{\xi<\gamma} f_\xi^{-1}(\alpha_\xi)$$

contains at least 2 elements for every $\gamma < \omega_1$.

Theorem 14. *The following are equivalent:*

(1) GRP
(2) II has a winning strategy in the game $\mathfrak{G}(\mathcal{F})$ *for every* $\mathcal{F} \subseteq {}^{\omega_2}\omega_1$ *of cardinality* $\aleph_2$.
(3) ω_2 *is generically weak compact by* ω*-closed forcing.*

Proof. (1) $\Longrightarrow$ (2): let $\mathcal{F}$ be any collection of functions from ω_2 to ω_1 of size $\aleph_2$ and take an ϵ-approachable $K \prec \mathrm{H}_\lambda$ that contains $\mathcal{F}$. We claim that player II wins the game $\mathfrak{G}^K(\mathcal{F})$: in the ξth move, he chooses $\alpha_\xi = f_\xi(\delta)$, the image of the point $\delta = K \cap \omega_2$. Since K is closed under countable sequences by the Remark following Lemma 5, the set $\bigcap_{\xi<\gamma} f_\xi^{-1}(\alpha_\xi)$ is in

K and contains δ for all $\gamma < \omega_1$. So this intersection is stationary in ω_2 by Lemma 12. By GRP, II has a winning strategy for the game $\mathfrak{G}(\mathcal{F})$.

(2) $\Longrightarrow$ (3): let $M \models \mathrm{ZFC}^-$ be of size $\aleph_2$ with $\omega_2 \in M$ and define $\mathcal{F} = M \cap {}^{\omega_2}\omega_1$. We fix a winning strategy σ for player II in the game $\mathfrak{G}(\mathcal{F})$. Now look at the game $\mathfrak{G}(\mathcal{F})$ in $M^{Col(\omega_1,\omega_2)}$: let I play an enumeration $\{f_\xi : \xi < \omega_1\}$ of $\mathcal{F}$ of order-type ω_1. The game proceeds:

I	f_0	f_1	$\dots$	f_ξ	$f_{\xi+1}$	$\dots$
II	α_0	α_1	$\dots$	α_ξ	$\alpha_{\xi+1}$	$\dots$

where the α_ξ's are played according to σ. Define the M-filter $\mathcal{U}$ by letting $\mathcal{U} = \{f_\xi^{-1}(\alpha_\xi) : \xi < \omega_1\}$. Since σ wins $\mathfrak{G}(\mathcal{F})$, we are given that $\mathcal{U}$ is an $\aleph_2$-complete ultrafilter with respect to M, i.e. $\mathcal{U}$ is closed under ω_1-sequences in M and for every set $A \subseteq \omega_2$ in M, either A or its complement is in $\mathcal{U}$. For non-triviality of $\mathcal{U}$ we use the fact that $\bigcap_{\xi<\gamma} f_\xi^{-1}(\alpha_\xi)$ contains at least 2 elements whenever $\gamma < \omega_1$. Hence, we can build the internal generic ultrapower ${}^{\omega_2}M/\mathcal{U}$ in M itself.[2] Note that this does not depend on the ultrafilter $\mathcal{U}$ being in M.

Claim. ${}^{\omega_2}M/\mathcal{U}$ is well-founded.

Proof. Assume that there is an infinite descending chain

$$[f_0]_\mathcal{U} > [f_1]_\mathcal{U} > [f_2]_\mathcal{U} > \dots$$

of ordinals in ${}^{\omega_2}M/\mathcal{U}$. Since our ultrapower is internal, all the witnesses $U_i = \{\alpha < \omega_2 : f_i(\alpha) > f_{i+1}(\alpha)\}$ will be in M. But $Col(\omega_1, \omega_2)$ is ω-closed, so $\langle U_i : i < \omega\rangle$ is in M. By $\aleph_2$-completeness of $\mathcal{U}$ for sequences in M, we conclude $\bigcap_{i<\omega} U_i \in \mathcal{U}$. Let $\alpha \in \bigcap_{i<\omega} U_i$, then

$$f_0(\alpha) > f_1(\alpha) > f_2(\alpha) > \dots$$

a contradiction. $\dashv$

With this claim, we get an embedding

$$j : M \longrightarrow N$$

where N is the transitive collapse of ${}^{\omega_2}M/\mathcal{U}$. The fact that $j \restriction \omega_2 = \mathrm{id} \restriction \omega_2$ follows from $\aleph_2$-completeness of $\mathcal{U}$. Of course, $j(\omega_2) > \omega_2$ by non-triviality.

(3) $\Longrightarrow$ (1): Let $\mathcal{A} \subseteq {}^{<\omega_1}\omega_2$ and assume that there is a set $C \subseteq \omega_2$ which is ω_1-club and such that II has a winning strategy in $\mathbb{G}^\alpha(\mathcal{A})$ for all $\alpha \in C$. Let M be the transitive collapse of $Sk(\mathcal{A}, C, \alpha)_{\alpha\le\omega_2}$. This M is a model of ZFC^-, contains ω_2 and has size $\aleph_2$. Thus, we find an elementary $j : M \longrightarrow N$ with $\mathrm{cp}(j) = \omega_2$ in some ω-closed extension $V^\mathbb{P}$. Let us work in this extension. We have:

[2] Arguments related to the equivalence between complete ultrafilters and elementary embddings are very well understood. For more details the reader is referred to the standard work [7, §5].

Claim. $\omega_2 \in j(C)$

Proof. Note that the following is true:

$$j''C = j(C) \cap \omega_2. \tag{3.1}$$

But the set $j(C)$ is closed under sequences of length less than $j(\omega_2)$, so we finished. ⊣

Using elementarity, the claim gives that II has a winning strategy for the game $\mathbb{G}^{\omega_2}(j(\mathcal{A}))$. Now we need the equation:

$$j''\mathcal{A} = j(\mathcal{A}) \cap {}^{<\omega_1}\omega_2. \tag{3.2}$$

(3.2) yields that II has a winning strategy for the game $\mathbb{G}^{\omega_2}(j''\mathcal{A})$ and since j is the identity on ω_2, a winning strategy for $\mathbb{G}^{\omega_2}(\mathcal{A}) = \mathbb{G}(\mathcal{A})$ follows. It is still necessary to pull this winning strategy back into the ground model, but this can be done by Lemma 9. □

Let us define *generically measurable by Γ* analogous to Definition 10 and say that κ has this property if there is $\mathbb{P} \in \Gamma$ such that $V^{\mathbb{P}}$ supports $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$. It is possible to drop the restriction on the size of the algebra $\mathcal{F}$ and modify the previous proof. We would be given the following fact:

Corollary 15. *The following are equivalent:*

(1) II has a winning strategy in the game $\mathfrak{G}({}^{\omega_2}\omega_1)$.
(2) II has a winning strategy in the game $\mathfrak{G}(\{f \in {}^{\omega_2}\omega_2 : f(\alpha) < \alpha\})$.
(3) ω_2 *is generically measurable by* ω*-closed forcing.*

Proof. (2) $\Longrightarrow$ (1) is trivial and (1) $\Longrightarrow$ (3) follows the corresponding lines of the proof of Theorem 14. (3) $\Longrightarrow$ (2) is similar to the proof of Theorem 14 too, so we only sketch it: we describe the winning strategy for player II in the game $\mathfrak{G}(\{f \in {}^{\omega_2}\omega_2 : f(\alpha) < \alpha\})$. If in the ξth move player I plays the regressive $f_\xi : \omega_2 \longrightarrow \omega_2$, look at the regressive function

$$j(f_\xi) : j(\omega_2) \longrightarrow j(\omega_2)$$

and answer with $\alpha_\xi = j(f_\xi)(\omega_2)$. By elementarity arguments, II is going to win this play. □

A very powerful way of constructing a generic embedding is to force with $\mathcal{I}^+$, the collection of all positive sets with respect to some precipitous ideal $\mathcal{I}$. Laver proved consistent (see [6]) that there be a normal ideal on ω_2 that is *σ-dense*, i.e. $\mathcal{I}^+$ contains a σ-closed dense set. This means, of course, that $\mathcal{I}^+$ is an ω-closed poset, so we have the following:

Corollary 16. *If there is a normal σ-dense ideal on ω_2 then ω_2 is generically measurable by ω-closed forcing and hence* GRP *holds.* □

Theorem 17. *The following are equivalent:*

(1) GRP$^+$
(2) For every regular λ*, II has a winning strategy in the game* $\mathfrak{G}(\mathcal{F}_\lambda)$*, where*

$$\mathcal{F}_\lambda = \{f : [\lambda]^{\omega_1} \longrightarrow \lambda \mid f(A) \in A \textit{ for every } A \textit{ in } [\lambda]^{\omega_1}\}.$$

(3) ω_2 *is generically supercompact by* ω*-closed forcing.*

Proof. (1) $\Longrightarrow$ (2): let λ be regular. Again, all we need is a winning strategy for player II in the game $\mathfrak{G}^K(\mathcal{F}_\lambda)$ whenever $K \prec \mathrm{H}_{(2^\lambda)^+}$ is ϵ-approachable and $\mathcal{F}_\lambda \in K$. Our strategy is similar: we answer the regressive function $f_\xi : [\lambda]^{\omega_1} \longrightarrow \lambda$ with $\alpha_\xi = f_\xi(K \cap \lambda)$. Such an answer is possible since $\alpha_\xi \in K$ by regressiveness of f_ξ. Note that every subset of $[\lambda]^{\omega_1}$ in K is unbounded if it contains the element $K \cap \lambda$. This makes our strategy winning just like in the proof of Theorem 14.

(2) $\Longrightarrow$ (3): we continue in the same fashion as before. By collapsing $|\mathcal{F}_\lambda|$ to ω_1 with the ω-closed poset $Col(\omega_1, |\mathcal{F}_\lambda|)$, we fix an enumeration $\{f_\xi : \xi < \omega_1\}$ of $\mathcal{F}_\lambda$ of order-type ω_1 in the generic extension. We let player I play all functions in this enumeration and take into account II's replies α_ξ that make him win. Define the filter $\mathcal{U} = \{f_\xi^{-1}(\alpha_\xi) : \xi < \omega_1\}$, this time on the underlying set $[\lambda]^{\omega_1}$. Now $\mathcal{U}$ is a V-normal ultrafilter, i.e. normal with respect to sequences in V. Let us build the generic ultrapower ${}^{[\lambda]^{\omega_1}}V/\mathcal{U}$ in V, yielding an elementary embedding[3]

$$j : V \longrightarrow M$$

where M is the transitive collapse of ${}^{[\lambda]^{\omega_1}}V/\mathcal{U}$. Note that the argument for the well-foundedness of ${}^{[\lambda]^{\omega_1}}V/\mathcal{U}$ was already given in Theorem 14. We check the properties of j: again, $j \restriction \omega_2 = \mathrm{id} \restriction \omega_2$ holds by $\aleph_2$-completeness of $\mathcal{U}$. It is an easily reviewed fact of ultrapower-embeddings with normal ultrafilters that $j''\lambda = [\mathrm{id}]_\mathcal{U}$, so $j''\lambda \in M$ is immediate. Finally, notice that $\mathrm{otp}([\mathrm{id}]_\mathcal{U}) < j(\omega_2)$ since this inequality holds really everywhere. But obviously,

$$\lambda = \mathrm{otp}(j''\lambda) = \mathrm{otp}([\mathrm{id}]_\mathcal{U}) < j(\omega_2).$$

(3) $\Longrightarrow$ (1): Let $\mathcal{A} \subseteq {}^{<\omega_1}\theta$ and assume that there is an ω_1-club $\mathcal{C} \subseteq [\theta]^{\aleph_1}$ such that II has a winning strategy in $\mathbb{G}^B(\mathcal{A})$ for all $B \in \mathcal{C}$. Choose $\lambda > \theta^{\aleph_1}$ and an ω-closed partial order $\mathbb{P}$ such that in $V^\mathbb{P}$ there is $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \omega_2$, $j(\omega_2) > \lambda$ and $j''\lambda \in M$. Set $B = j''\theta \in M$. From now on we work in $V^\mathbb{P}$.

Claim. The following two equations can be established:

$$j''\mathcal{A} = j(\mathcal{A}) \cap {}^{<\omega_1}B. \tag{3.3}$$

$$j''\mathcal{C} = j(\mathcal{C}) \cap [B]^{\aleph_1}. \tag{3.4}$$

[3] Basic facts about supercompact embeddings can be looked up in [7, §22].

Proof. Consider (3.3): if $y \in j(\mathcal{A})$, then y is of the form $j(x)$ if and only if $y \in {}^{<\omega_1}B$. But $x \in \mathcal{A}$ if and only if $y = j(x) \in j(\mathcal{A})$. ⊣

Claim. $B \in j(\mathcal{C})$

Proof. By elementarity we know that $j(\mathcal{C})$ is closed under sequences of length less than $j(\omega_2)$. From (3.4) we deduce that $j(\mathcal{C})$ is unbounded in $[B]^{\aleph_1}$, so B is the union of a directed system in $j(\mathcal{C})$ and $B \in j(\mathcal{C})$ holds. ⊣

By the second claim and elementarity, II wins the game $\mathbb{G}^B(j(\mathcal{A}))$. By (3.3) we have that II wins $\mathbb{G}^B(j''\mathcal{A})$. But $j : \theta \longrightarrow B$ is one-to-one, so II wins $\mathbb{G}^\theta(\mathcal{A}) = \mathbb{G}(\mathcal{A})$. All these arguments take place in $V^{\mathbb{P}}$, so this winning strategy might only live in the generic extension, but we are done by an application of Lemma 9. □

Corollary 18. *Let* $\mathbb{P} = \mathit{Coll}(\omega_1, < \kappa)$.

(a) If κ *is weakly compact then* $V^{\mathbb{P}} \models$ GRP.
(b) If κ *is supercompact then* $V^{\mathbb{P}} \models$ GRP$^+$.

Proof. By Theorems 11, 14 and 17. □

4. Applications of GRP

This section is devoted to applications of either GRP or GRP$^+$ respectively. Some of these implications are already known for quite some time if we take into account that both of these principles above are reformulations of generic compactness. We still give their proofs in the language of games.

We want to point out first that in view of the game-representation of clubs in $[\omega_2]^{\aleph_0}$, we can easily see that GRP implies the diagonal reflection of ω_2-many stationary subsets of $[\omega_2]^{\aleph_0}$ simultaneously:

Proposition 19. *Under* GRP, *for every sequence* $\mathcal{E}_\alpha$ $(\alpha < \omega_2)$ *of stationary subsets of* $[\omega_2]^{\aleph_0}$ *there is an* ϵ*-approachable*

$$M = \bigcup \langle M_\xi : \xi < \omega_1 \rangle$$

such that $\{\xi < \omega_1 : M_\xi \cap \omega_2 \in \mathcal{E}_\alpha\}$ *is stationary in* ω_1 *for every* $\alpha \in M \cap \omega_2$.

Proof. We play the following game:

I	γ, α_0	α_1	α_2	α_3	$\dots$
II	β_0	β_1	β_2	β_3	$\dots$

where γ and α_n, β_n $(n < \omega)$ are ordinals below ω_2. We let player II win this game if

$$\{\alpha_n : n < \omega\} \cup \{\beta_n : n < \omega\} \in \mathcal{E}_\gamma.$$

Note that player I has no winning strategy in this game, since all $\mathcal{E}_\alpha$'s are stationary. So there is an ϵ-approachable M for which he has no winning strategy. This finishes the proof. □

There is an analogous result for GRP^+. Simply play in some regular cardinal θ instead of ω_2 and the diagonal reflection for stationary subsets of $[\theta]^{\aleph_0}$ follows.

Now we are going to define a number of games that will be very useful for further applications of the Game Reflection Principle.

Definition 20. If $(\mathbb{P}, \leq)$ is a partial ordering, define the *cut-and-choose game* $\mathcal{G}_\kappa(\mathbb{P})$ in the following way: the plays of $\mathcal{G}_\kappa(\mathbb{P})$ look like this

Empty	p, A_0	A_1	A_2	A_3	$\ldots$
Nonempty	p_0	p_1	p_2	p_3	$\ldots$

where $p \in \mathbb{P}$, $p_n \in A_n$ $(n < \omega)$, A_0 is a maximal antichain below p and A_{n+1} a maximal antichain below p_n for $n < \omega$. Furthermore, all maximal antichains A_n $(n < \omega)$ are of size $\leq \kappa$. Nonempty wins if there is $q \in \mathbb{P}$ such that $q \leq p_n$ for every $n < \omega$.

The game $\mathcal{G}_\infty(\mathbb{P})$ is the same game as $\mathcal{G}_\kappa(\mathbb{P})$, except that there is no restriction on the sizes of the antichains A_n $(n < \omega)$.

We often want to play $\mathcal{G}$-games in Boolean algebras $\mathbb{B}$. We abuse notation and write $\mathcal{G}_\kappa(\mathbb{B})$ for $\mathcal{G}_\kappa(\mathbb{B}^+)$ in this case.

The first use of Definition 20 results in a well-known theorem of Gregory.

Proposition 21. GRP *implies that* ω_2*-Suslin-trees are essentially* σ*-closed.*

Proof. Assume that T is ω_2-Suslin. By results in [16], it is enough to show that Nonempty has a winning strategy in the game $\mathcal{G}_\infty(T)$.

Claim. If M is ϵ-approachable then Nonempty wins the game $\mathcal{G}^M_\infty(T)$.

Proof. Let $\delta = M \cap \omega_2$. To create his strategy, Nonempty picks any point $x \in T_\delta$. Note that by Suslinity, this point x is generic for M, i.e. the branch of all predecessors of x hits every maximal antichain in the model. Now it's easy for Nonempty to refine Empty's partitions along this branch and still remain in the tree. $\dashv$

This verifies a local winning strategy for Nonempty and we are done by an application of the Game Reflection Principle. $\square$

Rado's conjecture was first proved consistent by Todorčević in [14]. We will sketch a game-reflection-proof of this combinatorial statement.

Proposition 22. GRP^+ *implies Rado's conjecture.*

Proof. It's proved in [14] that Rado's conjecture is equivalent to the following statement: a tree T is the union of countably many antichains if and only if every subtree of size $\aleph_1$ is the union of countably many antichains. We finish the proof of this proposition by noting that there is a canonical game to characterize the notion of being the union of countably many antichains: player I plays points in the tree and player II answers with rationals. After ω_1-many steps, player II wins if the play determines a strictly increasing partial map from the tree into the rationals. $\square$

Let us turn to cut-and-choose games where antichains are restricted in size. We are going to see in the next theorem that GRP^+ has a strong influence on the ideal of non-stationary subsets of ω_2.

Theorem 23. *Under* GRP^+*, Nonempty wins* $\mathcal{G}_{\aleph_1}(\mathfrak{P}(\omega_2)/\mathrm{NS})$*.*

Proof. We investigate the game $\mathcal{G}^M_{\aleph_1}(\mathfrak{P}(\omega_2)/\mathrm{NS})$ for an ϵ-approachable M: it is helpful for Nonempty to consider the 'universal' club set

$$C_M = \omega_2 \setminus \bigcup(\mathrm{NS} \cap M).$$

By previous arguments we know that for all $E \subseteq \omega_2$ in M:

$$E \in \mathrm{NS}^+ \;\text{ iff }\; E \cap C_M \neq \emptyset. \tag{4.1}$$

Now Empty starts by playing a positive set E and an NS-partition P_0 of E of size $\aleph_1$. It is Nonempty's strategy to pick $\gamma \in E \cap C_M$ and fix it for the rest of the game.

Claim. There is $E_0 \in P_0$ such that $\gamma \in E_0$.

Proof. Assume that there isn't. In this case $\gamma \in E \setminus \bigcup P_0$. But then $E \setminus \bigcup P_0$ is stationary by (4.1) and disjoint from any member of the partition P_0. So P_0 is not maximal, a contradiction. ⊣

Let Nonempty play E_0 as an answer to P_0. Note that this is possible since $P_0 \subseteq M$ holds by the restricted size of P_0. Now Empty plays a partition P_1 of E_0 of size $\aleph_1$. Nonempty repeats proving the claim for P_1 and plays an E_1 as a response and so on. At the end of this game $\gamma \in \bigcap_{i<\omega} E_i$ will be true. Since M is closed under countable sequences, we know that $\bigcap_{i<\omega} E_i$ is a member of M and so it is positive, again by (4.1). We have established a winning strategy for Nonempty in the game $\mathcal{G}^M_{\aleph_1}(\mathfrak{P}(\omega_2)/\mathrm{NS})$.

From GRP^+ we can deduce that Nonempty has a winning strategy for the game $\mathcal{G}_{\aleph_1}(\mathfrak{P}(\omega_2)/\mathrm{NS})$. □

The just proved statement is an echo of Corollary 15. But there are two important differences between the games $\mathfrak{G}(\mathcal{F})$ and $\mathcal{G}_\kappa(\mathbb{P})$. On the one hand, Empty's freedom of choosing the set of ω-cofinal points at the start of the game complicates things somewhat. On the other hand, the later considered game has length ω only, so we might have to collapse ω_1 in order to construct a generic ultrafilter in the fashion of Theorems 14 or 17 just from the conclusion of Theorem 23. Nevertheless, it implies strong Chang's conjecture[4] heavy-handedly:

Proposition 24. *If Nonempty wins* $\mathcal{G}_{\aleph_1}(\mathfrak{P}(\omega_2)/\mathrm{NS})$*, then strong Chang's conjecture holds.*

[4] *Strong Chang's conjecture* was introduced by Shelah in [12]. It is equivalent to saying that 'Namba forcing is semiproper'. See also [13].

Proof. We show that for every countable substructure N, we can find an ordinal $\gamma \notin N$, $\omega_1 < \gamma < \omega_2$ such that $f(\gamma) \in N$ for all Skolem functions $f : \omega_2 \longrightarrow \omega_1 \in N$. We get such a γ by letting player I play all Skolem functions in N and pick a big enough ordinal in the outcome of this play. $\Box$

To have a further estimate on the strength of this winning strategy, we might also quote a result of Silver and Solovay, reproduced in [8, p.249], that still provides an inner model with a measurable cardinal (see also [16]). Their proof is actually a precursor of our Theorem 14 and shows that ω_2 is generically measurable by $Col(\omega, 2^{\omega_2})$.

We want to add that such a result is really optimal within the realm of cut-and-choose games for this particular algebra since player Empty can show up with a winning strategy for the more liberal game $\mathcal{G}_{\aleph_2}(\mathfrak{P}(\omega_2)/\mathrm{NS})$:

Theorem 25. *Empty wins* $\mathcal{G}_{\aleph_2}(\mathfrak{P}(\omega_2)/\mathrm{NS})$.

Proof. In the first move, Empty chooses to play on the ω-cofinals, i.e. he picks the positive set $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$. He goes on to fix increasing sequences α_n $(n < \omega)$ for every ω-cofinal α. In the nth move, Empty plays the partition $f_n : \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\} \longrightarrow \omega_2$ defined by $f_n(\alpha) = \alpha_n$. This function f_n is actually a *regressive partition.* Note that every regressive partition contains an NS-partition by an easy application of the Pressing Down Lemma. Now assume that Nonempty plays the ordinal β_n in his nth move indicating his choice, i.e. the preimage $f_n^{-1}(\beta_n)$. At the end of the day, the outcome will be

$$\bigcap_{n<\omega} f_n^{-1}(\beta_n) = \{\alpha < \omega_2 : f_n(\alpha) = \beta_n \text{ for all } n < \omega\} =$$
$$= \{\alpha < \omega_2 : \alpha_n = \beta_n \text{ for all } n < \omega\}.$$

But this set contains at most one element and we finished the proof. $\Box$

Theorems 23 and 25 show that there can be a huge difference between cut-and-choose games with partitions $f : \omega_2 \longrightarrow \omega_1$ on the one hand as opposed to regressive partitions $f : \omega_2 \longrightarrow \omega_2$. Notice that the exact opposite was true in Corollary 15.

Acknowledgements. The article is the second part of my thesis [10] which was produced under supervision of Hans-Dieter Donder and Stevo Todorčević.

I would like to thank the DFG for their generous support during three years of thesis work. I enjoyed extraordinary hospitality at the following institutes: the Department of Mathematics at the University of Toronto, Equipe de Logique Mathematique at Paris VII, the Mittag-Leffler institute in Djursholm and Institut für formale Logik in Vienna.

Finally, I want to thank Matthew Foreman and Boban Veličković for giving me the opportunity for very fruitful discussions.

References

1. James Baumgartner. A new class of order types. *Annals of Mathematical Logic*, 9:187–222, 1976.
2. Mohamed Bekkali. *Topics in set theory*, volume 1476 of *Lecture notes in Mathematics*. Springer-Verlag, 1992.
3. James Cummings. Large cardinal properties of small cardinals. In *Set theory (Curacao, 1995; Barcelona, 1996)*, pages 23–39. Kluwer Academic Publishers, 1998.
4. M. Foreman, Magidor M., and Shelah S. Martin's maximum, saturated ideals, and nonregular ultrafilters I. *Annals of Mathematics*, 127:1–47, 1988.
5. Matthew Foreman. Generic large cardinals: new axioms for mathematics? In *Proceedings of the International Congress of Mathematicians, Berlin*, volume II, pages 11–21, 1998.
6. F. Galvin, T. Jech, and M. Magidor. An ideal game. *Journal of Symbolic Logic*, 43:284–292, 1978.
7. Akihiro Kanamori. *The Higher Infinite*. Perspectives in Mathematical Logic. Springer-Verlag, 1997.
8. Akihiro Kanamori and Menachem Magidor. The evolution of large cardinal axioms in set theory. In *Higher set theory*, volume 669 of *Lecture notes in Mathematics*, pages 99–275. Springer-Verlag, 1978.
9. Alexander S. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, 1995.
10. Bernhard König. *Trees, Games and Reflections*. PhD thesis, LMU München, November 2001.
11. Kenneth Kunen. Saturated ideals. *Journal of Symbolic Logic*, 43:65–76, 1978.
12. Saharon Shelah. *Proper Forcing*, volume 940 of *Lecture notes in Mathematics*. Springer-Verlag, 1982.
13. Saharon Shelah. *Proper and Improper Forcing*. Perspectives in Mathematical Logic. Springer-Verlag, 1998.
14. Stevo Todorčević. On a conjecture of R. Rado. *Journal of the London Mathematical Society*, 27:1–8, 1983.
15. Stevo Todorčević. Conjectures of Rado and Chang and Cardinal Arithmetic. In Sauer et al., editors, *Finite and Infinite Combinatorics*, pages 385–398. Kluwer Academic Publishers, 1993.
16. Boban Veličković. Playful Boolean Algebras. *Transactions of the American Mathematical Society*, 296:727–740, 1986.
17. Boban Veličković. Forcing Axioms and Stationary Sets. *Advances of Mathematics*, 94:256–284, 1992.
18. W. Hugh Woodin. *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*. Walter de Gruyter & Co., Berlin, 1999.