Dense subtrees in complete Boolean algebras

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We characterize complete Boolean algebras with dense subtrees. The main results show that a complete Boolean algebra contains a dense tree if its generic filter collapses the algebra's density to its distributivity number and the reverse holds for homogeneous algebras.

Every good introduction to set-theoretic forcing starts with basic forcing algebras like "adding a Cohen real" or "adding a Cohen subset of ω_1". It seems that these algebras are particularly simple mainly because they contain a dense subtree, so we might as well pretend we are forcing with a tree. Such a forcing is especially easy to visualize as the generic object will be a branch through the tree and the notions of comparability and compatibility coincide.

The purpose of this note is to give a useful characterization of forcing algebras that contain such a dense subtree. As far as our knowledge goes, the following characterization has not been pointed out anywhere in the literature even though it seriously helps determining the isomorphism type of a forcing algebra.

1 Definitions

We use the term *forcing algebra* to refer to a complete Boolean algebra. These are denoted by letters like $\mathbb{B}$ or $\mathbb{C}$. We freely identify $\mathbb{B}$ with $\mathbb{B}^+$, the set of non-zero elements.

Because trees and Boolean algebras are intertwined in this paper, we adopt the convention that trees grow downward. When we say that a tree T is dense in a complete Boolean algebra $\mathbb{B}$ we mean that the identity mapping $\mathrm{id} : T \longrightarrow \mathbb{B}$ is a dense embedding in the sense of [3, p.221]. The common notation $\mathbb{B}_a$ is used to refer to the set $\{x \in \mathbb{B} : x \leq a\}$. The same notation T_a is used in terms of trees.

The *density* $\pi(\mathbb{B})$ of a forcing algebra $\mathbb{B}$ is the smallest cardinality of a dense subset $\mathcal{D}$. We call $\mathbb{B}$ *κ-distributive* if the intersection of less than κ many dense open subsets is again dense open. Note that this is equivalent to saying that less than κ many maximal antichains have a common refinement. The *distributivity number* $\mathrm{dist}(\mathbb{B})$ of $\mathbb{B}$ is the largest cardinal κ such that $\mathbb{B}$ is κ-distributive. Finally, $\mathbb{B}$ is called *nowhere κ^+-distributive* if $\mathbb{B}_a$ is not κ^+-distributive for any $a \in \mathbb{B}$.

Lemma 1.1 *$\mathbb{B}$ is nowhere κ^+-distributive if and only if there are dense open D_ξ $(\xi < \kappa)$ such that*

$$\bigcap_{\xi<\kappa} D_\xi = \emptyset.$$

P r o o f. $\Rightarrow$: it is forced that $\mathbb{B}$ adds a new κ-sequence, say $\dot{f} : \kappa \longrightarrow V$. For $\xi < \kappa$ let

$$D_\xi = \{b \in \mathbb{B} : b \text{ decides the value of } \dot{f}(\xi)\}.$$

$\Leftarrow$: let $a \in \mathbb{B}$, then

$$D^a_\xi = \{b \in D_\xi : b \leq a\} \ (\xi < \kappa)$$

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witnesses that $\mathbb{B}_a$ is not κ^+-distributive. □

Clearly, $\mathrm{dist}(\mathbb{B}) \leq \pi(\mathbb{B})$ holds for all atomless $\mathbb{B}$. Moreover, if we have $\mathrm{dist}(\mathbb{B}) = \pi(\mathbb{B})$ then $\mathbb{B}$ has a dense subtree [2].

$\mathbb{C}_\kappa$ is the forcing algebra that adds a Cohen subset of κ. In the terminology of [3], this is the completion of the poset $\mathrm{Fn}(\kappa, 2, \kappa)$, the set of partial functions from κ to 2 of cardinality less than κ. That poset can easily be seen to contain the dense subtree $(2^{<\kappa}, \supseteq)$. The algebra $\mathbb{C}_\kappa^\lambda$ adds λ-many Cohen subsets of κ and is the completion of the poset $\mathrm{Fn}(\kappa \times \lambda, 2, \kappa)$.

The following Lemma is used crucially.

Lemma 1.2 *Assume that $\mathbb{B}$ has a dense subtree and is nowhere κ^+-distributive. Then $\mathbb{B}$ contains a dense subtree of height at most κ.*

P r o o f. Let T be a dense subtree in $\mathbb{B}$ and D_ξ $(\xi < \kappa)$ a descending sequence of dense open subsets of T such that $\bigcap_{\xi<\kappa} D_\xi = \emptyset$. This exists by Lemma 1.1. Now define the maximal antichains

$$A_\xi = \{x \in D_\xi : w \notin D_\xi \text{ for all } w >_T x\}.$$

for all $\xi < \kappa$. Then $A = \bigcup_{\xi<\kappa} A_\xi$ is a tree of height at most κ, so the following claim finishes the proof.

Claim 1.2.1 *A is dense in T.*

P r o o f. Let $x \in T$, then there is $\xi < \kappa$ such that $x \notin D_\xi$. Find a z in the maximal antichain A_ξ such that z is comparable with x. Then $x >_T z$ since otherwise x would be in the open set D_ξ. □

□

2 Examples

All algebras of the form $\mathbb{C}_\kappa$ contain a dense subtree. It is simply $(2^{<\kappa}, \supseteq)$, the tree of binary sequences of length less than κ. The picture is different if we look at posets of the form $\mathbb{C}_\kappa^\lambda$ that add λ-many Cohen-subsets of κ. The first interesting example here is $\mathbb{C}_{\omega_1}^{\omega_2}$: it cannot contain a dense subtree under the Continuum Hypothesis. In fact, if κ is regular and $2^{<\kappa} < \lambda$, then $\mathbb{C}_\kappa^\lambda$ has no dense subtree:

> If $\mathbb{C}_\kappa^\lambda$ were to have a dense subtree then we could assume it to have height κ by Lemma 1.2. But $\mathbb{C}_\kappa^\lambda$ is known to have the $(2^{<\kappa})^+$-chain condition, so the dense subtree has cardinality $\kappa \cdot 2^{<\kappa}$ which is $< \lambda$ under the assumption $2^{<\kappa} < \lambda$. This is a contradiction since $\mathbb{C}_\kappa^\lambda$ adds λ-many Cohen subsets of κ.

The argument shows that $\mathbb{C}_\kappa^\lambda$ cannot contain a dense subtree if $2^{<\kappa} < \lambda$, so the question remains of what happens to $\mathbb{C}_\kappa^\lambda$ if the cardinal arithmetic is different?

Another interesting example is the forcing algebra that adds a fast club: $\mathbb{FC}$ is the regular open algebra of the poset $(\mathrm{FC}, \geq)$, where

$$\mathrm{FC} = \{(f, C) : C \subseteq \omega_1 \text{ is club}, f : \alpha \longrightarrow 2, \alpha < \omega_1\}$$

and $(f, C) \geq (g, D)$ iff

- $f \subseteq g$,
- $C \supseteq D$ and
- $\{\xi \in \mathrm{dom}(g - f) : g(\xi) = 1\} \subseteq C$.

This poset was first looked at by Jensen when he showed the consistency of SH + CH. $\mathbb{FC}$ adds a new club in ω_1 that is eventually inside (i.e. "faster than") every ground model club. One easily checks that FC is σ-closed. The generated algebra $\mathbb{FC}$ has the $(2^{\aleph_0})^+$-chain condition and collapses the continuum to $\aleph_1$. The density of $\mathbb{FC}$ depends on the particular model:

Lemma 2.1 $\aleph_2 \cdot 2^{\aleph_0} \leq \pi(\mathbb{FC}) \leq 2^{\aleph_1}$

P r o o f. The density of the club filter with the $\supseteq$-ordering is obviously very important for this since $\pi(\mathbb{FC})$ is exactly the maximum of both this cardinal and the continuum.

As for correctness of the bounds, the lower bound $2^{\aleph_0}$ and the upper bound $2^{\aleph_1}$ are trivial. Now for the lower bound $\aleph_2$, let C_ξ $(\xi < \omega_1)$ be a collection of clubs in ω_1 that is dense in the $\supseteq$-ordering, then there is $\zeta < \omega_1$ such that $C_\zeta \subseteq C := \triangle_{\xi<\omega_1} C_\xi$. This gives rise to

$$C_\zeta \subseteq C \subseteq^* \lim(C_\zeta),$$

which is impossible. □

Note that these bounds cannot really be improved, i.e. the density of the fast club is not necessarily $2^{\aleph_1}$: if we add λ-many Cohen reals to a model of GCH, then $2^{\aleph_1} \geq \lambda$, but the density of the club filter will remain $\aleph_2$, a consequence of the countable chain condition. If we add $\aleph_{\omega_1}$-many Cohen reals to a model of GCH we have the situation

$$\pi(\mathbb{FC}) = \aleph_2 \cdot 2^{\aleph_0} = \aleph_{\omega_1} < 2^{\aleph_1} = \aleph_{\omega_1+1}.$$

The forcing algebra $\mathbb{FC}$ does not seem to have a dense subtree. Here is a possible line of argument:

> Let us assume $2^{\aleph_0} < \pi(\mathbb{FC})$ for a moment. (This holds e.g. under CH, see Lemma 2.1.) If $\mathbb{FC}$ were to have a dense subtree, then again by Lemma 1.2 there would be a dense subtree of height ω_1. $\mathbb{FC}$ has the $(2^{\aleph_0})^+$-chain condition, so the size of this tree would be limited to $\aleph_1 \cdot 2^{\aleph_0} < \pi(\mathbb{FC})$. The latter is obviously impossible.

So we can ask a question similar than before: what happens to the fast club if it has density equal to the continuum (e.g. if $2^{\aleph_0} = 2^{\aleph_1}$)? Both of these questions will be answered later in the article.

3 A characterization

We need the following straightforward lemma:

Lemma 3.1 *Assume that $\mathbb{B}$ is a forcing algebra with a dense subtree of height κ. Further, let $\mathcal{D}$ be a κ-closed dense subset of $\mathbb{B}$.*
Then there is a dense subtree $T \subseteq \mathbb{B}$ such that T is κ-closed.

P r o o f. We may assume that there is a dense subtree S of height κ such that any x at a limit level is the infimum of its predecessors. With $\mathcal{D}$ being a κ-closed dense set and remembering that trees grow downward, we define

$$T = \{x \in S : \text{ for all } v > x \text{ in } S \text{ there is } w \in \mathcal{D} \text{ such that } v > w > x\}.$$

It's easy to see that T is κ-closed and dense. □

Loosely speaking, the next theorem says that if a forcing algebra collapses its own density to its distributivity number, then it must contain a dense subtree. One remark is in order here: when we say this we mean that $\mathrm{dist}(\mathbb{B})$ and $\pi(\mathbb{B})$ are computed in the *ground model*, but they have the same cardinality in $V^{\mathbb{B}}$.

Theorem 3.2 *Assume that the forcing algebra $\mathbb{B}$ is such that*

$$\mathbb{B} \Vdash |\mathrm{dist}(\mathbb{B})| = |\pi(\mathbb{B})|.$$

Then $\mathbb{B}$ contains a dense subtree. Moreover, if $\mathbb{B}$ has a κ-closed dense subset then $\mathbb{B}$ contains a κ-closed dense subtree.

Proof. Let $\mathbb{P}$ be a dense subset of size $\pi(\mathbb{B}) =: \lambda$. Set κ to be the distributivity number of $\mathbb{B}$ and assume that $\dot{\tau} : \kappa \longrightarrow \lambda$ is a $\mathbb{P}$-name for a mapping which is onto λ. Define

$$S = \{\|\dot{\tau} \restriction \xi = s\| : \xi < \kappa \text{ and } s : \xi \longrightarrow \lambda\}. \tag{3.1}$$

We are using that the distributivity number is κ: all functions of the form $s : \xi \longrightarrow \lambda$ from (3.1) are in the ground model.

Now let $S[p] = \{b \in S : b \wedge p \neq 0\}$.

Claim 3.2.1 *$S[p]$ has size λ for all $p \in \mathbb{P}$.*

Proof. Assume that $|S[p]| < \lambda$. Then there is $A \subseteq \lambda$ of size $< \lambda$ such that for all values $b = \|\dot{\tau} \restriction \xi = s\|$ in $S[p]$, the witness s is a function from ξ into A.

But then $p \Vdash \dot{\tau} : \kappa \longrightarrow A$, a contradiction since $|A| < \lambda$ and $\dot{\tau}$ is forced to be onto λ. □

Given the claim, we find a one-to-one function $\varphi : \mathbb{P} \longrightarrow S$ such that $\varphi(p) \in S[p]$ for all $p \in \mathbb{P}$.

For all $\alpha < \kappa$, refine S_α, the αth level of S, to a maximal antichain Q_α in the following way: Q_α is the union of the sets

- $S_\alpha \setminus \varphi"\mathbb{P}$,
- $\{\varphi(p) \wedge p : \varphi(p) \in S_\alpha\}$ and
- $\{\varphi(p) - p : \varphi(p) \in S_\alpha\}$.

Now define the tree T inductively. Assume that $\xi < \kappa$ and T_α $(\alpha < \xi)$ has been constructed. Then let T_ξ be a refinement of the collection

$$\{Q_\alpha, T_\alpha\}_{\alpha<\xi}$$

of maximal antichains. It is possible to find such a refinement as long as we proceed our induction below the distributivity number κ. T is clearly dense in $\mathbb{B}$ by the construction and T is a tree.

The 'moreover' in the statement of Theorem 3.2 now follows easily from Lemma 3.1. □

Theorem 3.2 adds to or improves over similar results in [1],[2] and [4]. We claim that the characterization of Theorem 3.2 is now optimal in the following sense:

Theorem 3.3 *Assume that $\mathbb{B}$ is nowhere $\operatorname{dist}(\mathbb{B})^+$-distributive, has everywhere density $\pi(\mathbb{B})$, and contains a dense subtree.*
Then $\mathbb{B} \Vdash |\operatorname{dist}(\mathbb{B})| = |\pi(\mathbb{B})|$.

Proof. Let T be the dense subtree inside of $\mathbb{B}$. Then we may assume that $|T| = \pi(\mathbb{B}) := \lambda$. Set κ to be the distributivity number of $\mathbb{B}$, so $\mathbb{B}$ is nowhere κ^+-distributive by assumption. Using Lemma 1.2, T can be assumed to have height κ. We construct a mapping $\pi : T \longrightarrow \lambda$ such that $\pi \restriction T_a$ is onto λ for all $a \in T$. This is possible with some book-keeping because T_a is dense in $\mathbb{B}_a$ and $\mathbb{B}_a$ itself has density λ. The latter statement is again by assumption. The following is now easy to check:

Claim 3.3.1 *For all $\xi < \lambda$, the set*

$$D_\xi := \{x \in T : \pi(x) = \xi\}$$

is dense in T. □

Now let $b : \kappa \longrightarrow \lambda$ be the generic branch through T enumerated by the function π. Given the claim, it follows that b is onto λ. □

Notice that the assumptions "nowhere $\operatorname{dist}(\mathbb{B})^+$-distributive" and "everywhere density $\pi(\mathbb{B})$" are true in case of a homogeneous $\mathbb{B}$. One might be tempted to drop them from the statement of Theorem 3.3, but it turns out that some assumption of homogeneity is necessary here because of pathological counterexamples of the following sort:

> Consider the disjoint union S of the trees $(2^{<\omega}, \supseteq)$ and $(2^{<\omega_1}, \supseteq)$ and let $\mathbb{B}(S)$ be the corresponding regular open algebra. Then density of $\mathbb{B}(S)$ is $2^{\aleph_0}$ and distributivity number is $\aleph_0$. But $\mathbb{B}(S)$ never collapses the continuum to $\aleph_0$, even though it has a dense subtree S.

4 Applications

Going back to our examples, we get the following Corollary of Theorem 3.2:

Corollary 4.1 *Assume κ is regular. The following are equivalent:*

1. $\mathbb{C}_\kappa^\lambda$ *has a dense subtree.*
2. $\mathbb{C}_\kappa^\lambda$ *is isomorphic to* $\mathbb{C}_\kappa$.
3. $\lambda \leq 2^{<\kappa}$.

Proof. (3) $\Rightarrow$ (1): by Theorem 3.2, since $\mathbb{C}_\kappa^\lambda$ has density $\lambda^{<\kappa}$ which is $2^{<\kappa}$ by (3). The poset is collapsing $2^{<\kappa}$ to κ though.

(1) $\Rightarrow$ (2): note that (1) implies $\lambda \leq 2^{<\kappa}$ by the remarks in the introduction, so we get $\lambda^{<\kappa} = 2^{<\kappa}$. To show (2), note that $\mathbb{C}_\kappa^\lambda$ is an algebra with a κ-closed dense set, a dense subtree of height κ, and splitting of cardinality $2^{<\kappa}$. So it must be forcing-isomorphic to $\mathbb{C}_\kappa$.

(2) $\Rightarrow$ (3): this is again by the remarks in the introduction since $\mathbb{C}_\kappa$ does have a dense subtree. □

It turns out that the fast club can have a dense subtree as well:

Corollary 4.2 $\mathbb{FC}$ *is isomorphic to* $\mathbb{C}_{\omega_1}$ *if and only if* $\pi(\mathbb{FC}) = 2^{\aleph_0}$.

Proof. If $\pi(\mathbb{FC}) = 2^{\aleph_0}$ then $\mathbb{FC}$ collapses its own density to $\aleph_1$ (cf. Section 2). So $\mathbb{FC}$ has a dense subtree by Theorem 3.2 and we argue similar to the proof of Corollary 4.1. □

We believe that more applications of Theorem 3.2 can be found by an interested reader. The last application we give is a result for products of Cohen-subsets:

Corollary 4.3 *Assume $\kappa \leq \lambda$. The following are equivalent:*

1. $\mathbb{C}_\kappa \times \mathbb{C}_\lambda$ *has a dense subtree.*
2. $\mathbb{C}_\kappa \times \mathbb{C}_\lambda$ *is isomorphic to* $\mathrm{Col}(\kappa, 2^{<\lambda})$.
3. $\lambda \leq 2^{<\kappa}$.

Proof. Note that $\pi(\mathbb{C}_\kappa \times \mathbb{C}_\lambda) = 2^{<\lambda}$.

(1) $\Rightarrow$ (2): an algebra with a dense subtree of height κ, density $2^{<\lambda}$ and a κ-closed dense subset must be isomorphic to $\mathrm{Col}(\kappa, 2^{<\lambda})$.

(2) $\Rightarrow$ (3): if $\lambda > 2^{<\kappa}$, then $\mathbb{C}_\kappa$ has λ-*cc* and the Lemma of Easton (see e.g. [3, p.265]) applies to show that $\mathbb{C}_\kappa \times \mathbb{C}_\lambda$ preserves λ. But this contradicts (2).

(3) $\Rightarrow$ (1): $\mathbb{C}_\kappa \times \mathbb{C}_\lambda$ collapses $2^{<\lambda}$ to λ and $2^{<\kappa}$ to κ, so if $\lambda \leq 2^{<\kappa}$, then $2^{<\lambda}$ is collapsed to κ overall. Now (1) follows from Theorem 3.2. □

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