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Local coherence

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Abstract

We characterize the tree of functions with finite support in terms of definability. This turns out to have various applications: a new kind of tree dichotomy for ω_1 on the one hand. On the other hand, we prove a reflection principle for trees on ω_2 under SPFA. This reflection of trees implies stationary reflection.

MSC: 03E05; 03E47

Keywords: Coherent trees; Aronszajn-trees; Reflection principles

0. Introduction

A question that really motivated most of the work in this paper is the following:

Question 0.1. *Is the isomorphism type of a tree completely determined by the isomorphism types of its bounded subtrees?*

In other words, can there be two non-isomorphic κ-trees T and S with $T_{<\alpha} \cong S_{<\alpha}$ for all $\alpha < \kappa$?

Note that the assertion saying 'there is no isomorphism between the κ-trees T and S' is Π^1_1 in $\langle H_\kappa, \in, T, S\rangle$. So if κ is weakly compact, there are no two different κ-trees with isomorphic initial segments. A drastic counterexample towards Question 0.1 is the fact that all normal ω_1-trees have isomorphic initial segments as long as they share

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the same splitting (see [13], or Corollary 4.3). We will investigate some instances of this problem for cardinals like ω_2 and restrict ourselves to a special sort of trees, namely the trivially coherent trees (see Definition 2.1). It turns out that this variation of the problem is still related to questions regarding large cardinals. If there is a lot of reflection in our universe, local properties of the tree will carry over to the global tree. If there is no such reflection, for example, in the constructible universe, local properties might not affect the global tree at all. The related problem of reflection of stationary sets was first looked at in [1] and further analyzed in [8].

In Section 2 of our article we will define the class of *trivially coherent* or *trivial* trees. We call them trivial because the fewest possible number of branches is extended at their limit levels: we take direct limits every single time. The pivotal Lemmas 2.17 and 2.18 will characterize trivial trees in terms of elementary substructures. As an application, we are going to find that every non-trivial ω_1-tree has a stationary antichain in PFA-models.

Although trivially coherent trees cannot be Aronszajn, there is a modified notion of a trivially coherent tree that allows this and we are going to call those trees *coherent*. Section 3 will provide a new construction of an ω_2-Aronszajn-tree of this kind from a weak version of a square-sequence.

Section 4 will deal with already mentioned reflection principles. We ask if it is consistent that every non-trivial tree is already non-trivial in some initial segment. Call this statement *reflection of non-triviality*. A positive answer towards its consistency will be given. This statement has a close relationship with the reflection of stationary sets (see Corollary 4.8), but the proofs indicate that it entails some more strength than simply stationary reflection as we still need the help of a forcing axiom in Theorem 4.19 (cf. Question 4.20).

1. Preliminaries

This section is devoted to the presentation of basic notions and propositions used in the paper. All other definitions will show up at the point where we need them. We will list a couple of standard facts and elementary lemmas that can also be found in introductory books on set theory such as for example [9] or [12]. The reader is referred to any of these two books in case that the author missed to define a couple of notions within the pages of Section 1. The educated reader, however, might want to skip these paragraphs if they happen to be too boring for him.

We start with very elementary notation: $f : A \overset{1-1}{\to} B$ means that f is a one-to-one mapping from A to B. $f : A \overset{\sim}{\to} B$ means that f is a one-to-one embedding of A into B that preserves structure. The reader could be confused by the fact that, in this case, f is not necessarily onto. If it is onto, we explicitly call f an *isomorphism* and denote the isomorphism relation by $\cong$.

Our shortcut $f''A$ for the image of A under f is very natural, as well as the notations ${}^{\delta}\gamma$ for the set of all functions from δ into γ and ${}^{<\delta}\gamma$ for the set of all functions from ordinals smaller than δ into γ. We use the symbol $[A]^{\lambda}$ for the set of all subsets of A with cardinality λ, $[A]^{<\lambda}$ is defined analogously. $\mathfrak{P}(A)$ denotes the power set of A, the set of all subsets of A.

We call a set $C \subseteq \kappa$ *club* if it is closed and unbounded in κ. We define the notion of a club for subsets of $[X]^{\aleph_0}$: $\mathscr{C}$ is *closed and unbounded* in $[X]^{\aleph_0}$ if

(i) for all $a \in [X]^{\aleph_0}$ there is a $b \in \mathscr{C}$ such that $a \subseteq b$,
(ii) whenever $\langle a_\xi\colon\ \xi < \omega\rangle \subseteq \mathscr{C}$ is an increasing sequence, then $\bigcup_{\xi<\omega} a_\xi \in \mathscr{C}$.

We confuse the notions of being club and containing a club. The set of all limit points of a given set of ordinals D is denoted by D'. The string $\mathrm{lh}(s)$ is a notation for the length of a sequence s. We write $M \prec N$ to say that M is an elementary submodel of N and $M \prec_{\omega_1} N$ to say that M is an elementary submodel of N such that $M \cap \omega_1 = N \cap \omega_1$. The continuous chain of models $\langle M_\xi\colon\ \xi < \lambda\rangle$ is an $\in$-*chain* if $M_\xi \in M_{\xi+1}$ for all $\xi < \lambda$. $Sk(X)$ is the Skolem closure of the set X when it should be clear from the context which superstructure we are working in, usually some H_θ. The reader has already noticed that we write symbols like κ, λ and θ for cardinals and $\alpha, \beta, \gamma, \zeta, \xi, \ldots$ for ordinals.

All trees are considered to be *normal*, for our purposes this means that they are trees of functions closed under initial segments with the property that every point splits and has successors of arbitrary height. An α-branch is identified with its union, a function on α. We use the following tree terminology: for any element t of a tree, let $t \restriction \alpha$ be the predecessor of t on the αth level. ht is the *height function* on a tree, T_α is the αth level of T and $T_{<\alpha} = \bigcup_{\xi<\alpha} T_\xi$. If t is element of a successor level, let *immpred*(t) be the immediate predecessor of t. We write $x \perp y$ as a shorthand for 'x is incomparable with y'. $x \wedge y$ is a symbol for the maximal $z \in T$ such that $z \leqslant_T x$ and $z \leqslant_T y$, the *infimum* of x and y. κ-*trees* are trees of height κ with the property that every level has size less than κ. A κ-*Aronszajn-tree* is a κ-tree that has no cofinal branch. We say that a κ-Aronszajn-tree T is *special* if we can associate to it a regressive function $f: T \to T$ such that the preimage of every point is a union of less than κ many antichains. Note that if κ is a successor cardinal, this is equivalent to the classical notion of speciality, saying that we can decompose the tree into less than κ many antichains.[3] In Definition 2.23 we introduce a more general notion of *special* trees that applies to trees with branches as well. Trees of height $\omega+1$ are called *Cantor-trees* if each finite level is countable, its limit level yet uncountable.

Some words on Forcing: $q \leqslant p$ means that q is a stronger condition than p. Names are denoted with dots on top (e.g. $\dot\tau$) but our notation shall not be too strict in this. If $\mathbb{P}$ is a partial order, G a $\mathbb{P}$-generic filter and M a model of a large enough fragment of set theory, we let

$$M[G] = \{\dot\tau[G]: \dot\tau \text{ is a } \mathbb{P}\text{-name in } M\},$$

where $\dot\tau[G]$ is the G-interpretation of the name $\dot\tau$. On the other hand, $V^{\mathbb{P}}$ is sometimes written for $V[G]$, but no confusion should arise.

[3] If $\kappa = \mu^+$, define an antichain decomposition $g: T \to \mu$ as follows: for $x \in T$ let $x = x_0 >_T x_1 >_T \cdots >_T x_n = \mathrm{root}$ be the regressive path induced by f. Set $g(x) = \langle \alpha_0, \ldots, \alpha_{n-1}\rangle$, where x_i is in the α_ith antichain of $f^{-1}(x_{i+1})$ (see [4, p. 63]).

We assume knowledge of the following axioms: $\square(\kappa)$ is the statement that there be a sequence C_α $(\alpha<\kappa)$ such that

1 C_α is club in α,
2 if α is a limit point of C_β then $C_\alpha = C_\beta \cap \alpha$,
3 there is no club $C \subseteq \kappa$ with $C_\alpha = C \cap \alpha$ for every limit point α of C.

If $\kappa=\lambda^+$, we might require a stronger condition than 3:

*3 the order-type of C_α is at most λ for all $\alpha<\kappa$.

If this occurs, we call C_α $(\alpha<\kappa)$ a $\square_\lambda$-sequence. Note that if λ is a regular cardinal, the non-existence of a $\square_\lambda$-sequence is equiconsistent with a Mahlo cardinal and the non-existence of a $\square(\lambda^+)$-sequence is equiconsistent with a weakly compact (basically a result of Jensen and Silver, see [10,25]).

We require familiarity with *proper* and *semiproper forcing*. One of the best introductions to proper forcing is [2], while [17] is an encyclopedic work about proper, semiproper and related forcing notions. The axioms PFA and PFA$^+$ can be found in [2], whereas SPFA was introduced for the first time in [8].

RP_2 denotes the following reflection of stationary sets: whenever θ is regular and $\mathscr{E}_0$, $\mathscr{E}_1$ a pair of stationary subsets of $[\mathrm{H}_\theta]^{\aleph_0}$, then there is an $\in$-chain $\langle M_\xi\colon\ \xi<\omega_1\rangle$ such that $\{\xi<\omega_1\colon\ M_\xi\in\mathscr{E}_i\}$ is stationary in ω_1 for $i=0,1$.

Finally, we give two important lemmas that are frequently used in the course of this work. We might sometimes even apply them without mentioning:

Lemma 1.1 (Stretching stationary sets). *Suppose that α is an ordinal of uncountable cofinality λ. If $f:\lambda\to\alpha$ enumerates a club set in α, then A is stationary in α if and only if $f^{-1}(A)$ is stationary in λ.*

Lemma 1.2 (Pressing Down Lemma). *Let κ be regular. If $f:S\to\kappa$ is a regressive function on a stationary set $S\subseteq\kappa$ then there exist a stationary set $S_0\subseteq S$ and $\gamma_0<\kappa$ such that $f(\alpha)=\gamma_0$ for all $\alpha\in S_0$.*

Proof. We prefer to give the non-classical proof: take an elementary sub-model $N\prec\mathrm{H}_\theta$ for some large enough regular θ such that N contains f as an element and $N\cap\kappa=\delta$ is an ordinal in S. Now $\gamma_0=f(\delta)\in N$ and therefore the set $S_0=f^{-1}(\gamma_0)$ is in N and contains δ. Thus, S_0 hits every club in N and by elementarity S_0 is stationary in κ. □

2. A class of minimal trees

2.1. Introducing coherence

The crucial moments in the construction of trees are usually the limit levels. As we mentioned in the Introduction, the most economical way to construct a tree is to take direct limits at every limit stage. If the tree is a tree of functions, this basically means we abandon all branches with infinite support and only extend branches with finite support. Let us give this tree a name:

Definition 2.1. We define a class of trees in the following fashion: if δ is an ordinal, we let

$$\mathbb{Q}^{\text{fin}}_{<\delta} = \{f \in {}^{<\delta}2\colon \ \text{supp}(f) \text{ is finite}\},$$

where $\text{supp}(f) = \{\alpha \in \text{dom}(f)\colon \ f(\alpha) \neq 0\}$. Normal subtrees of $\mathbb{Q}^{\text{fin}}_{<\delta}$ are called *trivially coherent* or *trivial*. Note that we will also call a tree trivially coherent if it is an isomorphic copy of a normal subtree of $\mathbb{Q}^{\text{fin}}_{<\delta}$.

The restriction to binary trees in the previous Definition 2.1 is just for notational simplicity and in no way essential for future results. Nevertheless, we want to stick to this restriction on binary splitting in the future. The only exceptions to our convention are Aronszajn-trees constructed in Section 3.

Remark 2.2. $\mathbb{Q}^{\text{fin}}_{\leqslant\delta}$ is a downward closed subtree of any normal tree of height $\delta + 1$.

Proof. Let T be a normal tree of height $\delta+1$. Using normality, we can find a mapping $b: T_{<\delta} \to T_\delta$ such that $x <_T b(x)$ and

$$x \leqslant_T y <_T b(x) \quad \text{implies} \quad b(x) = b(y).$$

Let $B_0 = b'' T_{<\delta}$ and define an even finer sublevel $B_1 \subseteq B_0$:

$$B_1 = \{t \in B_0\colon \ \forall \text{limit } \gamma < \delta \ \exists x \in T_{<\gamma} \ t \restriction \gamma <_T b(x)\}.$$

Let U be the downwards closure of B_1. We claim that U is the subtree of T that is isomorphic to $\mathbb{Q}^{\text{fin}}_{<\delta}$. Construct an isomorphism $\pi: U \xrightarrow{\sim} \mathbb{Q}^{\text{fin}}_{\leqslant\delta}$: if $x \in U_\beta$, define $\pi(x): \beta \to 2$ by letting:

$$\pi(x)(\alpha) = 0 \quad \text{iff} \quad x \restriction (\alpha + 1) <_T b(x \restriction \alpha)$$

for all $\alpha < \beta$. The function π maps into $\mathbb{Q}^{\text{fin}}_{\leqslant\delta}$ by the definition of B_1 and it can be seen to be onto. □

In other words, Remark 2.2 says that any normal tree with cofinal branches contains the trivially coherent tree.

Definition 2.3. Let T be a tree of height δ. If $f: T \restriction C \to T$ is regressive with $\text{Lim}(\delta) \subseteq C$ and $x \in T \restriction C$, define a regressive *trace* tr^f_x from x to the root of the tree as follows:

$$\text{tr}^f_x(0) = x,$$

$$\text{tr}^f_x(n+1) = \begin{cases} f(\text{tr}^f_x(n)) & \text{if } \text{tr}^f_x(n) \in T \restriction C, \\ \text{immpred}(\text{tr}^f_x(n)) & \text{if } \text{tr}^f_x(n) \notin T \restriction C. \end{cases}$$

Lemma 2.4. *The following are equivalent for any tree T of height δ:*

(1) *T is trivially coherent.*
(2) *$T = \bigcup_{n<\omega} S_n$, where each S_n has the properties:*
- *there are no triangles, i.e. there are no $x, y, z \in S_n$ such that $z \leqslant_T x, y$ and $x \perp y$,*
- *if $x \in S_n \cap T \restriction \mathrm{Lim}(\delta)$, then a final segment of the predecessors of x is in S_n.*

(3) *There is a regressive $f : T \restriction \mathrm{Lim}(\delta) \to T$ such that*

$$(\perp)_f \quad x_0 \perp x_1 \to \exists i \ f(x_i) >_T x_0 \wedge x_1.$$

Proof. (1)$\Rightarrow$(2) can be achieved by letting

$$x \in S_n \quad \text{iff} \quad |\mathrm{supp}(x)| = n.$$

(2)$\Rightarrow$(3): just let $f(x) = \min\{y <_T x\colon \ \exists n [y, x) \subseteq S_n\}$.

For (3)$\Rightarrow$(1) fix a function f as in (3). Now call an element x of a successor level a *zero point* whenever there is y in a limit level of T and $n < \omega$ such that

$$\mathrm{tr}_y^f(n+1) <_T x <_T \mathrm{tr}_y^f(n).$$

Note that by $(\perp)_f$, every point in the tree cannot have more than one zero point among its immediate successors and this means that we can find an isomorphism π defined on T, with the property that $\pi(x)(\mathrm{dom}(x) - 1) = 0$ for all zero points x. But then, as is easily established, $\pi'' T \subseteq \mathbb{Q}^{\mathrm{fin}}_{<\delta}$. □

These reformulations of trivial coherence set the stage for the following lemma, revealing a crucial property of trivially coherent trees. We need the following definitions:

Definition 2.5. Let T be a tree of height δ and $\mathrm{cf}(\delta) > \omega$. A subset $S \subseteq T$ is called *stationary* if $ht'' S$ is stationary in δ.

A subset $S \subseteq T$ *projects* 1–1 if there is $\gamma < \delta$ so that the projection mapping $\mathrm{pr}_\gamma :$ $S \to T$ defined by $s \mapsto s \restriction \gamma$ is 1–1.

An antichain $A \subseteq T$ is *non-trivial* if A is stationary and no stationary $A_0 \subseteq A$ projects 1–1.

Remark 2.6. If T is a κ-tree and κ regular, then an antichain in T is stationary if and only if it is non-trivial.

Lemma 2.7. *If $\mathrm{cf}(\delta) > \omega$, $\mathbb{Q}^{\mathrm{fin}}_{<\delta}$ has no non-trivial antichains.*

Proof. We assume towards a contradiction that A is a non-trivial antichain in $\mathbb{Q}^{\mathrm{fin}}_{<\delta}$. Let $E = \mathrm{ht}'' A$ and $t_\nu \in A \cap T_\nu$ $(\nu \in E)$. Define a regressive mapping $h : E \to \delta$ by setting

$$h(\nu) = \max\{\alpha < \nu\colon \ t_\nu(\alpha) \neq 0\} + 1.$$

The Pressing Down Lemma for singular ordinals (see e.g. [7, p. 36]) will provide an ordinal $\xi < \delta$ and a stationary $E_0 \subseteq E$ such that $h''E_0 \subseteq \xi$, i.e. the support of t_ν for $\nu \in E_0$ is below the ordinal ξ. Using the non-triviality of the antichain A, we have that $\{t_\nu\colon\ \nu \in E_0\}$ does not project 1–1, in particular not at the point ξ. As a consequence, there are distinct $\nu, \nu' \in E_0$ such that $t_\nu \restriction \xi = t_{\nu'} \restriction \xi$. But this makes t_ν and $t_{\nu'}$ comparable. A contradiction. □

Remark 2.6 yields:

Corollary 2.8. *If κ is regular, $\mathbb{Q}^{\mathrm{fin}}_{<\kappa}$ has no stationary antichains.*

So non-trivial antichains seem to be exactly what we need to contradict the trivial coherence of a given tree. Corollary 2.25 and Theorem 4.13 will show that Lemma 2.7 is in a sense optimal.

2.2. Characterizing coherence

This subsection is circling around Lemmas 2.17 and 2.18. Roughly speaking, they say that a tree T is trivially coherent if and only if every point $t \in T$ is definable from finitely many points below t. This is a much more intrinsic property than our original definition of trivially coherent trees.

In the following, let T be a tree. We will fix a regular cardinal θ much larger than the height of T. From now on, all elementary substructures $N \prec \mathrm{H}_\theta$ will be assumed countable, with a fixed well ordering $<_w$ of H_θ attached to them and, moreover, they contain T.

Definition 2.9. Let T be a tree of height δ. We will call a chain K through T an *M-chain* if $K \subseteq M$. The M-chains under consideration will usually be *converging*, which means that they have a limit in T. An M-chain K will be called *cofinal* (*in M*) if $\mathrm{dom}(\bigcup K) = \sup(M \cap \delta)$. If K is not cofinal, it is said to be *bounded* (*in M*).

A chain $K \subseteq T \cap N$ is *captured* by an elementary submodel N if there is a chain $L \in N$ such that $K \subseteq L$. A chain $K \subseteq T \cap N$ is *cofinally captured* by N if there is a cofinal chain $L \in N$ such that $K \subseteq L$. In these cases, we may also say that K is *captured* or *cofinally captured* by L, respectively.

Definition 2.10. $M \prec \mathrm{H}_\theta$ is called *T-simple* if all converging cofinal M-chains are captured by M. Note that such a cofinal M-chain will always be cofinally captured.

If a model $M \prec \mathrm{H}_\theta$ is not T-simple, we call it *T-complicated*. In this case we will always let t_M be the $<_w$-minimal element $t \in T$ that witnesses complicatedness, i.e. t is the limit of an uncaptured and cofinal chain.

$M \prec \mathrm{H}_\theta$ is called *locally T-trivial* if all converging bounded M-chains are captured by M. We call M *locally T-uniform* if all converging bounded M-chains are cofinally captured by M.

$M \prec \mathrm{H}_\theta$ is said to be *T-trivial* if it is locally T-trivial and T-simple. M is said to be *T-uniform* if it is locally T-uniform and T-simple.

Remark 2.11.

- $\mathrm{ht}(t_M) = \sup(M \cap \delta)$.
- If M is T-complicated, t_M is definable in $(\mathrm{H}_\theta, \in, <_w)$ with parameters M and T.

Definition 2.12.

$$\begin{aligned}
\mathscr{S}_T &= \{M \prec \mathrm{H}_\theta\colon\ M \text{ is } T\text{-simple}\},\\
\mathscr{C}_T &= \{M \prec \mathrm{H}_\theta\colon\ M \text{ is } T\text{-complicated}\} = \{M \prec \mathrm{H}_\theta\colon\ M \notin \mathscr{S}_T\},\\
\mathscr{L}\mathscr{T}_T &= \{M \prec \mathrm{H}_\theta\colon\ M \text{ is locally } T\text{-trivial}\},\\
\mathscr{L}\mathscr{U}_T &= \{M \prec \mathrm{H}_\theta\colon\ M \text{ is locally } T\text{-uniform}\},\\
\mathscr{T}\mathscr{C}_T &= \{M \prec \mathrm{H}_\theta\colon\ M \text{ is } T\text{-trivial}\} = \mathscr{L}\mathscr{T}_T \cap \mathscr{S}_T,\\
\mathscr{U}\mathscr{C}_T &= \{M \prec \mathrm{H}_\theta\colon\ M \text{ is } T\text{-uniform}\} = \mathscr{L}\mathscr{U}_T \cap \mathscr{S}_T.
\end{aligned}$$

Remark 2.13.

- $\mathscr{L}\mathscr{U}_T \subseteq \mathscr{L}\mathscr{T}_T$.
- $\mathscr{U}\mathscr{C}_T \subseteq \mathscr{T}\mathscr{C}_T \subseteq \mathscr{S}_T$.

Definition 2.14. If X is a subset of a tree T, the *closure* of X in T is

$$\overline{X} = X \cup \{t \in T\colon\ \text{there is a chain } K \subseteq X \text{ converging to } t\}.$$

Definition 2.15. Let T be a tree of height δ. The game $\mathbb{G}_T^{\mathrm{coh}}$ is played as follows:

I	x_0	x_1	x_2	x_3	$\dots$
II	b_0	b_1	b_2	b_3	$\dots$

where player I plays points $x_n \in T$ $(n<\omega)$, while player II answers with branches b_n through T. By a *branch* we mean a downward closed chain that is not necessarily cofinal (cf. Lemmas 2.17 and 2.18).

II wins this play of the game if $\overline{\{x_n : n<\omega\}} \subseteq \bigcup_{n<\omega} b_n$.

There is a slight and insignificant variation of $\mathbb{G}_T^{\mathrm{coh}}$ that can relieve some technical difficulties in the proofs of the important Lemmas 2.17 and 2.18:

Definition 2.16. The game $\mathbb{G}_T^{\mathrm{coh}*}$ is the same as the game $\mathbb{G}_T^{\mathrm{coh}}$ with the additional rules that:

(i) $x_n <_T x_{n+1}$ for all $n<\omega$, and
(ii) $x_n \in b_n$.

Lemma 2.17. *The following are equivalent for any tree T:*

(1) *T is trivially coherent.*
(2) *$\mathscr{T}\mathscr{C}_T$ is club in $[\mathrm{H}_\theta]^{\aleph_0}$.*
(3) *II has a winning strategy in the game $\mathbb{G}_T^{\mathrm{coh}}$.*
(4) *II has a winning strategy in the game $\mathbb{G}_T^{\mathrm{coh}*}$.*

Proof. Fix $\delta = \mathrm{ht}(T)$.

(1) ⇒ (2): pick $N \prec \mathrm{H}_\theta$ with $T \in N$. If $K \subseteq N \cap T$ is any converging chain in T, its support is finite so there is $\alpha \in N$ such that the support of each $t \in K$ is bounded by α. Define $F : \delta \to 2$ by

$$F(\xi) = \begin{cases} K(\xi) & \text{if } \xi < \alpha, \\ 0 & \text{if } \alpha \leqslant \xi < \delta. \end{cases}$$

Let L be the collection of all initial segments of F that are members of T. All this is definable in N, so $L \in N$ and L is a branch capturing K. We showed that N is T-trivial.

(2) ⇒ (3): in the first place, fix a pairing function $\langle\, , \rangle : \omega \times \omega \to \omega$ such that $k, l \leqslant \langle k, l \rangle$. Whenever $\vec{x} \in {}^{<\omega}T$, we will choose enumerations $S_{\vec{x}}$ of all the branches in the corresponding Skolem closure, i.e.

$$S_{\vec{x}} : \omega \to \{b \in Sk(\vec{x}) : b \text{ is a branch through } T\}.$$

By our assumption that $\mathscr{TC}_T$ is club, we can assume without loss of generality that all structures $Sk(\vec{x})$ are T-trivial. If $v_0, \ldots, v_{m-1}$ is the start of a play, and $\langle k, l \rangle = m$, then II responds

$$\sigma(v_0, \ldots, v_{m-1}) = S_{\vec{v} \upharpoonright k}(l).$$

Note that this is well-defined, since $m \geqslant k$.

Subclaim 2.17.1. *σ is winning for player II in the game $\mathbb{G}_T^{\mathrm{coh}}$.*

Proof. We assume that $K = \{v_{i(n)} : n < \omega\}$ converges and conclude the following:

$$K \subseteq Sk(v_n)_{n<\omega} = \bigcup_{n<\omega} Sk(v_0, \ldots, v_{n-1}).$$

But $Sk(v_n)_{n<\omega}$ is T-trivial since $\mathscr{TC}_T$ is club, so it contains a branch $b \supseteq K$. By definition of the strategy σ, this branch will finally be played. □

(3) ⇒ (4): let σ be a winning strategy for II in the game $\mathbb{G}_T^{\mathrm{coh}}$. We define a winning strategy τ for player II in the game $\mathbb{G}_T^{\mathrm{coh}*}$: whenever the sequence $x_0 <_T x_1 <_T \cdots <_T x_n$ is a finite chain through T, let

$$\tau(x_0, x_1, \ldots, x_n) = \begin{cases} \sigma(x_0, x_1, \ldots, x_n) & \text{if } x_n \in \sigma(x_0, x_1, \ldots, x_n), \\ \text{any } b \text{ with } x_n \in b & \text{otherwise.} \end{cases}$$

τ is actually a strategy for the game $\mathbb{G}_T^{\mathrm{coh}*}$ because $x_n \in \tau(x_0, x_1, \ldots, x_n)$ for all sequences $\langle x_0, x_1, \ldots, x_n \rangle$.

Subclaim 2.17.2. *τ is a winning strategy for player II in the game $\mathbb{G}_T^{\mathrm{coh}*}$.*

Proof. If there is a play:

I	x_0	x_1	x_2	x_3	…
II	b_0	b_1	b_2	b_3	…

according to τ and $x = \lim_{n<\omega} x_n$, set

$$B = \{\sigma(x_0,\ldots,x_n)\colon\ n < \omega\} \quad \text{and}$$
$$B_0 = \{b_n\colon\ n < \omega\} = \{\tau(x_0,\ldots,x_n)\colon\ n < \omega\}.$$

But if $b \in B \backslash B_0$, then there is an integer $m < \omega$ such that $x_m \notin b$, where $b = \sigma(x_0,\ldots,x_m)$. Hence, $x \notin b$. This argument shows that τ is winning for player II. □

(4) ⇒ (1): let τ be a winning strategy for II in the game $\mathbb{G}_T^{\mathrm{coh}*}$. Our aim is to construct an embedding $\pi : T \xrightarrow{\sim} \mathbb{Q}^{\mathrm{fin}}_{<\delta}$. So let $t \in T$ and inductively define t_n and α_n in the following way:

$$\alpha_0 = \mathrm{ht}(\tau(\mathrm{root}) \wedge t) \quad \text{and} \quad t_0 = t \restriction (\alpha_0 + 1),$$
$$\alpha_n = \mathrm{ht}(\tau(\mathrm{root}, t_0, \ldots, t_{n-1}) \wedge t) \quad \text{and} \quad t_n = t \restriction (\alpha_n + 1).$$

Clearly, $\alpha_n < \alpha_{n+1}$ as long as the process continues, i.e. as long as $\alpha_n < \mathrm{ht}(t)$. But actually, it will break down at some finite point:

Subclaim 2.17.3. *There is $k < \omega$ such that $\alpha_k = \mathrm{ht}(t)$.*

Proof. Assume not. Then there is a play

I	root	t_0	t_1	t_2	…
II	b_0	b_1	b_2	b_3	…

according to τ. But $t_n \notin b_n = \tau(t_0,\ldots,t_{n-1})$ by definition of t_n. Because of that, $x = \lim_{n<\omega} t_n \notin b_m$ for all $m < \omega$. This contradicts the fact that τ is winning for player II. □

We are in a position to define the embedding π: for $\xi < \mathrm{ht}(t)$ let

$$\pi(t)(\xi) = \begin{cases} 1 & \text{if } \xi = \alpha_n \text{ for some } n < k, \\ 0 & \text{else.} \end{cases}$$

By Subclaim 2.17.3, $\pi(t) \in \mathbb{Q}^{\mathrm{fin}}_{<\delta}$.

Subclaim 2.17.4. *π is one-to-one and preserves the tree relation.*

Proof. It can easily be seen that two incomparable elements differ in their π-values for the first time at the splitting node. This is enough to prove that π is one-to-one and preserves the tree relation. □ □

Rewriting the above proof of Lemma 2.17 yields:

Lemma 2.18. *The following are equivalent for any tree T of height δ:*

(1) $T \cong \mathbb{Q}^{\mathrm{fin}}_{<\delta}$.
(2) $\mathscr{UC}_T$ *is club in* $[\mathrm{H}_\theta]^{\aleph_0}$.

(3) *II wins the game* $\mathbb{G}_T^{\mathrm{coh}}$ *by playing cofinal branches.*
(4) *II wins the game* $\mathbb{G}_T^{\mathrm{coh}*}$ *by playing cofinal branches.*

Proof. The proof of Lemma 2.17 goes through here. The only additional argument occurs in the proof of (4)$\Rightarrow$(1) where we can now show that the isomorphism $\pi : T \xrightarrow{\sim} \mathbb{Q}^{\mathrm{fin}}_{<\delta}$ is onto.

Subclaim 2.18.1. π *is one-to-one, onto and preserves the tree relation.*

Proof. It was pointed out in the proof of Lemma 2.17 that π is one-to-one and preserves the tree relation. We show that π is onto:
if $q \in \mathbb{Q}^{\mathrm{fin}}_{\gamma}$ $(\gamma < \delta)$ has support $\{\gamma_0, \dots, \gamma_{m-1}\}$, define an ascending sequence in T, where $x_{n+1} \in T_{(\gamma_{n+1})}$ $(n < m)$:

$$x_0 = \text{root}$$
$$x_{n+1}(\xi) = \begin{cases} \tau(x_0, \dots, x_n)(\xi) & \text{if } \xi < \gamma_n, \\ 1 - \tau(x_0, \dots, x_n)(\xi) & \text{if } \xi = \gamma_n. \end{cases}$$

Note that this construction is possible only because τ provides us with cofinal branches. We go on to check that $\tau(x_0, \dots, x_m) \cap T_\gamma$ is the π-preimage of q. □ □

With this machinery developed, we make an interesting observation: since countable trees can be identified with countable dense linear orderings,[4] it follows that every countable dense linear ordering is in fact isomorphic to $(\mathbb{Q}, <)$ and we did not use Cantor's back-and-forth method to prove this. In another respect, we are able to view Lemmas 2.17 and 2.18 as an advance in Kurepa's classification of countable trees in [13]. Kurepa actually used Cantor's technique to construct isomorphisms between any two countable normal trees of the same height.

2.3. Substructure arguments

We are trying to apply what we just proved in a series of Corollaries. First, what is the structural difference between $\mathbb{Q}^{\mathrm{fin}}_{<\delta}$ and its normal subtrees, the trivially coherent trees? The answer is given by the next Lemma.

Lemma 2.19. *The following are equivalent for any tree* T *of height* δ:

(1) $T \cong \mathbb{Q}^{\mathrm{fin}}_{<\delta}$.
(2) (a) T *is trivially coherent*,
 (b) *for all* $x \in T$ *there is a cofinal branch* b *through* x *and*
 (c) T *is closed under chains of length less than* δ *of uncountable cofinality*.

Proof. (1)$\Rightarrow$(2): is immediate.

[4] If there is a tree, order it lexicographically to get a linear ordering. Starting from a linear ordering, we construct a *partition tree* by the method of *atomization*. Both operations lose almost no information. A good exposition of these procedures can be found in [20].

(2)$\Rightarrow$(1): we apply Lemma 2.18 and show that $\mathscr{UC}_T$ is club. We actually show that $\mathscr{TC}_T = \mathscr{UC}_T$ in this case. So pick any $N \in \mathscr{TC}_T$ and let $K \subseteq N \cap T$ be captured by $L \in N$, where $\lim(L)$ has minimal height. If $\lim(K)$ has height γ, then $\lim(L)$ has height $\gamma^* = \min(N \cap \mathrm{Ord} \setminus \gamma)$. Let us distinguish two cases:

(i): $\mathrm{cf}(\gamma^*) = \omega$. Now $\gamma = \gamma^*$ and therefore $\lim(K) \in N$. By (b), we can choose a cofinal branch $b \in N$ that goes through $\lim(K)$. This b cofinally captures the chain K.
(ii): $\mathrm{cf}(\gamma^*) > \omega$. If this happens, we know by (c) that L is converging in T. So let $t \in T$ be its limit. Again by (b), there is a cofinal branch $b \in N$ that contains t and hence, cofinally captures K. □

Continuing our applications, we see that Lemma 2.17 allows us to weaken characterization (3) of Lemma 2.4 in case $\delta = \omega_1$:

Corollary 2.20. *The following are equivalent for any tree T of height ω_1:*

(1) *T is trivially coherent.*
(2) *There is a regressive $f : T \restriction \mathrm{Lim}(\omega_1) \to T$ such that the preimage of every point is a chain.*

Proof. In view of Lemma 2.4, we only need to show (2)$\Rightarrow$(1). For this, we choose a regressive f witnessing (2) and show that $\mathscr{TC}_T$ is club: let $N \prec \mathrm{H}_\theta$ contain f as an element and take any converging N-chain $K \subseteq N \cap T$. If we set $t = \lim(K)$, we know that $f(t)$ has a countable height below the ordinal $N \cap \omega_1$, so $f(t) \in N$. But the f-preimage of $f(t)$ is a chain containing t, so its downward closure L is definable in N and will contain K. We proved that N is T-trivial. We are done by Lemma 2.17. □

We present another game closely tied to the notion of trivial coherence. Remember the convention that each node in the tree is a binary sequence.

Definition 2.21. The game $\mathbb{G}_2(T)$ is played as follows:

I	x_0	x_1	x_2	x_3	...
II	i_0	i_1	i_2	i_3	...

where $x_n \in T$, $i_n \in \{0,1\}$, $x_n{}^\frown i_n \leqslant_T x_{n+1}$ for all $n < \omega$ and

II wins iff x_n $(n < \omega)$ does not converge in T.

The next lemma has to be compared with Davis' characterization of either countable or perfect sets of reals in terms of games (see [5]). His game is a special instance of our game in case when the considered tree has height ω:

Lemma 2.22. *The following are equivalent for any tree T:*

(1) *T is trivially coherent.*
(2) *II has a winning strategy in the game $\mathbb{G}_2(T)$.*

Proof. (1)$\Rightarrow$(2): let $\sigma(x_0,\ldots,x_n)=1$. σ wins $\mathbb{G}_2(T)$ for player II.

(2)$\Rightarrow$(1): we show that if $N\prec \mathrm{H}_\theta$ contains a winning strategy σ, then it is T-trivial. This implies that $\mathcal{TC}_T$ is club in $[\mathrm{H}_\theta]^{\aleph_0}$. Fix such an N and a chain $K\subseteq N$ converging in T. By going to a cofinal subchain if necessary, we may assume that K has order-type ω.

Say that the sequence $\vec{x}=\langle x_1,\ldots,x_n\rangle\in N$ is *nice* if $\{x_l\}_{l\leqslant n}\subseteq K$ and moreover $x_{l+1}\geqslant_T x_l{}^\frown\sigma(x_1,\ldots,x_l)$ for all $l<n$.

Subclaim 2.22.1. *There is a nice sequence* $\vec{x}=\langle x_1,\ldots,x_n\rangle$ *with the property that* $x_n{}^\frown\sigma(\vec{x})\in K$ *and for all* $y>_T x_n$ *in* $K: y^\frown\sigma(\vec{x},y)\notin K$.

Proof. Assume that there isn't. In this case, there is a play

I	x_1	x_2	x_3	x_4	…
II	i_1	i_2	i_3	i_4	…

such that $i_l=\sigma(x_1,\ldots,x_l)$ $(l<\omega)$ and $\{x_l\}_{l<\omega}$ is a cofinal subchain of K. But σ is winning, so K cannot converge, a contradiction. □

Choose a nice sequence $\vec{x}=\langle x_1,\ldots,x_n\rangle$ as in Subclaim 2.22.1. Define the following branch b within N: b is maximal with the property that

$$b \text{ extends } x_n{}^\frown\sigma(\vec{x}) \text{ and for } \alpha > \mathrm{ht}(x_n)\text{: } b(\alpha) = 1-\sigma(\vec{x}, b\restriction\alpha).$$

The last subclaim gives that N is T-trivial.

Subclaim 2.22.2. $K\subseteq b$.

Proof. If not, let α_0 be the splitting point, i.e. the smallest point α such that $K(\alpha)\neq b(\alpha)$. Let $j=\sigma(\vec{x},b\restriction\alpha_0)=1-b(\alpha_0)=K(\alpha_0)$ by choice of b. We set $y=b\restriction\alpha_0=K\restriction\alpha_0$ and have $y>_T x_n$ but $y^\frown\sigma(\vec{x},y)=y^\frown j\in K$, in contradiction to Subclaim 2.22.1. □ □

2.4. A dichotomy for trees

We introduce a dichotomy for trees that strengthens Suslin's hypothesis and follows from PFA. The dichotomy says that whenever we try, by conventional means, to construct an ω_1-tree with more structure than direct limits, the tree will have a stationary antichain.

Definition 2.23. A tree T of height κ^+ is called *special* if there is a function $f:T\to\kappa$ such that if $s\leqslant t$, u and $f(s)=f(t)=f(u)$, then t and u are comparable.

Note that if the tree T has no cofinal branches, Definition 2.23 agrees with the notion of special for Aronszajn-trees as introduced before. The following has to be compared with Corollary 2.8.

Lemma 2.24. *For any special ω_1-tree T:*

either (1) $T \cong \mathbb{Q}^{\text{fin}}_{<\omega_1}$,
or (2) *T has a stationary antichain.*

Proof. Let $f : T \to \omega$ specialize T, i.e. if there are $s \leqslant t, u$ such that $f(s) = f(t) = f(u)$, then t and u are comparable.

Suppose T is an ω_1-tree that is not isomorphic to $\mathbb{Q}^{\text{fin}}_{<\omega_1}$. We know that there are three possibilities: either (a), (b) or (c) of Lemma 2.19(2) are false. (b) is obviously void and if (c) were false, T would have a special Aronszajn-subtree that can easily be seen to contain a stationary antichain.

So we may restrict ourselves to the case where T is not trivially coherent and by Lemma 2.17 we conclude that $\mathscr{T}\mathscr{C}_T$ does not contain a club. Note that $\mathscr{C}_T$ is generally a smaller set than the complement of $\mathscr{T}\mathscr{C}_T$. But $T \cap N$ is downward closed for any countable elementary N, so $\mathscr{S}_T = \mathscr{T}\mathscr{C}_T$ and $\mathscr{C}_T$ must be stationary. We may assume that $f, T \in N$ for every $N \in \mathscr{C}_T$. As a consequence of the stationarity of $\mathscr{C}_T$, the projection $E = \{N \cap \omega_1\colon N \in \mathscr{C}_T\}$ to countable ordinals is stationary in ω_1. For every $\xi \in E$, pick one $N_\xi \in \mathscr{C}_T$ such that $\xi = N_\xi \cap \omega_1$ and a witness t_ξ for the T-complicatedness of N_ξ.

Subclaim 2.24.1. *$f(t) \neq f(t_\xi)$ for all $t <_T t_\xi$ ($\xi \in E$).*

Proof. Assume that $f(t) = f(t_\xi)$ for some $t <_T t_\xi$ ($\xi \in E$). Working in N_ξ, we define

$$b = \{u \in T\colon\ t \leqslant_T u \text{ and } f(t) = f(u)\}.$$

Note that this is possible in the substructure, because the height of t is below ξ and t is therefore an element of N_ξ. Every two elements of b will be comparable by the properties of f. So $b \in N_\xi$ is a branch and $t_\xi \in b$. This means that t_ξ is captured by $b \in N_\xi$, a contradiction. □

Pressing Down via the specializing function f, we obtain a stationary $E_0 \subseteq E$ such that $f(t_\xi)$ is constant for all $\xi \in E_0$. But with Subclaim 2.24.1 accomplished, we know that $\{t_\xi\colon \xi \in E_0\}$ is in fact a stationary antichain in T. □

Corollary 2.25. *If every ω_1-tree is special, we have the following dichotomy for ω_1-trees T:*

either (1) $T \cong \mathbb{Q}^{\text{fin}}_{<\omega_1}$,
or (2) *T has a stationary antichain.*

This is true under PFA *but still consistent with* CH.

Proof. Note that PFA implies that every tree of height ω_1 is special (see [2, p. 951] for this). On the other hand, Shelah [17, p. 394] shows that every ω_1-tree being special is consistent with CH. □

3. Coherent Aronszajn-trees

3.1. Non-trivial coherent sequences

In the last section we generated a full tree by a single function in allowing finite changes of values of the function, or equivalently we looked at all the mappings with finite support. We cannot expect to construct an Aronszajn-tree of that kind, so we might weaken this requirement: let us define a *coherent* sequence of functions to be a sequence $f_\alpha : \alpha \to \alpha$ $(\alpha < \kappa)$ such that $f_\alpha =^* f_\beta \restriction \alpha$ for all $\alpha < \beta < \kappa$, where $=^*$ denotes equality modulo finite, i.e. $\{\xi < \alpha\colon f_\alpha(\xi) \neq f_\beta(\xi)\}$ is finite. The tree

$$T(f_\alpha\colon \alpha < \kappa) = \{f\colon \alpha \to \alpha \mid f =^* f_\alpha,\ \alpha < \kappa\}$$

will be the *coherent tree induced by* f_α $(\alpha < \kappa)$. We call these trees *uniformly coherent*. More generally, any normal subtree of a uniformly coherent tree is said to be *coherent*. Note that these trees can be Aronszajn and we will actually construct some of them later. But if we really want them to be without cofinal branches, we have to consider *non-trivial* coherent sequences, i.e. sequences for which there is no $f : \kappa \to \kappa$ such that $f_\alpha =^* f \restriction \alpha$ for all $\alpha < \kappa$. Otherwise, the whole sequence is generated by one branch of the tree and only trivially coherent trees will result:

Remark 3.1. Coherent trees with branches are trivially coherent.

Proof. Assume that $T \subseteq T(f_\alpha\colon \alpha < \kappa)$ for a coherent sequence f_α $(\alpha < \kappa)$ and that $g_\alpha \in T_\alpha$ $(\alpha < \kappa)$ is a branch through T. It is easy to see that $T^* = T(g_\alpha\colon \alpha < \kappa)$ is isomorphic to $\mathbb{Q}^{\text{fin}}_{<\kappa}$. But $T \subseteq T^*$ since if $t \in T_\alpha$, $t =^* f_\alpha$ but $f_\alpha =^* g_\alpha$, so $t =^* g_\alpha$. $\square$

On the other hand, non-triviality of the sequence is equivalent to the fact that $T(f_\alpha\colon \alpha < \kappa)$ is Aronszajn because of the following lemma:

Lemma 3.2. *Let κ be regular uncountable. If $T(f_\alpha\colon \alpha < \kappa)$ has a cofinal branch then every stationary $B \subseteq T(f_\alpha\colon \alpha < \kappa)$ contains a stationary chain $B_0 \subseteq B$.*

Proof. This is essentially just a reproof of Lemma 2.7 in the case where κ is regular to get a slightly stronger conclusion. Assume that B is stationary in the tree $T(f_\alpha : \alpha < \kappa)$ and set $E = \{\text{ht}(t)\colon t \in B\}$. We assume without restriction that B contains at most one point of every level of the tree, so let $t_\xi \in T_\xi$ be this point for all $\xi \in E$. Since $T(f_\alpha\colon \alpha < \kappa)$ has a cofinal branch, we can choose a function $f : \kappa \to \kappa$ such that $f_\alpha =^* f \restriction \alpha$ for all $\alpha < \kappa$. Define a regressive mapping $h : E \to \kappa$ by letting

$$h(\xi) = \max\{\gamma < \xi\colon t_\xi(\gamma) \neq f(\gamma)\} + 1.$$

Note that h is regressive for all limit ordinals $\xi \in E$ since $T(f_\alpha\colon \alpha < \kappa)$ is coherent and f is a branch through it. By an application of the Pressing Down Lemma, there is a stationary $E_0 \subseteq E$ and $\gamma_0 < \kappa$ such that $h(\xi) = \gamma_0$ for all $\xi \in E_0$. By a cardinality argument, there is another stationary $E_1 \subseteq E_0$ with $t_\zeta \restriction \gamma_0 = t_\xi \restriction \gamma_0$ for all $\zeta < \xi$ in E_1. Now, $B_0 = \{t_\xi\colon \xi \in E_1\}$ is the desired chain. $\square$

So if S is a normal subtree of a uniformly coherent tree T, then every cofinal branch through T induces a cofinal branch through S:

Corollary 3.3. *Trivially coherent trees of regular height must have a cofinal branch.*

3.2. An axiomatic approach to coherent Aronszajn-trees

We hope that no confusion arises from the fact that we give up on our binary-splitting convention for the purpose of Sections 3.2 and 3.3. The reason for this is really the proof of Theorem 3.9, where technical reasons forced us to do so (cf. the paragraph after Definition 2.1).

The next definition originates from [18], where the following notation is used:

$$T^{t_0} = \{t \in T\colon\ t \text{ is comparable with } t_0\}.$$

Definition 3.4. A κ-tree T is called *strongly homogeneous* if there is a family $\{h_{t_0,t_1}\colon t_0,t_1 \in T_\alpha, \alpha<\kappa\}$ of automorphisms with the following properties:

(1) h_{t_0,t_1} moves T^{t_0} to T^{t_1} and vice versa, so t_0 is mapped to t_1. h_{t_0,t_1} is the identity in all other parts of the tree. $h_{t,t}$ is the identity on T.
(2) (commutativity) $h_{s_0,s_2}(t_0) = h_{s_1,s_2}(h_{s_0,s_1}(t_0))$ holds for all $s_0,s_1,s_2 \in T_\alpha$ with $s_0 \leqslant t_0$.
(3) (uniformity) If $s_0,s_1 \in T_\alpha$ with $s_0 \leqslant t_0$ and $s_1 \leqslant h_{s_0,s_1}(t_0) = t_1$ then $h_{t_0,t_1} \restriction T^{t_0} = h_{s_0,s_1} \restriction T^{t_0}$.
(4) (transitivity) If α is a limit ordinal and $t_0,t_1 \in T_\alpha$, then there exist $s_0,s_1 \in T_{<\alpha}$ such that $h_{s_0,s_1}(t_0) = t_1$.

We are going to see in the next two theorems that conditions (1)–(4) of Definition 3.4 are a precise characterization of uniformly coherent trees. This means, (1)–(4) provide a structural description of this kind of tree. The result has already been quoted in [14, p. 79].

Remark 3.5. Every uniformly coherent tree is strongly homogeneous.

Proof. Let us assume that T is uniformly coherent, so it is induced by some sequence $(f_\alpha : \alpha \to \alpha \mid \alpha < \kappa)$. For $t_0,t_1 \in T_\alpha$, $s \in T^{t_0}$ of height β $(\alpha,\beta<\kappa)$ define

$$h_{t_0,t_1}(s) = t_1 * s,$$

where $(t_1 * s)\restriction \alpha = t_1$ and $(t_1 * s)(\xi) = s(\xi)$ for $\alpha \leqslant \xi < \beta$. Then $(T, \subseteq)$ will be strongly homogeneous via $\{h_{t_0,t_1}\colon t_0,t_1 \text{ in a } T_\alpha\}$. $\square$

Theorem 3.6. *Every strongly homogeneous tree is isomorphic to a uniformly coherent tree.*

Proof. From now on let $(T, \leqslant)$ be a strongly homogeneous κ-tree via the family $h_{t_0,t_1}\colon t_0,t_1 \in T_\gamma (\gamma<\kappa)$ of automorphisms. The splitting of T does not matter here, but we assume for simplicity that T is ω-splitting. We also assume without restriction that T has a root. Define $\pi: T \to \{f: \alpha \to \omega \mid \alpha < \kappa\}$ inductively:

$\alpha = 0: T_0 = \{\text{root}\}$, $\pi(\text{root}) = \emptyset$.
$\lim(\alpha)$: for a $t \in T_\alpha$ set $\pi(t) = \bigcup \pi''\{s \in T: s < t\}$.
$\beta = a + 1$: choose an $x \in T_\alpha$, let $\pi(x) = f: \alpha \to \omega$ and well-order the immediate successors of x by the enumeration $\{x_n: n < \omega\}$.

Define $\pi(x_n) = f \cup \{(\alpha, n)\}$ and for any other $s \in T_\beta$ set $r = s \restriction \alpha$ and if $h_{r,x}(s) = x_m$, define $\pi(s) = \pi(r) \cup \{(\alpha, m)\}$.

Subclaim 3.6.1. *$\pi(t)(\alpha) = \pi(h_{s_0,s_1}(t))(\alpha)$ holds whenever $\xi \leqslant \alpha < \beta$, $t \in T_\beta$ and $s_0, s_1 \in T_\xi$.*

Proof. We may assume that $s_0 \leqslant t$. Now define $t' = h_{s_0,s_1}(t)$. We choose a master point $x \in T_\alpha$ and a master enumeration x_n $(n < \omega)$ of the immediate successors of x.

Let $\pi(t)(\alpha) = \pi(t \restriction (\alpha + 1))(\alpha) = m$. Since $h_{s_0,s_1}(t \restriction \alpha + 1) = t' \restriction \alpha + 1$, an application of uniformity will yield

$$h_{t\restriction\alpha, t'\restriction\alpha}(t \restriction (\alpha + 1)) = t' \restriction (\alpha + 1).$$

By commutativity, we can deduce

$$h_{x, t'\restriction\alpha}(h_{t\restriction\alpha, x}(t \restriction (\alpha + 1))) = t' \restriction (\alpha + 1),$$

so $h_{x,t'\restriction\alpha}(x_m) = t' \restriction (\alpha + 1)$. But this last equation means in particular that $m = \pi(t' \restriction (\alpha + 1))(\alpha) = \pi(t')(\alpha)$ since $h_{t'\restriction\alpha, x} = h^{-1}_{x, t'\restriction\alpha}$ and so we proved Subclaim 3.6.1. □

Subclaim 3.6.2. *If $t_0, t_1 \in T_\delta$ then the set*

$$\{\alpha < \delta: \pi(t_0)(\alpha) \neq \pi(t_1)(\alpha)\}$$

is finite.

Proof. By induction on δ. This is obvious for successor steps, so let δ be limit: by transitivity choose $s_0, s_1 \in T_\eta$, $\eta < \delta$ such that $h_{s_0,s_1}(t_0) = t_1$. Subclaim 3.6.1 establishes the following equation:

$$\{\alpha < \delta: \pi(t_0)(\alpha) \neq \pi(t_1)(\alpha)\} = \{\alpha < \eta: \pi(s_0)(\alpha) \neq \pi(s_1)(\alpha)\}.$$

But this last set is finite by induction hypothesis. □

Subclaim 3.6.3. *$\pi: T \to \pi''T$ is an isomorphism.*

Proof. π is clearly order-preserving. To show that π is one-to-one, let $\pi(s_0) = \pi(s_1)$. We proceed by induction on β to show that $s_0, s_1 \in T_\beta$ implies $s_0 = s_1$. We can assume without restriction that β is a successor ordinal and $\mathit{immpred}(s_0) = \mathit{immpred}(s_1) = s$ (else use induction hypothesis). Then $h_{s,t}(s_0) = h_{s,t}(s_1)$ holds for any $t \in T_{\mathrm{ht}(s)}$ by the definition of π. But $h_{s,t}$ is an automorphism, so $s_0 = s_1$. □

All that is left to show is the following subclaim:

Subclaim 3.6.4. *$\pi''T$ is uniformly coherent.*

Proof. It suffices to show the following: whenever $t \in T_\alpha$, $f = \pi(t)$ and $f =^* f' : \alpha \to \omega$ then $f' \in \pi''T$. But this is clear by uniformity and the fact that the automorphisms are onto. □ □

3.3. Coherent Aronszajn-trees of larger height

Our aim is to construct coherent Aronszajn-trees of arbitrary height. In fact, the existence of a coherent ω_2-Suslin-tree has already been shown consistent by Veličković in [25] using a very strong combinatorial guessing principle that holds true in the constructible universe (called square with built-in-diamond). It is well-known and remarked in [18] that coherent ω_1-Suslin-trees are either constructed by $\diamondsuit$ or forced by a Cohen-real. Note also that there are various ZFC-constructions for coherent ω_1-Aronszajn-trees of the form

$$f_\nu : \nu \xrightarrow{1-1} \omega \quad (\nu < \omega_1)$$

(see e.g. [12, p. 70] or [21]). The following argument indicates that these constructions are not so easily generalized to ω_2.

Theorem 3.7. *There is no sequence $(f_\nu : \nu < \omega_2)$ with*

(i) $f_\nu : \nu \xrightarrow{1-1} \omega_1$ *and*
(ii) $f_\nu =^* f_\mu \restriction \nu$ *for* $\nu < \mu$.

Proof. Assume that $(f_\nu : \nu < \omega_2)$ satisfies (i) and (ii). For all $\alpha < \omega_1$ and $\nu < \omega_2$ define $F_\alpha(\nu) = \{\gamma < \nu : f_\nu(\gamma) < \alpha\}$ and $\tau_\alpha(\nu) = \text{otp}(F_\alpha(\nu)) < \omega_1$. It is clear that for $\nu \leqslant \mu, F_\alpha(\nu) \subseteq^* F_\alpha(\mu)$ holds and thus $\tau_\alpha(\nu) < \tau_\alpha(\mu) + \omega$ also. Note the following:

Subclaim 3.7.1. *If $\nu + \omega_1 \leqslant \mu$ then there exists $\delta_{\nu,\mu} < \omega_1$ such that for all $\alpha \geqslant \delta_{\nu,\mu}$: $\tau_\alpha(\nu) + \omega < \tau_\alpha(\mu)$.*

Proof. To prove this, choose $\delta < \omega_1$ such that $\text{otp}(\delta \cap f''_\mu[\nu, \mu)) \geqslant \omega + 1$. But then for all $\alpha \geqslant \delta$:

$$\begin{aligned} \tau_\alpha(\nu) + \omega &= \text{otp}(\{\gamma < \nu :\ f_\mu(\gamma) < \alpha\}) + \omega \\ &< \text{otp}(\{\gamma < \mu :\ f_\mu(\gamma) < \alpha\}) = \tau_\alpha(\mu). \end{aligned} \tag{3.1}$$

This proves the subclaim. □

Now there is a fixed $\delta < \omega_1$ such that $\delta = \delta_{\lambda,\lambda+\omega_1}$ for stationarily many ω_1-cofinal λ's. Then for $\lambda < \lambda'$ such:

$$\tau_\delta(\lambda + \omega_1) < \tau_\delta(\lambda') + \omega < \tau_\delta(\lambda' + \omega_1), \tag{3.2}$$

but this contradicts $\tau_\alpha(\nu) < \omega_1$ for all $\alpha < \omega_1, \nu < \omega_2$. □

Another limitation is given by the next observation:

Remark 3.8. There are no coherent ω_2-Aronszajn-trees in the Levy-Collapse of a weakly compact cardinal to ω_2. Hence, CH does not imply the existence of a coherent ω_2-Aronszajn-tree.

Proof. By [23], every ω_2-Aronszajn-tree in this model contains the complete binary tree ${}^{<\omega_1}2$. This violates coherence in a strong fashion. $\square$

Remember that there are two canonical ways to construct an ω_2-Aronszajn-tree. One is using CH and the other one $\square(\omega_2)$. We just mentioned that CH is not enough, but as we shall see, there is a construction of a coherent ω_2-Aronszajn-tree from the other assumption, a $\square(\omega_2)$-sequence.

Theorem 3.9. *If $\square(\kappa)$ holds then there is a coherent κ-Aronszajn-tree.*

Proof. Let C_ν $(\nu<\kappa)$ be a $\square(\kappa)$-sequence. Of course, we can choose a square sequence with $1=\min(C_\nu)$ for all $\nu<\kappa$.

Inductively construct functions $f_\nu:\nu\to\nu$ $(\nu\in\operatorname{Lim}\kappa)$ with the following properties:

(i) $f_\lambda =^* f_\nu\restriction\lambda$ for all $\lambda<\nu<\kappa$,
(ii) all f_ν's are *almost one-to-one*, i.e. if $f_\nu(\alpha)=f_\nu(\beta)\neq 0$ with $\alpha<\beta<\nu$, then $\alpha=\beta$,
(iii) $C_\nu=\{\nu_\delta:\delta<\operatorname{otp}(C_\nu)\}$, where the ν_δ's are defined inductively:

$$\nu_0 = 1,$$
$$\nu_{\delta+1} = \text{the } \alpha > \nu_\delta \text{ with } f_\nu(\alpha)=\nu_\delta,$$
$$\nu_\lambda = \sup_{\delta<\lambda}\nu_\delta \ (\text{as long as } \nu_\lambda < \nu),$$

(iv) $\lambda\in C_\nu' \leftrightarrow f_\lambda = f_\nu\restriction\lambda$.

Note that the right-to-left direction of (iv) follows from (iii). We distinguish the following cases:

Case A. $\sup C_\nu'=\nu$.

This is simple, just let $f_\nu=\bigcup_{\lambda\in C_\nu'} f_\lambda$. (i)–(iv) will be maintained.

Case B. $\lambda=\sup C_\nu'<\nu$ (in particular cf $\nu=\omega$).

Case Ba. $\nu=\delta+\omega$ for a limit δ.

Let $C_\nu = C_\lambda \cup \{\lambda_n\}_{n<\omega}$, where $\lambda = \lambda_0 < \lambda_1 < \cdots < \nu$. We define f_ν by extending f_λ in the following way:

$$f_\nu(\alpha) = \begin{cases} f_\lambda(\alpha) & \text{if } \alpha < \lambda, \\ \lambda_n & \text{if } \alpha = \lambda_{n+1}, \\ f_\delta(\alpha) & \text{if } \alpha \in [\lambda, \delta) \setminus \{\lambda_n\}_{1 \leqslant n < \omega} \text{ and} \\ & f_\delta(\alpha) \notin \mathrm{rng}(f_\lambda) \cup \{\lambda_n\}_{n<\omega}, \\ 0 & \text{else.} \end{cases}$$

Now $f_\lambda =^* f_\delta \restriction \lambda$ leads to $f_\delta =^* f_\nu \restriction \delta$, because the second case of the definition of f_ν is violated only finitely many times whenever $\alpha \in [\lambda, \delta)$ (except for the trivial case $f_\delta(\alpha) = f_\nu(\alpha) = 0$). (ii), (iii) and (iv) are immediately seen to hold by construction.

Case Bb. $\nu \neq \delta + \omega$ for any limit δ (i.e. ν is a limit of limits).

Again, let $C_\nu = C_\lambda \cup \{\lambda_n\}_{n<\omega}$ with $\lambda = \lambda_0 < \lambda_1 < \cdots < \nu$. Choose an increasing sequence δ_n $(n<\omega)$ of limit ordinals with $\delta_0 = \lambda$, $\sup_n \delta_n = \nu$. We want to define f_ν to have the following properties:

(a) $f_\nu : \nu \to \nu$ is almost one-to-one,
(b) $f_\nu \restriction \lambda = f_\lambda$,
(c) $f_\nu =^* f_{\delta_n}$ for all $n<\omega$,
(d) $f_\nu(\lambda_{n+1}) = \lambda_n$.

Choose an $a_0 \subseteq [\lambda, \nu)$ cofinal in ν with order-type ω such that $g : (\nu \setminus a_0) \to \nu$ defined by

$$g = \bigcup_{n<\omega} f_{\delta_n} \restriction ([\delta_{n-1}, \delta_n) \setminus a_0)$$

is almost one-to-one, where we let $\delta_{-1} = 0$. We will set $a_1 = \{\lambda_n\}_{n<\omega}$ and $a_2 = \{\alpha < \nu : g(\alpha) \in a_1\}$. Now define $A = a_0 \cup a_1 \cup a_2 \subseteq [\lambda, \nu)$.

Subclaim 3.9.1. *A is cofinal in ν and has order-type ω.*

Proof. Since the subclaim is surely true for $a_0 \cup a_1$, it suffices to show that a_2 is either finite or ν-cofinal with order-type ω. Without any restriction, $a_0 \cap a_2 = \emptyset$ can be assumed for this. So suppose that a_2 is infinite. If a_2 was bounded in ν or had order-type bigger than ω, then in either case there would be $\beta < \nu$ and an infinite subset $\hat{a}_1 \subseteq a_1$ such that $g''(a_2 \cap \beta) = \hat{a}_1$. If this is so, pick $m < \omega$ such that $\delta_m > \beta$. By definition of g, $g \restriction \beta =^* f_{\delta_m} \restriction (\beta \setminus a_0)$ and therefore $f''_{\delta_m}(a_2 \cap \beta) =^* \hat{a}_1$. Since $\hat{a}_1$ is always cofinal in ν, this contradicts the fact that $f_{\delta_m} : \delta_m \to \delta_m$. $\square$

By Subclaim 3.9.1, it is sufficient to change g on points in A in order to make (d) true and still maintain property (c). To see this, note that $\hat{g} = g \restriction (\nu \setminus A)$ is such that $a_1 \cap \mathrm{dom}(\hat{g}) = a_1 \cap \mathrm{rng}(\hat{g}) = \emptyset$, so we may extend $\hat{g}$ to fulfill (a) and (d).

Finally define $f_\nu : \nu \to \nu$ by letting

$$f_\nu(\alpha) = \begin{cases} g(\alpha) & \text{if } \alpha \in (\nu \setminus A), \\ \lambda_n & \text{if } \alpha = \lambda_{n+1}, \\ 0 & \text{else.} \end{cases}$$

Now f_ν satisfies (a)–(d) and thus (i)–(iv). This finishes the construction of $(f_\nu : \nu \in \operatorname{Lim} \kappa)$.

Subclaim 3.9.2. $(f_\nu : \nu < \kappa)$ *induces a coherent* κ*-Aronszajn-tree.*

Proof. Note property (iv) of the construction:

$$\lambda \in C'_\nu \leftrightarrow f_\lambda = f_\nu \restriction \lambda.$$

Since C_ν $(\nu < \kappa)$ is a $\Box(\kappa)$-sequence, the following statement is a consequence of (iv):

$$\text{there is no unbounded } D \subseteq \operatorname{Lim} \kappa \text{ such that for all } \lambda < \nu \in D\text{: } f_\lambda \subseteq f_\nu. \tag{3.3}$$

Otherwise $C = \bigcup_{\lambda \in D} C_\lambda$ is a club in κ such that $C \cap \alpha = C_\alpha$ for all limit points α of C. This would contradict the fact that C_ν $(\nu < \kappa)$ is a $\Box(\kappa)$-sequence. (3.3) says that there is no cofinal branch through the trunk $\{f_\nu : \nu < \kappa\}$. But Lemma 3.2 gives that even the induced tree $T(f_\nu : \nu < \kappa)$ does not have a cofinal branch. □ □

The author does not know of a way to construct a coherent Aronszajn-tree on ω_2 with a ϱ-function. [5]

We can deduce by Lemma 3.2 that coherent Aronszajn-trees do not contain Aronszajn-trees of smaller height. Of course, they do not contain Cantor-subtrees either. This provides us with:

Corollary 3.10. *The following are equiconsistent under ZFC*:

(1) *there is a weakly compact cardinal*,
(2) *every* ω_2*-Aronszajn-tree does contain either an* ω_1*-Aronszajn-subtree or a Cantor-subtree*.

Proof. If we have a weakly compact we can destroy all ω_2-Aronszajn-trees in a suitable collapse [15]. Of course, the Levy-Collapse would also suffice to show that (1) implies (2) because of the results in [23]. For (2) to (1), we assume that every ω_2-Aronszajn-tree does contain an ω_1-Aronszajn-subtree or a Cantor-subtree. Then $\neg\Box(\omega_2)$ must hold by our Theorem 3.9 and the observation above. From this we know that ω_2 is weakly compact in L (see [10,21]). □

[5] Todorčević's method of ϱ-functions is not explicitly used in this paper, the interested reader is referred to [21] or [24].

This last corollary answers a question raised in [24, p. 244], but as Todorčević noted much earlier in unpublished work, this problem can also be solved with the help of a ϱ-function.

We already noted that we had to give up on our binary-splitting convention in the previous theorems for technical reasons. We can still have a coherent binary Aronszajn-tree if we wish:

Corollary 3.11. *If $\square(\kappa)$ holds then there is a coherent κ-Aronszajn-tree given by a sequence of functions of the form $f_\alpha : \alpha \to 2$ $(\alpha < \kappa)$.*

Proof. We start with the coherent Aronszajn-tree induced by the sequence $f_\alpha : \alpha \to \alpha$ $(\alpha < \kappa)$ constructed in Theorem 3.9. Now we define a new sequence $g_\alpha : \alpha \to 2$ $(\alpha \in C)$, this time on the club $C \subseteq \kappa$ of ordinals closed under some pairing function $\langle\,,\rangle$:

$$g_\alpha(\langle \zeta, \xi \rangle) = 1 \quad \text{iff} \quad f_\alpha(\zeta) = \xi.$$

We can look at the g_α's as being a code for the graph of f_α. It is easy to check that g_α $(\alpha \in C)$ is a coherent and non-trivial sequence with these properties inherited from the old sequence f_α $(\alpha < \kappa)$. $\square$

We finally investigate the impact speciality has on coherent trees. It turns out that we have both special and non-special coherent Aronszajn-trees.

Definition 3.12. If C_ν $(\nu < \kappa)$ is a $\square(\kappa)$-sequence, we define the associated tree $(\kappa, <^2)$ by letting

$$\alpha <^2 \beta \text{ if and only if } \alpha \in C'_\beta.$$

In this context, C_ν $(\nu < \kappa)$ is called *special* if the associated tree $(\kappa, <^2)$ is special.

The following lemma can be found in [4, p. 65], but we include the proof of it for convenience.

Lemma 3.13. *$\square_\lambda$ is equivalent to the existence of a special $\square(\lambda^+)$-sequence.*

Proof. For the left-to-right implication, we assume that C_α $(\alpha < \lambda^+)$ is a $\square_\lambda$-sequence. Our task is to specialize the tree $(\lambda^+, <^2)$, so we define a function $f : \lambda^+ \to \lambda + 1$ by letting $f(\alpha) = \mathrm{otp}(C_\alpha)$. Obviously, if $\mathrm{otp}(C_\alpha) = \mathrm{otp}(C_\beta)$ then $\alpha \not<^2 \beta$.

Now we concentrate on the other direction, i.e. right-to-left: let $f : \lambda^+ \to \lambda$ witness the speciality of the tree $(\lambda^+, <^2)$ with respect to the $\square(\lambda^+)$-sequence C_ν $(\nu < \lambda^+)$. Construct a continuous sequence $\beta_\nu(\xi)$ $(\xi \leqslant \theta_\nu)$ for every limit ordinal $\nu < \lambda^+$:

$$\begin{aligned} \beta_\nu(0) &= \text{the minimal limit point of } C_\nu, \\ \beta_\nu(\xi + 1) &= \text{the first limit point } \beta > \beta_\nu(\xi) \text{ in } C_\nu \\ &\quad \text{with minimal possible } f\text{-value.} \end{aligned}$$

It is important to note that f strictly increases as ξ increases because f is a specializing function. When this process stops at the point $\beta_\nu(\theta_\nu)$, the final segment $C_\nu \cap (\beta_\nu(\theta_\nu), \nu)$ is a set of order-type at most ω, so we can define

$$\bar{C}_\nu = \{\beta_\nu(\xi): \xi \leqslant \theta_\nu\} \cup (C_\nu \cap (\beta_\nu(\theta_\nu), \nu)).$$

Now the order-type of $\bar{C}_\nu$ is $\leqslant \lambda$ and $(\bar{C}_\nu: \nu < \lambda^+)$ is a $\Box_\lambda$-sequence by the uniform definition of the sequence $\beta_\nu(\xi)$ $(\xi \leqslant \theta_\nu)$. □

Theorem 3.14. *In the construction of Theorem* 3.9, *the following are equivalent*:

(1) $T(f_\nu: \nu < \kappa)$ *is special*,
(2) C_ν $(\nu < \kappa)$ *is special*.

Proof. For (1) ⇒ (2), choose a function $g: T \to \kappa$ that specializes the constructed tree $T(f_\nu: \nu < \kappa)$ and is regressive in the sense that $g(t) < \mathrm{ht}(t)$ for all $t \in T$. Now simply define a function $h: \kappa \to \kappa$ by letting $h(\nu) = g(f_\nu)$. This function is regressive: $h(\nu) = g(f_\nu) < \mathrm{ht}(f_\nu) = \nu$. By property (iv), i.e. $\lambda \in C'_\nu \leftrightarrow f_\lambda = f_\nu \restriction \lambda$, we know that h specializes the tree $(\kappa, <^2)$.

In (2) ⇒ (1), let the regressive $g: \kappa \to \kappa$ specialize the tree $(\kappa, <^2)$ and choose a pairing function $\langle\, , \rangle: \kappa \times \kappa \leftrightarrow \kappa$. We claim that the following function specializes the tree $T(f_\nu: \nu < \kappa)$:

$$h(f) = \langle g(\mathrm{ht}(f)), \Delta(f) \rangle,$$

where $\Delta(f)$ codes the finitely many values in which f differs from f_ν. To see that h is specializing, let f, f' be elements of T with $\nu = \mathrm{ht}(f) < \mathrm{ht}(f') = \nu'$ and moreover, let $h(f) = h(f') = \langle \varepsilon_0, \varepsilon_1 \rangle$. In this case ε_1 codes a finite subset of ν and $\varepsilon_0 = g(\nu) = g(\nu')$ holds. But if f and f' would be compatible, then $f <_T f'$ would hold and since we coded properly the differences from the trunk $\langle f_\nu: \nu < \kappa \rangle$, $f_\nu <_T f_{\nu'}$ would result. By (iv), we conclude that ν is a limit point of $C_{\nu'}$ and obtain a contradiction to the fact that g specializes the sequence C_ν $(\nu < \kappa)$. This finishes the proof. □

Corollary 3.15.
(a) *If* $\Box_\lambda$ *holds, there is a special coherent* λ^+*-Aronszajn-tree.*
(b) *If* $\lambda \geqslant \omega_2$ *and* $\Box(\lambda)$ *holds, there is a non-special coherent* λ*-Aronszajn-tree.*

Proof. (a) is provided by Lemma 3.13.

For (b) we note that Todorčević has shown a way to construct a non-special $\Box(\lambda)$-sequence from any given $\Box(\lambda)$-sequence in [22]. □

4. Reflecting non-coherence

4.1. Non-coherence might not reflect

Definition 4.1. A tree T *is locally coherent* if $T_{<\gamma}$ is trivially coherent for every $\gamma < \mathrm{ht}(T)$.

We show in this subsection that non-coherence does not necessarily reflect at the cardinality ω_2, i.e. there are locally coherent ω_2-trees that are not coherent. Remember that we already constructed a locally coherent tree that is not trivially coherent in Theorem 3.9. That tree, however, is still coherent. Now we present non-coherent trees of this sort.

Lemma 4.2. *The following are equivalent for any tree T:*

(1) *T is locally coherent.*
(2) *$\mathscr{LT}_T$ is club in $[\mathrm{H}_\theta]^{\aleph_0}$.*

Proof. Fix $\delta = \mathrm{ht}(T)$.

(1)$\Rightarrow$(2): take $N \prec \mathrm{H}_\theta$ that knows of our tree T. N will be locally T-trivial, since N contains embeddings $\psi_\gamma : T_{<\gamma} \xrightarrow{\sim} \mathbb{Q}^{\mathrm{fin}}_{<\gamma}$ for all $\gamma \in N \cap \delta$.

(2)$\Rightarrow$(1): if $\gamma < \delta$, pick $N \in \mathscr{LT}_T$ that contains the ordinal γ. But N is locally T-trivial, so N is $T_{<\gamma}$-trivial. $T_{<\gamma}$ is trivially coherent by Lemma 2.17. $\square$

Corollary 4.3. *ω_1-trees are locally coherent.*

Proof. Assume T is an ω_1-tree and if $K \subseteq N \cap T$ a converging bounded chain for some substructure N. Then $\gamma = \mathrm{ht}(\bigcup K) < N \cap \omega_1$, so $\bigcup K$ is an element of $T_\gamma \subseteq N$ that captures K. $\square$

Lemma 4.4. *The following are equivalent for any tree T:*

(1) *T is trivially coherent.*
(2) *T is locally coherent and $\mathscr{S}_T$ is club in $[\mathrm{H}_\theta]^{\aleph_0}$.*

Proof. By Lemmas 2.17 and 4.2. $\square$

The first interesting example of a locally coherent non-coherent tree can be constructed by utilizing the next definition which is taken from [20, Section 4]. The idea of the construction is to start with the tree $\mathbb{Q}^{\mathrm{fin}}_{<\kappa}$ and enrich it. We pick branches that do not converge and add new limit points on top of them. If we have the right resources, we can keep the initial segments of the tree thin, while obtaining a large antichain in the global tree.

Let $S^\omega_\kappa = \{\xi < \kappa : \mathrm{cf}(\xi) = \omega\}$.

Definition 4.5. Fix converging sequences $\eta_\xi = \langle \eta_\xi(n) : n < \omega \rangle$ for every $\xi \in S^\omega_\kappa$. For technical reasons, let $\eta_0 = \emptyset$. If $A \subseteq S^\omega_\kappa$, define

$$T(A) = \{ f \in {}^{<\kappa}2 : \mathrm{supp}(f) = F \cup \{\eta_\xi(n) : n < \omega\} \text{ for some } \xi \in A \cup \{0\} \text{ and finite } F \subseteq \kappa \setminus \xi \}.$$

Remark 4.6. For $\xi \in S^\omega_\kappa$, define $t_\xi \in {}^\xi 2$, by $\mathrm{supp}(t_\xi) = \{\eta_\xi(n) : n < \omega\}$. Then $\{t_\xi : \xi \in A\}$ is an antichain in $T(A)$.

Theorem 4.7. *Let κ be a regular cardinal. If $E \subseteq S^\omega_\kappa$ is stationary non-reflecting, then $T(E)$ is locally coherent non-coherent.*

Proof. Note that $T(E)$ has the stationary antichain $\{t_\xi : \xi \in E\}$. By Corollary 2.8 it is not trivially coherent. But $T(E)$ has a cofinal branch, so it is non-coherent by Remark 3.1.

In order to prove local coherence, pick $N \prec \mathrm{H}_\theta$ such that $E \in N$:

Subclaim 4.7.1. *$N \in \mathscr{LT}_{T(E)}$, therefore $\mathscr{LT}_{T(E)}$ is club.*

Proof. Take any bounded chain $K \subseteq N$ that converges in $T(E)$ and let $\lim(K) = x \in T_\gamma$. We may assume that $x = t_\gamma$, so that $\gamma \in E$. Now define

$$\gamma^* = \min((N \cap \mathrm{Ord}) \backslash \gamma) < \kappa.$$

Case 1. $\mathrm{cf}(\gamma^*) = \omega$. Thus, $\gamma = \gamma^* \in N$, so t_γ is definable in N. But this means that K is captured by N.

Case 2. $\mathrm{cf}(\gamma^*) > \omega$. In N, there is a club $C \subseteq \gamma$ disjoint from E. Hence, $\gamma \notin E$, a contradiction. □

This means we are done by Lemma 4.2. □

Corollary 4.8. *If every locally coherent ω_2-tree is trivially coherent, then every stationary subset of $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ reflects.*

We remark that a similar construction works for $\square(\kappa)$-sequences:

Theorem 4.9. *Let κ be a regular cardinal. If $\square(\kappa)$ holds then there is a locally coherent non-coherent κ-tree.*

Proof. Fix a $\square(\kappa)$-*sequence* $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa)\rangle$. We define the tree $T(\vec{C})$ as follows:

$$T(\vec{C}) = \{f \in {}^{<\kappa}2 : \mathrm{supp}(f) = F \cup C_\alpha \text{ for some limit ordinal } \alpha < \kappa \text{ and finite } F \subseteq \kappa \backslash \alpha\}.$$

First, we note that $T(\vec{C})$ is really a normal tree when ordered by inclusion: if $f \in T(\vec{C})$ and $\gamma < \mathrm{dom}(f)$ we know that $f \restriction \gamma \in T(\vec{C})$ by coherence of $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa)\rangle$.

Subclaim 4.9.1. *$T(\vec{C})$ is locally coherent.*

Proof. Let $N \prec \mathrm{H}_\theta$ with $\vec{C} \in N$. If $K \subseteq N \cap T$ is a convergent chain, set $f = \lim(K)$ and $\gamma = \mathrm{ht}(f)$. In this context, we may assume that $\mathrm{supp}(f)$ is unbounded in γ, otherwise a capturing branch is easily defined within the substructure N. So the definition of $T(\vec{C})$ leaves only one remaining choice, i.e. $\mathrm{supp}(f) = C_\gamma$. In this case, define

$$\gamma^* = \min((N \cap \mathrm{Ord}) \backslash \gamma) < \kappa.$$

We may assume that $\mathrm{cf}(\gamma^*)>\omega$, so by elementarity $\gamma\in C'_{\gamma^*}$. We conclude that $C_\gamma = C_{\gamma^*}\cap\gamma$ by coherence. Let us define $g:\gamma^*\to 2$ by $\mathrm{supp}(g)=C_{\gamma^*}$. Then $g\in N$ and $f\subseteq g$. This proves the subclaim. $\square$

Now assume that $T(\vec{C})$ is trivially coherent. A Pressing Down argument similar to the proof of Lemma 3.2 would yield an uncountable chain $\langle f_\alpha:\alpha\in B\rangle$, where $\mathrm{supp}(f_\alpha)=C_\alpha$. But then B' is a club trivializing the $\square(\kappa)$-sequence $\langle C_\alpha: \alpha\in\mathrm{Lim}(\kappa)\rangle$. So $T(\vec{C})$ is not trivially coherent and since the zero-sequence is a cofinal branch, it is non-coherent, again by Remark 3.1. $\square$

4.2. *An* SPFA-*result*

The following arguments show that the constructions of Theorems 4.7 and 4.9 are not possible in ZFC alone. We use SPFA, the semiproper forcing axiom, to prove the consistency of the statement ‘every locally coherent ω_2-tree is trivially coherent’.

A good introduction to the following techniques is [19].

Definition 4.10. We call a finite $\in$-chain of submodels *continuous* if it can be extended to a continuous $\in$-chain of length ω_1.

Again, let us fix a tree T and assume that every substructure referred to will contain this tree without further saying. Define the two posets $\mathscr{P}_T$ and $\mathscr{Q}_T$:

$$\mathscr{P}_T=\{p:\ p \text{ is a finite continuous } \in\text{-chain of models } N\prec \mathrm{H}_\theta\},$$

$$\mathscr{Q}_T=\{p:\ p \text{ is a finite continuous } \in\text{-chain of models } N\prec \mathrm{H}_\theta \text{ such that either } \begin{array}{l}\bullet\ N \text{ is } T\text{-complicated, or}\\ \bullet \text{ every } M\succ_{\omega_1} N \text{ is } T\text{-simple}\}.\end{array}$$

Both posets are ordered in the same way: let $q\leqslant p$ iff $q\supseteq p$ and

$$M\in q\backslash p, N\in p, M\in N \text{ implies } t_M\not\leqslant t_N, \text{ whenever } M,N\in\mathscr{C}_T.$$

Lemma 4.11. *For any tree* T:

(a) $\mathscr{P}_T$ *is proper*.
(b) $\mathscr{Q}_T$ *is semiproper*.

Proof. Fix $\delta=\mathrm{ht}(T)$.

(a): Pick an elementary $N\prec\mathrm{H}_\lambda$ for a large enough regular λ and a condition $p\in\mathscr{P}_T\cap N$. The following extension of p will be generic:

$$q=p\cup\{(\gamma,N\cap\mathrm{H}_\theta)\},\quad \text{where } \gamma=N\cap\omega_1.$$

Subclaim 4.11.1. q *is* $(N,\mathscr{P}_T)$*-generic.*

Proof. Choose $\mathscr{D}\in N$ dense open and extend $q\geqslant r\in\mathscr{D}$. Now define $n=|r\backslash N|$, $p_0=r\cap N$ and let $r\backslash N=\{(\gamma_l,N_l):\ 1\leqslant l\leqslant n\}$. For notational simplicity, we assume that all

$N_1,\dots,N_n$ are T-complicated. Working in N and using elementarity we construct a tree $(S,\leqslant_S)$ of height $n+1$ with the following properties:

(a) elements of S are either $\mathrm{root}_S=p_0$ or tuples $s=(\alpha_s,K_s)$, where α_s is a countable ordinal and K_s a T-complicated substructure,
(b) if $s=(\alpha_s,K_s)\in S$ and $\{(\alpha_x,K_x)\}_{x\in\mathscr{I}_s}$ is the set of immediate successors of s, then $\{K_x\}_{x\in\mathscr{I}_s}$ is unbounded in $[\mathrm{H}_\theta]^{\aleph_0}$,
(c) every branch through S is a condition in $\mathscr{D}$.

Such a tree exists in N since $r\in\mathscr{D}$ is a condition in the universe that guarantees the existence of arbitrary large versions of N_i in the substructure N for every $1\leqslant i\leqslant n$ whose collection together with p_0 is in $\mathscr{D}$. Still in N, construct a path $B=\{p_0,s_1,\dots,s_n\}$ through S inductively: if s_l has been fixed for some $l<n$, let us consider the set $X_l=\{(\alpha_x,K_x)\}_{x\in\mathscr{I}_l}$ of all immediate successors of s_l. Now apply the Dilworth-decomposition theorem (see [6]) to the partial order $\mathbb{A}=\{t_{K_x}\colon x\in\mathscr{I}_l\}$ with the inherited tree ordering: if there is an antichain of size $n+1$ in $\mathbb{A}$, choose $s_{l+1}=(\alpha_{l+1},K_{l+1})\in X_l$ such that $t_{K_{l+1}}\not\leqslant t_{N_i}$ for all $1\leqslant i\leqslant n$. If there is no such antichain, represent

$$\mathbb{A}=\bigcup\{\mathscr{C}_1,\dots,\mathscr{C}_n\}$$

as a union of n-many chains $\mathscr{C}_i$. Now by unboundedness of $\{K_x\}_{x\in\mathscr{I}_l}$, there is $x\in\mathscr{I}_l$ with $K=K_x\supseteq\{\mathscr{C}_0,\dots,\mathscr{C}_{n-1}\}$. Fix $1\leqslant i\leqslant n$ such that $t_K\in\mathscr{C}_i$. Let b be the downward closure of $\mathscr{C}_i$ in the tree T. We know that $b\in K$ and b captures the chain below t_K. But this contradicts T-complicatedness of K.

So we can construct a path $B=\{p_0,s_1,\dots,s_n\}$ through S such that $t_{K_j}\not\leqslant t_{N_i}$ for all $1\leqslant j\leqslant n$ and $1\leqslant i\leqslant n$. We are thus done by the fact that $B\in\mathscr{D}\cap N$ and $B\cup r$ is a condition in $\mathscr{P}_T$ that extends both B and r. □

(b): Pick an elementary $N\prec\mathrm{H}_\lambda$ for a large enough regular λ and a condition $p\in\mathscr{Q}_T\cap N$. If every $M\succ_{\omega_1}N\cap\mathrm{H}_\theta$ is T-simple, set $N^*=N$. Otherwise choose $M\succ_{\omega_1}N\cap\mathrm{H}_\theta$ for which t_M is defined and let

$$N^*=Sk_{\mathrm{H}_\lambda}(N\cup(M\cap\delta)).$$

Subclaim 4.11.2. (i) $N^*\cap\delta=M\cap\delta$
(ii) $N^*\succ_{\omega_1}N$
(iii) t_M *witnesses* T*-complicatedness of* $N^*\cap\mathrm{H}_\theta$, *i.e.* t_M *is not captured by* $N^*\cap\mathrm{H}_\theta$.

Proof. For (i), let $\gamma\in N^*\cap\delta$. Then there are a formula φ and parameters $\vec{a}\in N^{<\omega}$, $\vec{\beta}\in(M\cap\delta)^{<\omega}$ such that $\varphi(\vec{a},\vec{\beta},\xi)$ defines γ in H_λ. Define the function

$$f:(M\cap\delta)^{<\omega}\to\delta$$

by letting $f(\vec{\zeta})=\xi$ if there is exactly one $\xi<\delta$ such that $\mathrm{H}_\lambda\models(\vec{a},\vec{\zeta},\xi)$. Otherwise, let $f(\vec{\zeta})=0$. But $f\in N$ since $\vec{a}\in N^{<\omega}$, so f is in the superstructure M as well. Finally, $\gamma=f(\vec{\beta})\in M\cap\delta$.

(ii) follows easily from (i), whereas (iii) is the same definability argument as above: instead of ordinals, we look for a capturing branch b in the range of our Skolem-functions and find that every such $b \in N^*$ is already in M. □

Now define the condition

$$q = p \cup \{(\gamma, N^* \cap \mathrm{H}_\theta)\}, \quad \text{where } \gamma = N^* \cap \omega_1.$$

The proof of the following subclaim follows the lines of the proof of subclaim 4.11.1:

Subclaim 4.11.3. *q is $(N^*, \mathcal{Q}_T)$-generic, hence $(N, \mathcal{Q}_T)$-semigeneric.* □

Corollary 4.12. *If $\mathscr{C}_T$ is club then $\mathcal{Q}_T$ is forcing-equivalent to $\mathscr{P}_T$ and thence proper.*

Theorem 4.13. *Under* SPFA, *if* $\mathrm{cf}(\delta) \geqslant \omega_1$ *and $\mathscr{C}_T$ is stationary for a tree of height δ, then there is an ω_1-cofinal ordinal $\gamma \leqslant \delta$ such that $T_{<\gamma}$ contains a non-trivial antichain. Moreover, $\gamma = \delta$ if* $\mathrm{cf}(\delta) = \omega_1$.

Proof. We apply the semiproper forcing axiom to the poset $\mathcal{Q}_T$: fix an enumeration $\phi_N : \omega \to N$ of every substructure N and write $N(n)$ for $\phi_N(n)$. Now define the dense sets B_α, D^n_γ for all $\alpha < \omega_1$, $n < \omega$ and $\gamma \in \mathrm{Lim}(\omega_1)$:

$$B_\alpha = \{p \in \mathcal{Q}_T:\ \alpha \in \mathrm{dom}(p)\}, \tag{4.1}$$

$$D^n_\gamma = \{p \in \mathcal{Q}_T:\ \gamma \in \mathrm{dom}(p) \text{ and there is } \beta \in \gamma \cap \mathrm{dom}(p) \text{ such that } p(\gamma)(n) \in p(\beta)\}. \tag{4.2}$$

We show that B_α is dense: fix $\alpha < \omega_1$ and let $p = \{(\alpha^l, M^l):\ l < m\}$ be a condition in $\mathcal{Q}_T$ that can be extended to the continuous $\in$-chain $\langle M_\nu:\ \nu < \omega_1\rangle$. Let $k < m$ be the smallest integer such that $\alpha \leqslant \alpha_k$. We may assume that $\alpha < \alpha_k$ and that all M^l $(k \leqslant l < m)$ are T-complicated. Now argue in M_α: like in the proof of Subclaim 4.11.1, we find a T-complicated structure K in M_α such that $M^l \in K$ for all $l < k$ and $t_K \not\leqslant t_{M^l}$ for all l with $k \leqslant l < m$. Then $q = p \cup \{(\alpha, K)\}$ extends p and $q \in B_\alpha$. The argument for D^n_γ is similar.

If we choose a generic filter for all B_α, D^n_γ defined above, we get a continuous $\in$-chain $\langle N_\xi:\ \xi < \omega_1\rangle$.

Subclaim 4.13.1. *$E = \{\xi < \omega_1:\ N_\xi$ is T-complicated$\}$ is stationary.*

Proof. Assume that E is nonstationary. Then for every $M \prec \mathrm{H}_\theta$ with $\langle N_\xi:\ \xi < \omega_1\rangle \in M$, there is a club $C \subseteq \omega_1$ in M such that $C \cap E = \emptyset$. Hence

$$\gamma = M \cap \omega_1 = \sup(M \cap C) \in C,$$

i.e. N_γ is T-simple. We have that $N_\gamma = \bigcup\{N_\xi\}_{\xi<\gamma} \subseteq M$ because M knows of the sequence $\langle N_\xi:\ \xi < \omega_1\rangle$. Of course $\gamma \subseteq N_\gamma$, so $N_\gamma \cap \omega_1 = \gamma = M \cap \omega_1$, therefore $N_\gamma \prec_{\omega_1} M$. By the definition of $\mathcal{Q}_T$ we get that M is T-simple. Thus, there is a club $\mathcal{D}=\{M\prec \mathrm{H}_\theta$: $\langle N_\xi$: $\xi<\omega_i\rangle\in M\}\subseteq[\mathrm{H}_\theta]^{\aleph_0}$ inside of $\mathcal{S}_T$, in contradiction to the assumption that $\mathcal{C}_T$ is stationary in $[\mathrm{H}_\theta]^{\aleph_0}$. □

Subclaim 4.13.2. *There is a stationary $E'\subseteq E$ such that $t_{N_\eta}\not\leqslant t_{N_\xi}$ for all $\eta<\xi\in E'$.*

Proof. First, check that the set $\{\alpha\in\xi\cap E$: $t_{N_\alpha}<t_{N_\xi}\}$ is finite for all $\xi\in E$: choose a condition p forcing that N_ξ is in the generic sequence. From the definition of $\leqslant_{\mathcal{Q}_T}$, all structures in $\{N_\alpha$: $\alpha\in\xi\cap E$ and $t_{N_\alpha}<t_{N_\xi}\}$ are in p. So let $\beta_\xi<\xi$ be a code for $\{\alpha\in\xi\cap E : t_{N_\alpha}<t_{N_\xi}\}$. By Pressing Down, we obtain a stationary E_0 and β such that $\beta_\xi=\beta$ for all $\xi\in E_0$. The statement of the subclaim then holds for $E'=E_0\setminus(\beta+1)$. □

We set $\gamma_\xi=\sup(N_\xi\cap\delta)$ and $\gamma=\sup_{\xi<\omega_1}\gamma_\xi$. This is the time to note that $\gamma=\delta$ if $\mathrm{cf}(\delta)=\omega_1$ and $\gamma<\delta$ otherwise.

Subclaim 4.13.3. *$\{t_{N_\xi}$: $\xi\in E'\}$ is a non-trivial antichain in $T_{<\gamma}$.*

Proof. Note that $A=\{t_{N_\xi}$: $\xi\in E'\}$ is a stationary antichain in $T_{<\gamma}$ by Subclaim 4.13.2. It remains to show that no stationary subset $A_0\subseteq A$ projects 1–1 into a level $\beta<\gamma$, so fix $A_0\subseteq A$ stationary and $\beta<\gamma$. Now enumerate

$$N_{\omega_1}=\bigcup_{\xi<\omega_1} N_\xi=\{x_\alpha\}_{\alpha<\omega_1}$$

and assume without further restriction that $N_\xi=\{x_\alpha\}_{\alpha<\xi}$ for all ordinals $\xi\in E'$. Via this enumeration we are able to define a regressive mapping on the stationary set $\{\xi\in E'$: $t_{N_\xi}\in A_0\}\setminus(\beta+1)$:

$$h(\xi)=\text{some } t<_T t_{N_\xi} \text{ in } N_\xi \text{ above the level } \beta.$$

Pressing Down, we get a point $t_0\in T$ above the level β and stationarily many points t_{N_ξ} with $t_0<_T t_{N_\xi}$. This shows that A_0 does not project 1–1 into the level $\mathrm{ht}(t_0)$, so it does not project 1–1 into the level β. □ □

Corollary 4.14. *Under* SPFA, *if* $\mathrm{cf}(\delta)\geqslant\omega_2$ *then every locally coherent tree of height δ is trivially coherent.*

Proof. Assume T is not trivially coherent. By Lemma 4.4, we are able to assume that $\mathcal{C}_T$ is stationary in $[\mathrm{H}_\theta]^{\aleph_0}$. Theorem 4.13 states that there is $\gamma<\delta$ such that $T_{<\gamma}$ contains a non-trivial antichain. But this contradicts local coherence by Lemma 2.7. □

Now we see why there is no proper poset that does the job of $\mathcal{Q}_T$: Corollaries 4.14 and 4.8 show that we need to assume stationary reflection of subsets of $\{\alpha<\omega_2$: $\mathrm{cf}(\alpha)=\omega\}$ to reach the conclusion of Theorem 4.13. But it is proved in [3] that PFA does not reflect such stationary sets.

It is shown in the author's thesis [11] that the conclusion of Corollary 4.14, the reflection of non-triviality, holds true in the Levy-Collapse of a supercompact cardinal to ω_2.[6] But note that the stronger statement, the conclusion of Theorem 4.13, implies the dichotomy for ω_1-trees from 2.25 if we let $\delta=\omega_1$. So the conclusion of Theorem 4.13 can never be valid in the Levy-Collapse by a result of Baumgartner (see [16]), saying that every σ-closed forcing which adds a subset of ω_1 also adds a $\diamondsuit$-sequence on ω_1. In particular, there is always an ω_1-Suslin-tree after Levy-Collapsing a large cardinal to ω_2.

The proof of Theorem 4.13 has a number of other corollaries. The first one is achieved by using Corollary 4.12:

Theorem 4.15. *Under* PFA, *if* $\mathrm{cf}(\delta)\geqslant\omega_1$ *and* $\mathscr{C}_T$ *is club for a tree of height* δ, *then there is an* ω_1*-cofinal ordinal* $\gamma\leqslant\delta$ *such that* $T_{<\gamma}$ *contains a non-trivial antichain.*

Another observation is that SPFA can be replaced by PFA$^+$.

Theorem 4.16. *Under* PFA$^+$, *if* $\mathrm{cf}(\delta)\geqslant\omega_1$ *and* $\mathscr{C}_T$ *is stationary for a tree of height* δ, *then there is an* ω_1*-cofinal ordinal* $\gamma\leqslant\delta$ *such that* $T_{<\gamma}$ *contains a non-trivial antichain. Moreover*, $\gamma=\delta$ *if* $\mathrm{cf}(\delta)=\omega_1$.

Proof. Forcing with the proper poset $\mathscr{P}_T$, we get a generic sequence of models $\langle M_\xi: \xi<\omega_1\rangle$ in $V^{\mathscr{P}_T}$.

Subclaim 4.16.1.

$$V^{\mathscr{P}_T}\models \dot{E}=\{\xi<\omega_1 : M_\xi \textit{ is } T\textit{-complicated}\} \textit{ is stationary.}$$

Proof. Assume that $p\Vdash\dot{C}$ is club. Pick a T-complicated $N\prec \mathrm{H}_\theta$ such that $p, \dot{C}\in N$. This can be done by stationarity of $\mathscr{C}_T$. Now let $\gamma=N\cap\omega_1$ and $q=p\cup\{(\gamma,N)\}$. Since $q\Vdash\gamma=N\cap\omega_1=\sup(N\cap\dot{C})\in\dot{C}$, we have $q\Vdash\gamma\in\dot{E}\cap\dot{C}$. This proves the subclaim. □

With 4.16.1 accomplished, we use PFA$^+$ to get a filter $G=\langle N_\xi: \xi<\omega_1\rangle$, generic for B_α, D^n_γ ($\alpha<\omega_1$, $n<\omega$, $\gamma\in\mathrm{Lim}(\omega_1)$) (as defined in (4.1), (4.2)) and with the additional property that the computed set $\dot{E}[G]$ is stationary in ω_1. Now continue exactly like in the proof of Theorem 4.13, i.e. repeat the proofs of Subclaims 4.13.2 and 4.13.3. □

Like in the case of Corollary 4.14, we get:

Corollary 4.17. *Under* PFA$^+$, *if* $\mathrm{cf}(\delta)\geqslant\omega_2$ *then every locally coherent tree of height* δ *is trivially coherent.*

With an argument similar to the previous techniques, we can show that RP_2 determines the game $\mathbb{G}^{\mathrm{coh}}_T$:

[6] If we require this reflection only at the point ω_2 instead of all ordinals of cofinality at least ω_2, we need just a weakly compact.

Lemma 4.18. *Assume that* RP_2 *holds and* $\mathrm{cf}(\delta)\geqslant\omega_2$. *If* T *is a locally coherent tree of height* δ *with* $\mathscr{S}_T$ *stationary, then* T *is trivially coherent.*

Proof. Assume that $\mathscr{C}_T$ is stationary. Then RP_2 will provide us with an $\in$-chain $\langle M_\xi : \xi\leqslant\omega_1\rangle$ such that the set $E=\{\xi<\omega_1\colon M_\xi$ is T-complicated$\}$ is stationary, co-stationary.

Let $\gamma_\xi=\sup(M_\xi\cap\delta)$ and $\gamma=\sup_{\xi<\omega_1}\gamma_\xi$. We have that $\gamma<\delta$ by the large cofinality of δ.

Subclaim 4.18.1. $\{t_{M_\xi}\colon \xi\in E\}$ *contains a chain of order-type* ω_1.

Proof. By local coherence, we can pick an embedding $\pi: T_{<\gamma}\xrightarrow{\sim}\mathbb{Q}^{\mathrm{fin}}_{<\gamma}$ and define $\varepsilon_\xi=\{\alpha<\gamma_\xi\colon \pi(t_{M_\xi})(\alpha)\neq 0\}$. Like in preceding arguments, we Press Down with respect to a fixed enumeration of M_{ω_1}. For this let

$$g(\xi)=\text{some } x\in M_\xi\cap T \text{ such that } x<_T t_{M_\xi} \text{ and } \mathrm{ht}(x)>\sup\varepsilon_\xi.$$

The Pressing Down Lemma gives a stationary $E_0\subseteq E$ and $x\in T$ such that $t_{M_\xi}=x^\frown\vec{0}$ for all $\xi\in E_0$. Thus, $\langle t_{M_\xi}\colon \xi\in E_0\rangle$ is an uncountable chain. □

But this contradicts the following subclaim which will therefore finish the proof:

Subclaim 4.18.2. $\{\xi\in E\colon t_{M_\xi}\in b\}$ *is countable for all branches* $b\subseteq T$.

Proof. Assume that b is a branch through T hitting uncountably many t_{M_ξ}'s. Let us first check that the set

$$C=\{\xi<\omega_1\colon\ b\cap M_\xi \text{ is cofinal in } M_\xi\}$$

is closed and unbounded in ω_1. It is obviously closed and unboundedness holds since $E\cap C$ is uncountable. For all ξ in the stationary set $C\setminus E$, define

$$h(\xi)=\text{some } L\in M_\xi \text{ such that } b\restriction\gamma_\xi\subseteq L.$$

Note that this is possible as M_ξ is T-simple and $\xi\in C$. Pressing Down again, there is $\zeta<\omega_1$ and a δ-branch $L\in M_\zeta$ such that $b\restriction\gamma\subseteq L$. Using the assumption about b, we pick a T-complicated M_η above M_ζ with $t_{M_\eta}\in b$. But now $L\in M_\zeta\subseteq M_\eta$ contradicts the fact that t_{M_η} witnesses the T-complicatedness of M_η. □ □

Theorem 4.15 and Lemma 4.18 lead us to the following strengthening of both Corollaries 4.14 and 4.17:

Theorem 4.19. *Under* $\mathrm{PFA}+\mathrm{RP}_2$, *if* $\mathrm{cf}(\delta)\geqslant\omega_2$ *then every locally coherent tree of height* δ *is trivially coherent.*

We find the following question still interesting.

Question 4.20. *Is the reflection of non-triviality equivalent to some well-known reflection principle for stationary sets*?

As we pointed out in the introduction, this seems unlikely but not much is known about it. The consistency strength of such a reflection principle has to be at least weakly compact by Theorem 4.9. The reflection of non-triviality at ω_2 has strength exactly weakly compact (cf. footnote 6) and therefore is definitely stronger than the reflection of stationary subsets of $S^{\omega}_{\omega_2}$ which has the consistency strength of a Mahlo cardinal.

As a last remark, let us go back to Question 0.1. If κ is regular, the assumptions above do not just imply the reflection of non-triviality but really the reflection of the tree $\mathbb{Q}^{\text{fin}}_{<\kappa}$ itself.

Theorem 4.21. *Let κ be regular. Under* PFA + RP$_2$, *if T is a κ-tree with*

$$T_{<\gamma} \cong \mathbb{Q}^{\text{fin}}_{<\gamma} \quad \textit{for all } \gamma < \kappa,$$

then $T \cong \mathbb{Q}^{\text{fin}}_{<\kappa}$.

Proof. By Theorem 4.19, T is trivially coherent. We use Lemma 2.19 to show that $T \cong \mathbb{Q}^{\text{fin}}_{<\kappa}$. First, note that by the assumption of T being locally isomorphic to $\mathbb{Q}^{\text{fin}}_{<\kappa}$, every chain of uncountable cofinality converges in T. Note that as a subtree of T, the tree T^x is also trivially coherent for every $x \in T$. So since κ is regular we can apply Corollary 3.3 to T^x. Hence every point can be cofinally extended. So (b) and (c) of Lemma 2.19 are true and we are done. □

Note also that the examples from 4.7 and 4.9 are not just locally coherent but even *locally isomorphic* to $\mathbb{Q}^{\text{fin}}_{<\omega_2}$, i.e. $T_{<\gamma} \cong \mathbb{Q}^{\text{fin}}_{<\gamma}$ for all $\gamma < \omega_2$. This can easily be checked by using Lemma 2.18. So Question 0.1 is independent of ZFC for $S = \mathbb{Q}^{\text{fin}}_{<\omega_2}$, but it seems still open for arbitrary ω_2-trees:

Question 4.22. *Is it consistent that every two locally isomorphic ω_2-trees are isomorphic*?

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