

**“TOPICS IN ALGEBRA” COURSE NOTES
AUTUMN 2003**

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PREFACE

These are the notes for the course “Topics in Algebra” — really a course on algebraic stacks — given by Professor Robert Kottwitz during the Autumn 2003 quarter at the University of Chicago. This is my first draft of the notes, and I expect they read as such. The exposition is unpolished in many places, and the diagrams could stand considerable improvement. I have followed almost exactly the organization of Professor Kottwitz’s lectures, with only a few of minor changes made where I felt it was natural to do so. For sure, the large-scale structure of the course remains intact. I have made only a minimal effort beyond that of Professor Kottwitz to be systematic with notation. No effort has been made to be systematic with references. Amenities such as a bibliography or an index will have to wait until future revisions. Any mathematical errors are almost certainly my own.

The notes are freely available on the web at <http://www.math.uchicago.edu/~bds/>. I welcome any comments, suggestions, and — especially — corrections at bds@math.uchicago.edu.

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NOTATIONS AND CONVENTIONS

All categories considered in these notes are assumed small unless noted otherwise. For any category \mathcal{C} , we write \mathcal{C}^{opp} to denote the opposite category.

For any categories \mathcal{C} and \mathcal{D} , a functor $\mathcal{C} \rightarrow \mathcal{D}$ always means a covariant functor. Thus we denote a contravariant functor from \mathcal{C} to \mathcal{D} by $\mathcal{C}^{\text{opp}} \rightarrow \mathcal{D}$. $\text{Fun}(\mathcal{C}, \mathcal{D})$ is

the category whose objects are covariant functors $\mathcal{C} \rightarrow \mathcal{D}$ and whose morphisms are the natural transformations between these functors.

We write (Sets) to denote the category of sets and functions. We write (Top) to denote the category of topological spaces and continuous maps. We write (Comm) to denote the category of commutative rings with identity and homomorphisms sending $1 \mapsto 1$. We write (A -Mod) to denote the category of A -modules and A -linear homomorphisms. We write (Sch) to denote the category of schemes and morphisms of locally ringed spaces. We write (Cat) to denote the category of categories and covariant functors.

Given a scheme S and a point $s \in S$, we write $\kappa(s)$ for the residue field at s .

1. GROTHENDIECK TOPOLOGIES AND SHEAVES

1.1. **A Motivating Example.** Consider the n th-power map

$$\begin{aligned} f: \mathbb{C}^\times &\rightarrow \mathbb{C}^\times \\ z &\mapsto z^n. \end{aligned}$$

For nonzero $n \in \mathbb{Z}$, f is a covering map: it is locally trivial on the base space, in that there exists an open cover $\{U_i\}$ of \mathbb{C}^\times such that each $f^{-1}(U_i)$ is isomorphic to a disjoint union of copies of U_i . Said differently, the base change $V := \coprod_i U_i \rightarrow \mathbb{C}^\times$ induces a diagram

$$\begin{array}{ccc} \mathbb{C}^\times \times_{\mathbb{C}^\times} V & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathbb{C}^\times & \xrightarrow{f} & \mathbb{C}^\times \end{array}$$

in which the top row is isomorphic to the projection $V \times \{1, \dots, |n|\} \rightarrow V$. Thus the base change transforms the bottom row into a trivial covering map.

In the above example, we can just as well regard f as a morphism of complex varieties $\mathbb{G}_m \rightarrow \mathbb{G}_m$. However, f is no longer locally trivial in the above sense for the Zariski topology. Now consider the base change $\mathbb{G}_m \xrightarrow{f} \mathbb{G}_m$:

$$\begin{array}{ccc} \mathbb{G}_m \times_{\mathbb{G}_m} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow f \\ \mathbb{G}_m & \xrightarrow{f} & \mathbb{G}_m. \end{array}$$

Here the top row is isomorphic (as varieties) to the projection $\mathbb{G}_m \times \{\mu_n\} \rightarrow \mathbb{G}_m$, where $\{\mu_n\}$ is the set of n th roots of unity. Thus base change by f makes the bottom row trivial, in a sense. Comparing to the case of the previous paragraph, this suggests that if we want to think of the bottom row as a covering map, then we should somehow think of the right column as an open cover of \mathbb{G}_m . Grothendieck topologies will provide such a means.

1.2. **Presheaves.** In this and the next few sections, we define presheaves and sheaves on arbitrary categories. As a guiding example to bear in mind throughout, for any topological space X we define $\text{Open}(X)$ to be the category whose objects are the open subsets of X and whose morphisms are the inclusions between these subsets in X . Explicitly, if $U, V \subset X$, then $\text{Hom}_{\text{Open}(X)}(U, V)$ contains a single element if $U \subset V$ and is empty otherwise.

Now let \mathcal{C} be a category. For a topological space X , a presheaf (of sets, say) on X is a just a contravariant functor from $\text{Open}(X)$ to (Sets) . So we make the following general definition.

Definition 1.2.1. A *presheaf* (of sets) on \mathcal{C} is a functor $\mathcal{C}^{\text{opp}} \rightarrow (\text{Sets})$. We define $\text{Presh}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{opp}}, (\text{Sets}))$.

Of course, we can also speak of presheaves taking values in other categories, for example, abelian groups. When we speak simply of presheaves without specifying the target category, we always mean presheaves of sets.

Definition 1.2.2. Let $\beta: F \rightarrow G$ be a morphism of presheaves. We define $\text{im } \beta$ to be the presheaf on \mathcal{C}

$$S \mapsto \text{im} \left[F(S) \xrightarrow{\beta} G(S) \right].$$

We remark that in $\text{Presh}(\mathcal{C})$, all small limits and colimits exist. They are computed “objectwise.” Explicitly, for a system of presheaves F_i indexed by the set I ,

$$\left(\varprojlim_{i \in I} F_i\right)(S) = \varprojlim_{i \in I} (F_i(S)),$$

and similarly for colimits.

We next turn to morphisms in $\text{Presh}(\mathcal{C})$.

Theorem 1.2.3. *The morphism $F \rightarrow G$ is an epimorphism in $\text{Presh}(\mathcal{C})$ if and only if for all $S \in \text{ob } \mathcal{C}$, we have $F(S) \twoheadrightarrow G(S)$.*

We first prove a lemma.

Lemma 1.2.4. *Let \mathcal{C} be any category (not necessarily small).*

- (1) $F \rightarrow G$ is an epimorphism if and only if the two canonical maps $G \rightrightarrows G \amalg_F G$ are equal if and only if either map is an isomorphism.
- (2) $F \rightarrow G$ is a monomorphism if and only if the two canonical projections $F \times_G F \rightrightarrows F$ are equal if and only if either map is an isomorphism.

Proof. This is immediate from the definitions. For (1), if $F \rightarrow G$ is an epimorphism, then G has the same universal property as $G \amalg_F G$, and conversely. We get (2) by dualizing. \square

Proof of Theorem 1.2.3. By the previous lemma, $F \rightarrow G$ is an epimorphism if and only if $G \xrightarrow{\sim} G \amalg_F G$. For presheaves, $G \xrightarrow{\sim} G \amalg_F G$ if and only if

$$(*) \quad G(S) \xrightarrow{\sim} (G \amalg_F G)(S) = G(S) \amalg_{F(S)} G(S) \quad \text{for all objects } S.$$

By the lemma again, (*) holds if and only if $F(S) \twoheadrightarrow G(S)$ for all S . \square

Theorem 1.2.5. *The morphism $F \rightarrow G$ is a monomorphism in $\text{Presh}(\mathcal{C})$ if and only if for all $S \in \text{ob } \mathcal{C}$, we have $F(S) \hookrightarrow G(S)$.*

Proof. Using the monomorphism version of Lemma 1.2.4, we argue analogously to the proof of Theorem 1.2.3. \square

1.3. Yoneda’s Lemma. For any \mathcal{C} , there is a canonical covariant functor

$$\begin{aligned} h: \mathcal{C} &\rightarrow \text{Presh}(\mathcal{C}) \\ S &\mapsto \text{Hom}_{\mathcal{C}}(-, S). \end{aligned}$$

We call $h_S := h(S)$ the *presheaf represented by S* , and we say a presheaf is *representable* if it is isomorphic to some h_S . The weak version of Yoneda’s lemma states that h is fully faithful.

For the strong version of Yoneda’s lemma, let F be any presheaf and consider the natural map

$$\text{Hom}_{\text{Presh}(\mathcal{C})}(h_S, F) \rightarrow F(S)$$

given by evaluation on $\text{id}_S \in h_S(S)$. The strong version now states that this map is a bijection of sets.

We obtain the weak version of Yoneda’s lemma from the strong simply by taking F to be a representable presheaf.

1.4. **Sieves.** To guide our intuition, we first treat the simple case of $\text{Open}(X)$. Let $U \subset X$ be open. A *sieve* on U is a collection \mathcal{U} of open subsets of U with the following property: if $V \in \mathcal{U}$ and W is an open subset contained in V , then $W \in \mathcal{U}$. We say \mathcal{U} is a *covering sieve* if

$$\bigcup_{V \in \mathcal{U}} V = U.$$

For the general definition, let \mathcal{C} be a (small, by convention) category, and let $S \in \text{ob } \mathcal{C}$.

Definition. A *sieve* on S is a collection \mathcal{U} of morphisms, all with target S , with the following property:

(Sve) if $f: T \rightarrow S$ is in \mathcal{U} and $g: U \rightarrow T$ is any morphism with target T , then the composite $fg: U \rightarrow T \rightarrow S$ is in \mathcal{U} .

Note that this definition of a sieve agrees with the one given in the special case of $\text{Open}(X)$.

Given a sieve \mathcal{U} on S , we define, for each $T \in \text{ob } \mathcal{C}$,

$$h_{\mathcal{U}}(T) := \{f \in \text{Hom}(T, S) \mid f \in \mathcal{U}\} \subset h_S(T).$$

The sieve property means precisely that $h_{\mathcal{U}}$ is a subpresheaf of h_S . Conversely, given a subpresheaf $F \subset h_S$, we get a sieve

$$\bigcup_{T \in \text{ob } \mathcal{C}} F(T).$$

These two processes are clearly inverse to each other. Therefore, to give a sieve on S is to give a subpresheaf of h_S . From now on, we use the same symbol \mathcal{U} for both the sieve (as in the sense of the sieve definition) and the corresponding subpresheaf of h_S , provided no confusion can arise.

We shall see that the subpresheaf notion of a sieve, though perhaps more abstract, will be the more convenient version of the definition for our purposes.

Two remarks are in order. First, the sieves on a given object S are naturally partially ordered by inclusion (whether we regard them as collections of morphisms or as subpresheaves; it makes no difference), and the sieve h_S is always the unique maximal sieve on S .

Second, given a sieve \mathcal{U} on S and a morphism $f: T \rightarrow S$ in \mathcal{C} , there is a natural way to pull back \mathcal{U} to a sieve on T . Namely, we define the *pullback* $f^*(\mathcal{U})$ of \mathcal{U} (along f) to be

$$f^*(\mathcal{U}) := \{\text{morphisms } g \text{ with target } T \mid fg \in \mathcal{U}\}.$$

To define $f^*(\mathcal{U})$ as a subpresheaf, observe that the morphism $f: T \rightarrow S$ corresponds to a morphism of presheaves (that we also call f)

$$f: h_T \rightarrow h_S,$$

and the sieve \mathcal{U} gives an inclusion of presheaves $\mathcal{U} \hookrightarrow h_S$. We therefore get a diagram

$$\begin{array}{ccc} h_T \times_{h_S} \mathcal{U} & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ h_T & \xrightarrow{f} & h_S. \end{array}$$

Since $\mathcal{U} \hookrightarrow h_S$, we have $h_T \times_{h_S} \mathcal{U} \hookrightarrow h_T$, and one easily checks that $h_T \times_{h_S} \mathcal{U}$ is precisely the presheaf corresponding to the sieve $f^*(\mathcal{U})$. We therefore also write $f^*(\mathcal{U})$ for the presheaf $h_T \times_{h_S} \mathcal{U}$.

1.5. Grothendieck Topologies.

Definition 1.5.1. A *Grothendieck topology* on a category \mathcal{C} consists of, for each $S \in \text{ob } \mathcal{C}$, a collection $\text{Cov}(S)$ of sieves on S , called *covering sieves*, satisfying the following properties.

- (GT1) For every $S \in \text{ob } \mathcal{C}$, we have $h_S \in \text{Cov}(S)$.
- (GT2) For every morphism $f: T \rightarrow S$ and every sieve $\mathcal{U} \in \text{Cov}(S)$, we have $f^*(\mathcal{U}) \in \text{Cov}(T)$.
- (GT3) Suppose $\mathcal{V} \in \text{Cov}(S)$ and \mathcal{U} is a sieve on S such that, for all morphisms $g: T \rightarrow S$ in \mathcal{V} , we have $g^*(\mathcal{U}) \in \text{Cov}(T)$. Then $\mathcal{U} \in \text{Cov}(S)$.

A *site* is a category equipped with a particular Grothendieck topology.

Informally, (GT2) says that if a sieve is a covering sieve, then it is locally a covering sieve. (GT3) says the converse.

Example 1.5.2. Take $\mathcal{C} = \text{Open}(X)$. Then the covering sieves defined in §1.4 on \mathcal{C} form a Grothendieck topology.

Exercise 1.5.3. Let \mathcal{C} be a site.

- (1) Show that if $\mathcal{U} \hookrightarrow \mathcal{V} \hookrightarrow h_S$ and $\mathcal{U} \in \text{Cov}(S)$, then $\mathcal{V} \in \text{Cov}(S)$.
- (2) Show that if $\mathcal{U}, \mathcal{V} \in \text{Cov}(S)$, then $\mathcal{U} \cap \mathcal{V} \in \text{Cov}(S)$.
- (3) Let $\mathcal{U} \in \text{Cov}(S)$, and for each morphism $f: T \rightarrow S$ in \mathcal{U} , let $\mathcal{V}_f \in \text{Cov}(T)$. Define $f\mathcal{V}_f := \{fg \mid g \in \mathcal{V}_f\}$. Show that

$$\bigcup_{f \in \mathcal{U}} f\mathcal{V}_f$$

is a covering sieve on S .

Exercise (1) allows us to slightly weaken condition (GT1): it suffices to require only that for each object S , $\text{Cov}(S)$ is nonempty.

Exercise (2) tells us that $\text{Cov}(S)$ is a directed set.

Definition 1.5.4. Let $S \in \text{ob } \mathcal{C}$. We define the *category of objects over S* , denoted \mathcal{C}/S , to be the category whose objects are morphisms $f: T \rightarrow S$, where the morphisms between $f: T \rightarrow S$ and $g: U \rightarrow S$ are the morphisms $T \rightarrow U$ that make

$$\begin{array}{ccc} T & \longrightarrow & U \\ f \downarrow & & \downarrow g \\ S & \longlongequal{\quad} & S \end{array}$$

commute.

Example 1.5.5. When \mathcal{C} is a site, \mathcal{C}/S has a natural induced Grothendieck topology: there is an obvious bijection between sieves on $T \rightarrow S$ in \mathcal{C}/S and sieves on T in \mathcal{C} , and we use this bijection to define the covering sieves on $T \rightarrow S$.

Example 1.5.6 (Zariski topology). Define $(\text{Aff}) := (\text{Comm})^{\text{opp}}$. Of course, (Aff) is equivalent to the category of affine schemes—and we even refer to its objects as affine schemes—but for our purposes we just regard it formally as the opposite

category. Let Spec denote the canonical (contravariant) functor from (Comm) to (Aff) .

We now define the Zariski topology on (Aff) . First let R be a commutative ring and let $X = \text{Spec}(R)$. Recall that for $f \in R$, we have $R_f = S^{-1}R$, where $S = \{1, f, f^2, \dots\}$. Let $X_f = \text{Spec}(R_f)$. Applying Spec to the canonical map $R \rightarrow R_f$, we get a morphism $X_f \rightarrow X$ corresponding to the usual notion of a principal open subset of an affine scheme. We say a sieve on X is a standard covering sieve if it is generated by finitely many morphisms

$$X_{f_i} \rightarrow X, \quad i = 1, \dots, r,$$

with $(f_1, \dots, f_r) = R$. We define the topology by saying that an arbitrary sieve \mathcal{U} on X is a covering sieve if \mathcal{U} contains a standard covering sieve; that is, if

$$\{f \in R \mid \text{the morphism } X_f \rightarrow X \text{ is in } \mathcal{U}\}$$

generates the unit ideal in R .

We leave it as an exercise to check that our definition gives a Grothendieck topology on (Aff) . (GT1) and (GT2) are easy to check. (GT3) is easy to check via an argument using Zorn’s lemma. We give a sketch. For $X = \text{Spec}(R)$ and S any ring, we have the functor of points $X(S) := \text{Hom}(R, S)$. For a field k , $x \in X(k)$, and $f \in R$, we write $f(x)$ for the element $x(f) \in k$. It follows from the universal property of R_f that $X_f(k) = \{x \in X(k) \mid f(x) \neq 0\}$. Now by Zorn’s lemma, every proper ideal in a nonzero ring is contained in some maximal ideal. It follows immediately that for elements $f_1, \dots, f_r \in R$,

$$(*) \quad (f_1, \dots, f_r) = R \iff X_{f_1}(k) \cup \dots \cup X_{f_r}(k) = X(k) \text{ for all fields } k.$$

(GT3) is now easy to check using (*).

Alternatively, one can check (GT3) directly, without using the axiom of choice.

1.6. Sheaves.

Definition 1.6.1. For the presheaf F on \mathcal{C} and the sieve \mathcal{U} on $S \in \text{ob } \mathcal{C}$, we define the \mathcal{U} -local sections $F(\mathcal{U})$ of F over S to be

$$F(\mathcal{U}) := \text{Hom}_{\text{Presh}(\mathcal{C})}(\mathcal{U}, F).$$

The \mathcal{U} -local sections of F can be defined somewhat more concretely as follows. Say $\mathcal{U} = \{f: U_f \rightarrow S\}$ as a collection of morphisms. A \mathcal{U} -local section is then an element

$$s = (s_f) \in \prod_{f \in \mathcal{U}} F(U_f)$$

satisfying the compatibility condition

$$(*) \quad (Fg)(s_f) = s_{fg}$$

for all $f \in \mathcal{U}$ and all g with target U_f .

In analogy with the classical case of sheaf theory on a topological space, the compatibility condition (*) on the composition

$$V \xrightarrow{g} U \xrightarrow{f} S,$$

can be written abusively as $s_U|_V = s_V$. In the classical case, one would typically demand as a compatibility condition that sections agree on overlaps, for example $s_U|_{U \cap V} = s_V|_{U \cap V}$. The sieve property ensures that such a condition is automatically built into (*).

It is a routine matter of working through the definitions to verify that the above two notions of \mathcal{U} -local sections are equivalent.

Given a sieve \mathcal{U} on S and a presheaf F , there is a natural restriction map $F(S) \rightarrow F(\mathcal{U})$: Yoneda's lemma identifies $F(S)$ with $\text{Hom}(h_S, F)$, and the inclusion $\mathcal{U} \hookrightarrow h_S$ induces the restriction map $\text{Hom}(h_S, F) \rightarrow \text{Hom}(\mathcal{U}, F) = F(\mathcal{U})$. More generally, given sieves $\mathcal{V} \hookrightarrow \mathcal{U} \hookrightarrow h_S$, we get a restriction map $F(\mathcal{U}) \rightarrow F(\mathcal{V})$.

Definition 1.6.2. Let \mathcal{C} be a site. We say the presheaf F is a *sheaf* (respectively, *separated*) if for all $S \in \text{ob } \mathcal{C}$ and for all $\mathcal{U} \in \text{Cov}(S)$, the restriction map

$$F(S) \rightarrow F(\mathcal{U})$$

is a bijection (respectively, an injection). We let $\text{Sh}(\mathcal{C})$ denote the full subcategory of $\text{Presh}(\mathcal{C})$ consisting of the sheaves on \mathcal{C} .

Example 1.6.3. Let \mathcal{C} be the trivial category — that is, a category with a single object $*$ and a single morphism id_* . Then every presheaf is a sheaf, and $\text{Presh}(\mathcal{C}) = \text{Sh}(\mathcal{C})$ is equivalent (actually, isomorphic) to (Sets) .

Example 1.6.4. Let \mathcal{C} be a category, and endow it with the trivial topology: for all objects S , $\text{Cov}(S) = \{h_S\}$. Then every presheaf on \mathcal{C} is a sheaf. Thus anything we prove about sheaves can also be proved for presheaves.

Example 1.6.5. Take $\mathcal{C} = \text{Open}(X)$ with its usual topology (1.5.2). Then our notion of a sheaf on \mathcal{C} is equivalent to the usual one for the topological space X .

Example 1.6.6. Let G be a group, and let X be a left G -set — that is, a set on which G acts on the left. We define $G \backslash X$ to be the category whose objects are the elements of X , and whose morphisms $x_1 \rightarrow x_2$ are the set $\{g \in G \mid gx_1 = x_2\}$. Composition is given by the group law on G . Thus $G \backslash X$ is a groupoid.

Write $G \backslash *$ for the groupoid $G \backslash \{*\}$, where G acts trivially on the singleton set $\{*\}$. Thus $G \backslash *$ is the groupoid consisting of a single object $*$ with automorphism group $\text{Aut}_{G \backslash *}(*) = G$. Give $G \backslash *$ the trivial topology. Then a sheaf on $G \backslash *$ is just a presheaf on $G \backslash *$, and either is equivalent to a G -set. Similarly, a sheaf of abelian groups on $G \backslash *$ is equivalent to a G -module.

Example 1.6.7. Take $\mathcal{C} = (\text{Aff})$ (1.5.6), and let S be an affine scheme. Then $(\text{Aff})/S$ has a natural Grothendieck topology (1.5.5), and the notions of a sheaf on $(\text{Aff})/S$ and of a sheaf on the underlying topological space of S are equivalent (precisely, we have an equivalence of categories).

We conclude the subsection with a definition that doesn't seem to fit anywhere else (yet?).

Definition 1.6.8. Let F be a sheaf on the site \mathcal{C} . We define a *global section* of F to be an element of

$$\varprojlim_{\mathcal{C}} F(S).$$

When \mathcal{C} has a final object $*$, a global section is just an element of $F(*)$. Even when \mathcal{C} does not have a final object, the global sections can be realized as follows: the presheaf $e: S \mapsto \{\emptyset\}$ is always a final object in $\text{Presh}(\mathcal{C})$, and the global sections are then the set $\text{Hom}_{\text{Presh}(\mathcal{C})}(e, F)$. We leave it as an exercise to work out the equivalences.

1.7. Sheafification. Given a presheaf F on the site \mathcal{C} , we now construct the associated sheaf F^{++} . The process is analogous to the classical case of presheaves and sheaves on a topological space, and we omit most of the proofs.

Definition 1.7.1. Given a presheaf F and an object S , we define

$$F^+(S) := \varinjlim_{\mathcal{U} \in \text{Cov}(S)} F(\mathcal{U}).$$

Lemma 1.7.2. F^+ is a presheaf in a natural way.

Proof. We just explain how the morphism $f: T \rightarrow S$ gives a map $F^+(S) \rightarrow F^+(T)$, and we omit the rest. Given a sieve $\mathcal{U} \in \text{Cov}(S)$, we get a pullback morphism $f^*(\mathcal{U}) \rightarrow \mathcal{U}$. Applying F , we get

$$F(\mathcal{U}) \rightarrow F(f^*(\mathcal{U})) \rightarrow \varinjlim_{\mathcal{V} \in \text{Cov}(T)} F(\mathcal{V}) = F^+(T).$$

It is easy to check that the maps from different $F(\mathcal{U})$ to $F^+(T)$ are compatible, and we therefore get a map from the colimit

$$F^+(S) = \varinjlim_{\mathcal{U} \in \text{Cov}(S)} F(\mathcal{U}) \rightarrow F^+(T).$$

□

There is a natural map $F \rightarrow F^+$, and thus a natural map $\text{Hom}(F^+, G) \rightarrow \text{Hom}(F, G)$ for any presheaf G .

Lemma 1.7.3. If G is a sheaf, then the natural map $F \rightarrow F^+$ induces a bijection $\text{Hom}(F^+, G) \xrightarrow{\sim} \text{Hom}(F, G)$.

Thus if F is a sheaf, we get a canonical isomorphism $F \xrightarrow{\sim} F^+$.

Lemma 1.7.4. F^+ is a separated presheaf.

Lemma 1.7.5. If F is a separated presheaf, then F^+ is a sheaf.

Thus for any presheaf F , F^{++} is a sheaf.

Definition 1.7.6. We call F^{++} the *sheaf associated to F* , or the *sheafification of F* .

1.8. Colimits and Limits in $\text{Sh}(\mathcal{C})$. Applying (1.7.3) twice, we get a bijection

$$\text{Hom}_{\text{Sh}(\mathcal{C})}(F^{++}, G) \xrightarrow{\sim} \text{Hom}_{\text{Presh}(\mathcal{C})}(F, G).$$

Letting ι denote the inclusion functor $\text{Sh}(\mathcal{C}) \hookrightarrow \text{Presh}(\mathcal{C})$, we thus have an adjoint pair $(^{++}, \iota)$. Since $^{++}$ is a left adjoint, it follows formally that $^{++}$ preserves all colimits. That is,

$$(\dagger) \quad \varinjlim_{\mathcal{C}} F_i^{++} = \left(\varinjlim_{\mathcal{C}} F_i \right)^{++}$$

for any system of presheaves F_i , provided $\varinjlim F_i$ exists in $\text{Presh}(\mathcal{C})$ (which it will in case the indexing category is small, for example). If the F_i are sheaves, then each $F_i \cong F_i^{++}$, and so the sheaf colimit is the sheafification of the presheaf colimit:

$$(\ddagger) \quad \varinjlim_{\mathcal{C}} F_i = \left(\varinjlim_{\mathcal{C}} F_i \right)^{++}.$$

We summarize our results in the following theorem.

Theorem 1.8.1.

- (1) The functor $^{++}$ preserves arbitrary colimits (\dagger).
- (2) All small colimits exist in $\text{Sh}(\mathcal{C})$. They are computed as in (\ddagger).

We next turn to limits in $\text{Sh}(\mathcal{C})$.

Theorem 1.8.2. *The functor $^{++}$ preserves finite limits.*

We first prove a lemma.

Lemma 1.8.3. *For the sieve \mathcal{U} and the presheaves F_i indexed by I ,*

$$\left(\varprojlim_{i \in I} F_i\right)(\mathcal{U}) = \varprojlim_{i \in I} F_i(\mathcal{U}).$$

Proof.

$$\left(\varprojlim_{i \in I} F_i\right)(\mathcal{U}) = \text{Hom}\left(\mathcal{U}, \varprojlim_{i \in I} F_i\right) = \varprojlim_{i \in I} \text{Hom}(\mathcal{U}, F_i) = \varprojlim_{i \in I} F_i(\mathcal{U}).$$

□

Proof of Theorem 1.8.2. First recall that in (Sets), filtered colimits commute with finite limits. Explicitly, if I is a filtered category and J is a finite indexing set, then

$$\varprojlim_{i \in I} \varprojlim_{j \in J} S_{ij} = \varprojlim_{j \in J} \varprojlim_{i \in I} S_{ij}$$

for sets S_{ij} .

Now let F_i be presheaves indexed by the finite set I . Then

$$\begin{aligned} \left(\varprojlim_{i \in I} F_i\right)^+(S) &= \varprojlim_{\mathcal{U} \in \text{Cov}(S)} \left(\varprojlim_{i \in I} F_i\right)(\mathcal{U}) \\ &= \varprojlim_{\mathcal{U} \in \text{Cov}(S)} \varprojlim_{i \in I} F_i(\mathcal{U}) && \text{(Lemma 1.8.3)} \\ &= \varprojlim_{i \in I} \varprojlim_{\mathcal{U} \in \text{Cov}(S)} F_i(\mathcal{U}) && \text{(Cov}(S) \text{ is a directed set)} \\ &= \varprojlim_{i \in I} F_i^+(S). \end{aligned}$$

□

Theorem 1.8.4. *For the system of sheaves F_i indexed by I ,*

$$\varprojlim_{i \in I} \text{Presh}(\mathcal{C}) F_i$$

is a sheaf. Therefore, arbitrary small limits exist in $\text{Sh}(\mathcal{C})$ and are equal to the corresponding presheaf limits.

The proof follows without difficulty from the definitions. We omit the details.

1.9. Monomorphisms and Isomorphisms in $\text{Sh}(\mathcal{C})$. In this subsection we characterize, to some extent, the monomorphisms and isomorphisms in $\text{Sh}(\mathcal{C})$. We treat epimorphisms in the next subsection. Let \mathcal{C} be a site throughout.

Definition 1.9.1. Let $\beta: F \rightarrow G$ be a morphism of sheaves. We define the *image sheaf* $\text{im } \beta$ to be the sheaf associated to the presheaf $\text{im}_{\text{Presh}(\mathcal{C})} \beta$ (1.2.2).

Our first result describes monomorphisms.

Theorem 1.9.2. *The morphism of sheaves $\beta: F \rightarrow G$ is a monomorphism if and only if for all $S \in \text{ob } \mathcal{C}$, we have $F(S) \xrightarrow{\beta} G(S)$.*

Proof. By (1.2.4), β is a monomorphism if and only if the two projections $F \times_G F \rightrightarrows F$ are equal and give an isomorphism in $\text{Sh}(\mathcal{C})$. By (1.8.4), the sheaf pullback $(F \times_G F)_{\text{Sh}(\mathcal{C})}$ is the same as the presheaf pullback $(F \times_G F)_{\text{Presh}(\mathcal{C})}$. Thus β is a monomorphism of presheaves, and we apply (1.2.5) to finish. \square

Our second result describes isomorphisms.

Theorem 1.9.3. *The morphism of sheaves $\beta: F \rightarrow G$ is an isomorphism if and only if it is a monomorphism and an epimorphism.*

We first prove a lemma.

Lemma 1.9.4. *Consider the diagram of presheaves*

$$\begin{array}{ccc} F & \xrightarrow{\beta_2} & G_2 \\ \beta_1 \downarrow & & \downarrow \\ G_1 & \longrightarrow & G_1 \amalg_F G_2. \end{array}$$

Suppose that β_1 and β_2 are monomorphisms, F is a sheaf, and G_1 and G_2 are separated. Then $H := G_1 \amalg_F G_2$ is separated.

Proof. We just give a sketch. We have to show that for any $S \in \text{ob } \mathcal{C}$ and any distinct $g_1, g_2 \in H(S)$, the images of g_1 and g_2 in $H(\mathcal{U})$ are distinct for any $\mathcal{U} \in \text{Cov}(S)$.

There are three cases. If $g_1, g_2 \in G_1(S)$, then we use that G_1 is separated. If $g_1, g_2 \in G_2(S)$, then we use that G_2 is separated. If $g_1 \in G_1(S)$ and $g_2 \in G_2(S)$, then we first note that they become equal in $H(S)$ if and only if there exists some (unique, because β_1 and β_2 are monomorphisms) $f \in F$ with $\beta_1(f) = g_1$ and $\beta_2(f) = g_2$. We now proceed with a patching argument, using that F is a sheaf, to finish the proof. \square

Proof of Theorem 1.9.3. The implication \implies is clear, so we assume that $\beta: F \rightarrow G$ is a monomorphism and an epimorphism in $\text{Sh}(\mathcal{C})$. Let H be the presheaf pushout $(G \amalg_F G)_{\text{Presh}(\mathcal{C})}$, and let H' be the sheaf pushout $(G \amalg_F G)_{\text{Sh}(\mathcal{C})} = H^{++}$. We have a diagram

$$G \rightarrow H \rightarrow H^{++} = H'.$$

By (1.2.4), the composite $G \rightarrow H'$ is an isomorphism. By the previous lemma, H is separated, and therefore the map $H \rightarrow H'$ is a monomorphism of presheaves. Therefore $G \rightarrow H$ is monomorphism of presheaves, and hence an isomorphism. Thus by (1.2.4) again, β is an epimorphism of presheaves. By (1.2.3), we have that $F(S) \twoheadrightarrow G(S)$ for all objects S , and the result follows. \square

1.10. Epimorphisms in $\text{Sh}(\mathcal{C})$.

Lemma 1.10.1. *If $\beta: F \rightarrow G$ is an epimorphism in $\text{Sh}(\mathcal{C})$, then $\text{im } \beta \xrightarrow{\sim} G$.*

Proof. β factors in $\text{Presh}(\mathcal{C})$ as

$$F \twoheadrightarrow \text{im}_{\text{Presh}(\mathcal{C})} \beta \hookrightarrow G.$$

Sheafifying $\text{im}_{\text{Presh}(\mathcal{C})} \beta \hookrightarrow G$, we get a monomorphism $\text{im } \beta \hookrightarrow G$ ((1.2.4) and (1.8.2)). Thus β factors as

$$F \rightarrow \text{im } \beta \hookrightarrow G.$$

Since β is an epimorphism of sheaves, so is $\text{im } \beta \hookrightarrow G$. The lemma now follows from (1.9.3). \square

We say the morphism of sheaves $\beta: F \rightarrow G$ is *locally surjective* if for all $S \in \text{ob } \mathcal{C}$ and all $s \in G(S)$, there exists $\mathcal{U} \in \text{Cov}(S)$ such that $s|_U$ is in the image of $F(U) \xrightarrow{\beta} G(U)$ for all $U \in \mathcal{U}$. Our notation is somewhat abusive: strictly speaking, \mathcal{U} is a collection of morphisms $f: U_f \rightarrow S$, but here we have suppressed f . We will often do so the future to lighten the notation.

Theorem 1.10.2. *The morphism of sheaves $\beta: F \rightarrow G$ is an epimorphism if and only if β is locally surjective.*

Proof. We just prove that an epimorphism is locally surjective; the converse is straightforward. As in the proof of (1.10.1), β factors in $\text{Presh}(\mathcal{C})$ as

$$F \twoheadrightarrow \text{im}_{\text{Presh}(\mathcal{C})} \beta \rightarrow \text{im } \beta \xrightarrow{\sim} G.$$

Since $F \twoheadrightarrow \text{im}_{\text{Presh}(\mathcal{C})} \beta$ is surjective, it suffices to show $\text{im}_{\text{Presh}(\mathcal{C})} \beta \rightarrow \text{im } \beta$ is locally surjective. But since $\text{im}_{\text{Presh}(\mathcal{C})} \beta$ is a subpresheaf of the sheaf G , $\text{im}_{\text{Presh}(\mathcal{C})} \beta$ is separated. Thus $\text{im } \beta = (\text{im}_{\text{Presh}(\mathcal{C})} \beta)^+$. Since $F \rightarrow F^+$ is always locally surjective for any presheaf F , we're done. \square

We next prove that epimorphisms in $\text{Sh}(\mathcal{C})$ are always “good” epimorphisms, in a certain sense.

Definition 1.10.3. Let \mathcal{C} be any category (not necessarily small). We say the morphism $f: S' \rightarrow S$ is a *strict epimorphism* if

- (1) $S' \times_S S'$ exists in \mathcal{C} , and
- (2) $f: S' \rightarrow S$ is the cokernel of $S' \times_S S' \rightrightarrows S'$. That is, given a test object T and a morphism $g: S' \rightarrow T$ such that the two composites

$$S' \times_S S' \rightrightarrows S' \xrightarrow{g} T$$

are equal, there exists a unique morphism $h: S \rightarrow T$ such that $S' \xrightarrow{g} T$ factors through

$$S' \xrightarrow{f} S \xrightarrow{h} T.$$

Exercise 1.10.4.

- (1) Show that a strict epimorphism is an epimorphism.
- (2) Show that in (Sets), every epimorphism is a strict epimorphism.
- (3) Show that in (Top), a morphism is an epimorphism if and only if it's surjective, and that it's a strict epimorphism if and only if it's a quotient map.

- (4) Show that in an Abelian category, every epimorphism is a strict epimorphism.
- (5) Take $\mathcal{C} = (\text{Comm})$.
 - (a) Show that $\mathbb{Z} \rightarrow \mathbb{Q}$ is an epimorphism, but not a strict epimorphism.
 - (b) Show that $f: A \rightarrow B$ is a strict epimorphism if and only if f is surjective.

Theorem 1.10.5. *The morphism $\beta: F \rightarrow G$ in $\text{Sh}(\mathcal{C})$ is an epimorphism if and only if β is a strict epimorphism.*

Proof. Let H denote the sheaf cokernel of the diagram $F \times_G F \rightrightarrows F$. Thus H is the sheaf associated to the presheaf

$$S \mapsto \text{cok}[F(S) \times_{G(S)} F(S) \rightrightarrows F(S)].$$

But this presheaf is just $\text{im}_{\text{Presh}(\mathcal{C})} \beta$. Thus $H = \text{im} \beta$, and since β is an epimorphism, $\text{im} \beta \xrightarrow{\sim} G$ (1.10.1). \square

1.11. The Standard Topology on $\text{Sh}(\mathcal{C})$. For any category \mathcal{C} , we have a Yoneda embedding $\mathcal{C} \hookrightarrow \text{Presh}(\mathcal{C})$. In practice, when \mathcal{C} has a topology, the image often lands in $\text{Sh}(\mathcal{C})$, and typically $\text{Sh}(\mathcal{C})$ is in some sense a better behaved category. It thus becomes desirable to endow $\text{Sh}(\mathcal{C})$ with its own topology.

For any site \mathcal{C} , we define a natural Grothendieck topology on $\text{Sh}(\mathcal{C})$, called the *standard topology*, as follows. Given a sheaf F , we say that the sieve $\mathcal{U} = \{f: F_f \rightarrow F\}$ is a covering sieve if for every sheaf G and every pair of distinct morphisms $F \rightrightarrows G$, there exists $f \in \mathcal{U}$ such that composition with f gives distinct morphisms $F_f \rightrightarrows G$.

A word of caution: $\text{Sh}(\mathcal{C})$ is never a small category, regardless of \mathcal{C} . Intuitively, the definition of a covering sieve says that

$$\coprod_f F_f \rightarrow F$$

is an epimorphism in the category of sheaves, but \mathcal{U} is almost never a set; rather, it's a category. Therefore the above coproduct needn't exist.

More seriously, our handling of sieves, Grothendieck topologies, and sheaves implicitly (and sometimes explicitly) assumes that almost everything in sight is a set. It is therefore desirable to have more flexible notions of topologies and sheaves that allow us to circumvent such limitations. Our solution here is simply to ignore the problem.

We now roughly sketch a proof that the definition of the standard topology on $\text{Sh}(\mathcal{C})$ satisfies axioms (GT1)–(GT3). (GT1) is trivial, and (GT3) is easy. Only (GT2) requires much work. The following lemma is useful.

Lemma 1.11.1. *Suppose we have a morphism of sheaves $G \rightarrow F$ and a system of compatible morphisms from the inductive system of sheaves $\{F_i\}$ into F . Then*

$$(*) \quad \varinjlim_i (G \times_F F_i) \xrightarrow{\sim} G \times_F \varinjlim_i F_i.$$

Proof. We only sketch the proof. The general outline will be useful for a number of future proofs. We proceed in three steps.

Step 1: $(*)$ holds in (Sets). We leave the proof as an exercise. The facts that $F \times G \cong \coprod_F G$ in (Sets) and that all colimits commute are helpful.

Step 2: () holds in $\text{Presh}(\mathcal{C})$.* One uses step 1. We again omit the details.

Step 3: () holds in $\text{Sh}(\mathcal{C})$.* By step 2, we have

$$\varinjlim_i (G \times_F F_i) \xrightarrow{\sim} G \times_F \varinjlim_i F_i,$$

where all limits and colimits are taken in $\text{Presh}(\mathcal{C})$. We now sheafify both sides and use that $^{++}$ commutes with arbitrary colimits and finite limits ((1.8.1) and (1.8.2)). \square

In checking (GT2), one applies the lemma in the case that the colimit is a coproduct.

Now let F be a sheaf on \mathcal{C} . By Yoneda's lemma, we have a fully faithful embedding $\text{Sh}(\mathcal{C}) \hookrightarrow \text{Presh}(\text{Sh}(\mathcal{C}))$ sending $F \mapsto h_F = \text{Hom}_{\text{Sh}(\mathcal{C})}(-, F)$.

Theorem 1.11.2. *Endow $\text{Sh}(\mathcal{C})$ with its standard topology. Then h_F is a sheaf on $\text{Sh}(\mathcal{C})$.*

Proof. Let $G \in \text{ob}(\text{Sh}(\mathcal{C}))$, and let $\{G_i \rightarrow G\}$ be a covering sieve on G , so that

$$H := \prod_i G_i \xrightarrow{f} G$$

is an epimorphism of sheaves. Applying h_F , we get a diagram of sets

$$(*) \quad \text{Hom}(G, F) \xrightarrow{f^*} \text{Hom}(H, F) \begin{array}{c} \xrightarrow{\text{pr}_1^*} \\ \xrightarrow{\text{pr}_2^*} \end{array} \text{Hom}(H \times_G H, F),$$

where pr_1 and pr_2 are the canonical projections $H \times_G H \rightrightarrows H$. By (1.10.5), f is a strict epimorphism, and so (*) is an exact diagram of sets; that is, f^* injects $\text{Hom}(G, F)$ onto the equalizer of pr_1^* and pr_2^* .

Applying (1.11.1) twice, we see $H \times_G H \cong \prod_{i,j} (G_i \times_G G_j)$. Since

$$\text{Hom}(H, F) = \prod_i \text{Hom}(G_i, F)$$

and

$$\text{Hom}\left(\prod_{i,j} (G_i \times_G G_j), F\right) = \prod_{i,j} \text{Hom}(G_i \times_G G_j, F),$$

we conclude from (*) that

$$\text{Hom}(G, F) \xrightarrow{f^*} \prod_i \text{Hom}(G_i, F) \rightrightarrows \prod_{i,j} \text{Hom}(G_i \times_G G_j, F)$$

is exact. But this is precisely to say h_F satisfies the sheaf property. \square

Of course, we've said nothing about how the topology of \mathcal{C} interacts with that of $\text{Sh}(\mathcal{C})$. See SGA for the full story.

2. FIBERED CATEGORIES AND STACKS

In this section we generalize the notions of presheaves and sheaves of sets developed in §1 to get corresponding notions of presheaves and sheaves of categories — that is, of fibered categories and stacks, respectively.

2.1. Presheaves of Categories. Let \mathcal{C} be a category. Our first definition is a straight generalization of a presheaf of sets (1.2.1).

Definition 2.1.1. A *strict presheaf of categories* on \mathcal{C} is a functor $\mathcal{C}^{\text{opp}} \rightarrow (\text{Cat})$.

Thus, to be quite explicit, a strict presheaf of categories \mathcal{F} on \mathcal{C} associates to each $S \in \text{ob } \mathcal{C}$ a category $\mathcal{F}(S)$, and to each morphism $f: T \rightarrow S$ in \mathcal{C} a covariant functor $f^* := \mathcal{F}(f): \mathcal{F}(S) \rightarrow \mathcal{F}(T)$, such that

- (1) for all objects S , we have $\mathcal{F}(\text{id}_S) = \text{id}_{\mathcal{F}(S)}$, and
- (2) for every composition

$$S'' \xrightarrow{g} S' \xrightarrow{f} S,$$

we have an equality of functors $(fg)^* = g^*f^*$.

The notion of a strict presheaf of categories is in some sense unnatural: it is rare to find examples where the functors $(fg)^*$ and g^*f^* are equal on the nose. The following example is more typical. Take $\mathcal{C} = (\text{Top})$, and for each topological space S , let $\mathcal{F}(S)$ be the category of (complex, say) rank n vector bundles over S . A continuous map $T \rightarrow S$ gives a functor $\mathcal{F}(S) \rightarrow \mathcal{F}(T)$ via the usual pullback construction. However, given a composition of continuous maps

$$S'' \xrightarrow{g} S' \xrightarrow{f} S,$$

the functors $(fg)^*$ and g^*f^* aren't equal (they don't give the same sets), but only canonically isomorphic. We thus weaken the definition of a strict presheaf of categories as follows.

Definition 2.1.2. A *presheaf of categories* \mathcal{F} on \mathcal{C} consists of the data

- (1) for all $S \in \text{ob } \mathcal{C}$, a category $\mathcal{F}(S)$;
- (2) for all morphisms $f: T \rightarrow S$ in \mathcal{C} , a pullback functor $f^* := \mathcal{F}(f): \mathcal{F}(S) \rightarrow \mathcal{F}(T)$;
- (3) for all $S \in \text{ob } \mathcal{C}$, an isomorphism of functors $\varphi_S: \text{id}_{\mathcal{F}(S)} \xrightarrow{\sim} \text{id}_S^*$; and
- (4) for every composition $S'' \xrightarrow{g} S' \xrightarrow{f} S$ in \mathcal{C} , an isomorphism of functors $\varphi_{f,g}: g^*f^* \xrightarrow{\sim} (fg)^*$,

these data subject to the following constraints:

- (1) (Left identity) for every morphism $f: T \rightarrow S$, the composition

$$f^* = \text{id}_{\mathcal{F}(T)} f^* \xrightarrow{\varphi_T f^*} \text{id}_T^* f^* \xrightarrow{\varphi_f, \text{id}_T} f^*$$

is the identity natural transformation on f^* ;

- (2) (Right identity) for every morphism $f: T \rightarrow S$, the composition

$$f^* = f^* \text{id}_{\mathcal{F}(S)} \xrightarrow{f^* \varphi_S} f^* \text{id}_S^* \xrightarrow{\varphi_{\text{id}_S, f}} f^*$$

is the identity natural transformation on f^* ; and

(3) (Associativity) for every composition

$$V \xrightarrow{h} U \xrightarrow{g} T \xrightarrow{f} S$$

in \mathcal{C} , the diagram

$$\begin{array}{ccc} h^* g^* f^* & \xrightarrow{\varphi_{g,h} f^*} & (gh)^* f^* \\ h^* \varphi_{f,g} \downarrow & & \downarrow \varphi_{f,gh} \\ h^* (fg)^* & \xrightarrow{\varphi_{fg,h}} & (fgh)^* \end{array}$$

commutes.

The associativity axiom says that the various φ provide a single “canonical” isomorphism $h^* g^* f^* \xrightarrow{\sim} (fgh)^*$. More generally, given a composition

$$S_n \xrightarrow{f_{n-1}} S_{n-1} \rightarrow \cdots \rightarrow S_2 \xrightarrow{f_1} S_1,$$

it follows that the φ give a single canonical isomorphism

$$f_{n-1}^* \cdots f_1^* \xrightarrow{\sim} (f_1 \cdots f_{n-1})^*.$$

The situation is analogous to that of an associative multiplication law on a set: once we know that $(ab)c = a(bc)$ for all elements a , b , and c , we can omit the parentheses in more complicated expressions without ambiguity.

We remark that since $\varphi_{fg,h} \circ (h^* \varphi_{f,g}) = \varphi_{f,gh} \circ (\varphi_{g,h} f^*)$, φ behaves formally just like a 2-cocycle in group cohomology.

Example 2.1.3. Take $\mathcal{C} = (\text{Top})$. For each $S \in \text{ob } \mathcal{C}$, let $\mathcal{F}(S)$ denote the category of sheaves (of sets, say) on S . Then the usual pullback functor of sheaves f^* associated to each continuous map f makes \mathcal{F} into a presheaf of categories, since we have a canonical isomorphism $g^* f^* \xrightarrow{\sim} (fg)^*$ for any f and g . (\mathcal{F} is not a strict presheaf, since these functors are not equal.)

To verify that \mathcal{F} satisfies the presheaf axioms, one can use that the direct image functor f_* is right adjoint to the pullback functor f^* , and that we always have an equality of functors $(fg)_* = f_* g_*$.

Example 2.1.4. Let \mathcal{C} be any category in which fibered products exist, and for each $S \in \text{ob } \mathcal{C}$, let $\mathcal{F}(S) = \mathcal{C}/S$ (1.5.4). Given a morphism $f: S' \rightarrow S$ in \mathcal{C} and an object $(T \rightarrow S)$ in \mathcal{C}/S , we get a cartesian diagram

$$\begin{array}{ccc} S' \times_S T & \longrightarrow & T \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S. \end{array}$$

We define f^* to be the functor

$$(T \rightarrow S) \mapsto (S' \times_S T \rightarrow S'),$$

and so make \mathcal{F} into a presheaf of categories, since we again have a canonical isomorphism $g^* f^* \xrightarrow{\sim} (fg)^*$ for each f and g .

One can verify that \mathcal{F} satisfies the presheaf axioms in essentially the same way as in the previous example. Given a morphism $f: S' \rightarrow S$, we define the functor $f_*: \mathcal{C}/S' \rightarrow \mathcal{C}/S$ by sending $(T \rightarrow S')$ to the composite $(T \rightarrow S' \xrightarrow{f} S)$. Then (f_*, f^*) is an adjoint pair, and $(fg)_* = f_* g_*$ for all f and g .

Alternatively, one can check the axioms by considering fibered categories, which we begin discussing in §2.3.

2.2. Cartesian Morphisms. Before defining fibered categories, we introduce some preliminary notation and definitions. By a category *over* \mathcal{C} , we mean a category \mathcal{D} equipped with a covariant functor $a: \mathcal{D} \rightarrow \mathcal{C}$. Given a fixed category \mathcal{D} over \mathcal{C} , we write

$$\begin{array}{ccc} D' & \xrightarrow{\phi} & D \\ | & & | \\ S' & \xrightarrow{f} & S \end{array}$$

to indicate that the diagram $D' \xrightarrow{\phi} D$ in \mathcal{D} lies over the diagram $S' \xrightarrow{f} S$ in \mathcal{C} . Given a diagram

$$\begin{array}{ccc} D' & & D \\ | & & | \\ S' & \xrightarrow{f} & S, \end{array}$$

we write $\text{Hom}_f(D', D)$ for the subset of $\text{Hom}_{\mathcal{D}}(D', D)$ lying over f . Given $S \in \text{ob } \mathcal{C}$, we write $\mathcal{D}(S)$ for the subcategory of \mathcal{D} of objects over S and morphisms over id_S .

Definition 2.2.1. Let \mathcal{D} be a category over \mathcal{C} , and suppose we have a diagram

$$\begin{array}{ccc} D' & \xrightarrow{\phi} & D \\ | & & | \\ S' & \xrightarrow{f} & S. \end{array}$$

We say ϕ is *cartesian* if for every morphism $g: S'' \rightarrow S'$ in \mathcal{C} and every object D'' in \mathcal{D} over S'' , ϕ satisfies the following property: for every $\psi: D'' \rightarrow D$ lying over fg , there exists a unique $\rho: D'' \rightarrow D'$ over g such that ψ factors as

$$D'' \xrightarrow{\rho} D' \xrightarrow{\phi} D;$$

that is, composition with ϕ induces a bijection

$$\text{Hom}_g(D'', D') \xrightarrow{\sim} \text{Hom}_{fg}(D'', D).$$

In pictures, ϕ is cartesian if for every test diagram

$$(*) \quad \begin{array}{ccccc} D'' & & D' & \xrightarrow{\phi} & D \\ | & & | & & | \\ S'' & \xrightarrow{g} & S' & \xrightarrow{f} & S, \end{array}$$

ϕ induces a bijection between morphisms $D'' \rightarrow D'$ over g and morphisms $D'' \rightarrow D$ over fg .

There is an obvious analogy between diagrams $(*)$ and usual pullback diagrams—hence the name cartesian. When ϕ is cartesian, we call D' a *pullback* of D (over

f), and we write

$$\begin{array}{ccc} D' & \xrightarrow{\phi} & D \\ | & \square & | \\ S' & \xrightarrow{f} & S. \end{array}$$

Exercise 2.2.2. For any category \mathcal{C} , let $\text{Mor}(\mathcal{C})$ denote the category of morphisms in \mathcal{C} : the objects of $\text{Mor}(\mathcal{C})$ are morphisms $T \rightarrow S$ in \mathcal{C} , and the morphisms in $\text{Mor}(\mathcal{C})$ from $(T' \rightarrow S')$ to $(T \rightarrow S)$ are pairs (g, f) of morphisms in \mathcal{C} such that

$$\begin{array}{ccc} T' & \xrightarrow{g} & T \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

commutes.

Now suppose all fibered products exist in \mathcal{C} . Show that a morphism in $\text{Mor}(\mathcal{C})$ is cartesian if and only if it corresponds to a pullback square in \mathcal{C} .

The following two lemmas will be useful in the next few sections. The proofs follow easily from the definitions, and we leave the details as exercises.

Lemma 2.2.3. *The composition of cartesian morphisms is cartesian.* □

Lemma 2.2.4. *Suppose we have diagrams*

$$\begin{array}{ccc} D'_1 & \xrightarrow{\phi_1} & D_1 \\ | & & | \\ S' & \xrightarrow{f} & S \end{array} \quad \text{and} \quad \begin{array}{ccc} D'_2 & \xrightarrow{\phi_2} & D_2 \\ | & \square & | \\ S' & \xrightarrow{f} & S, \end{array}$$

where ϕ_2 is cartesian. Then for every morphism $\psi: D_1 \rightarrow D_2$ in $\mathcal{D}(S)$, there exists a unique morphism $\psi': D'_1 \rightarrow D'_2$ in $\mathcal{D}(S')$ such that the diagram

$$\begin{array}{ccc} D'_1 & \xrightarrow{\phi_1} & D_1 \\ \psi' \downarrow & & \downarrow \psi \\ D'_2 & \xrightarrow{\phi_2} & D_2 \end{array}$$

commutes. □

2.3. Fibered Categories. In §2.1, we saw that the notion of a strict presheaf of categories is in some sense too rigid, and so we were led to the weaker definition of a presheaf of categories. Example 2.1.4 suggests that even this definition is somewhat unnaturally rigid: in defining the functor f^* , we had to implicitly choose a pullback to complete each diagram

$$\begin{array}{ccc} & & T \\ & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

to a cartesian square. The situation is analogous to the case in group cohomology when one chooses a set-theoretic section of an epimorphism. It is desirable to work in a framework in which such choices are not built into the data.

To this end, we get a new version of (2.1.4) by considering $\text{Mor}(\mathcal{C})$ (2.2.2). $\text{Mor}(\mathcal{C})$ has a canonical covariant functor to \mathcal{C} sending $(T \rightarrow S) \mapsto S$ and $(g, f) \mapsto f$, and the data of $\text{Mor}(\mathcal{C}) \rightarrow \mathcal{C}$ contains all the data of (2.1.4), except the choices of pullbacks.

We shall see that $\text{Mor}(\mathcal{C})$ is a fibered category over \mathcal{C} . The above example illustrates what is true in general: fibered categories contain all the data of presheaves except fixed choices of pullback functors, and therefore fibered categories may be regarded as the more natural and flexible of the two.

Definition 2.3.1. We say $\mathcal{D} \rightarrow \mathcal{C}$ is a *fibered category* if every test diagram

$$\begin{array}{ccc} & & D \\ & & | \\ S' & \longrightarrow & S \end{array}$$

can be completed to a cartesian square

$$\begin{array}{ccc} D' & \longrightarrow & D \\ | & \square & | \\ S' & \longrightarrow & S. \end{array}$$

Example 2.3.2. Suppose that all fibered products exist in \mathcal{C} . Then, after (2.2.2), $\text{Mor}(\mathcal{C})$ is a fibered category over \mathcal{C} .

Example 2.3.3. Let G be a group and $N \trianglelefteq G$, so that $G \twoheadrightarrow G/N$. Then the induced functor $G \setminus * \rightarrow (G/N) \setminus *$ (1.6.6) is a fibered category, and every morphism in $G \setminus *$ is cartesian.

To make (2.3.3) into a presheaf of categories on $(G/N) \setminus *$, we have to choose a set-theoretic section of $G \twoheadrightarrow G/N$. Again we see the general principle: a presheaf of categories is equivalent (in a sense we make precise in the next two subsections) to a fibered category with a “choice of section.”

2.4. Presheaves of Categories Associated to a Fibered Category. In this subsection and the next, we make explicit and precise the relationship between fibered categories and presheaves of categories. First, suppose we have a fibered category $a: \mathcal{D} \rightarrow \mathcal{C}$. In this subsection, we obtain a presheaf of categories from the given fibered category. The construction uses a categorical form of the axiom of choice and is therefore not canonical. However, different choices result in equivalent presheaves.

We let \mathcal{F} denote the presheaf of categories to be constructed. On objects, we define $\mathcal{F}(S) := \mathcal{D}(S)$.

To define the pullback functors, let $f: S' \rightarrow S$ be a morphism in \mathcal{C} . We get a functor $f^*: \mathcal{F}(S) \rightarrow \mathcal{F}(S')$ as follows. Since \mathcal{D} is a fibered category, we can choose (using a suitable version of the axiom of choice) a pullback over f for each object D over S ; define $f^*(D)$ to be this pullback. We define f^* on morphisms by using Lemma 2.2.4. In the notation of the lemma, $f^*(\psi) = \psi'$.

To define the isomorphisms $\varphi_{f,g}$ for every composition

$$S'' \xrightarrow{g} S' \xrightarrow{f} S$$

in \mathcal{C} , observe first that we have a composition of morphisms

$$(*) \quad g^* f^* D \rightarrow f^* D \rightarrow D$$

for every object D over S . Hence there exists a unique morphism $g^* f^* D \rightarrow (fg)^* D$ over id_S such that

$$\begin{array}{ccc} g^* f^* D & \longrightarrow & D \\ \downarrow & & \parallel \\ (fg)^* D & \longrightarrow & D \end{array}$$

commutes. Since the morphisms in $(*)$ are cartesian, and since the composition of cartesian morphisms is cartesian (2.2.3), $g^* f^* D$ is also a pullback of D . Thus $g^* f^* D \rightarrow (fg)^* D$ is an isomorphism, and we use this isomorphism to define $\varphi_{f,g}$.

One uses a similar approach to define the isomorphisms $\varphi_S: \text{id}_{\mathcal{F}(S)} \xrightarrow{\sim} \text{id}_S^*$ for each $S \in \text{ob } \mathcal{C}$.

We leave it as an exercise to verify that the associativity and two identity axioms for a presheaf of categories are satisfied.

2.5. The Fibered Category Associated to a Presheaf of Categories. Given a presheaf of categories \mathcal{F} on \mathcal{C} , we now construct a fibered category $a: \mathcal{D} \rightarrow \mathcal{C}$. The construction is canonical, unlike in the previous subsection. To illustrate the general procedure, we begin with an example.

Take $\mathcal{C} = \text{Open}(X)$ and let $\mathcal{F}(U)$ be the category of (complex, say) vector bundles over U for each open set $U \subset X$. The inclusion $j: V \hookrightarrow U$ induces the usual pullback functor $j^*: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$. Given vector bundles E over U and F over V ,

$$\begin{array}{ccc} F & & E \\ \downarrow & & \downarrow \\ V & \xrightarrow{j} & U, \end{array}$$

a map $F \rightarrow E$ should just be a vector bundle map $F \rightarrow j^* E$.

We mimic this example in the general case. Given \mathcal{F} , let \mathcal{D} denote the category to be constructed. We define the objects of \mathcal{D} to be

$$\text{ob } \mathcal{D} := \coprod_{S \in \text{ob } \mathcal{C}} \text{ob } \mathcal{F}(S).$$

That is, $\text{ob } \mathcal{D}$ consists of all pairs (S, D) with $S \in \text{ob } \mathcal{C}$ and $D \in \text{ob } \mathcal{F}(S)$. It is easy to define the functor $a: \mathcal{D} \rightarrow \mathcal{C}$ on objects: $a(S, D) := S$.

To define the morphisms in \mathcal{D} , suppose we have $f: S' \rightarrow S$ in \mathcal{C} and a diagram

$$\begin{array}{ccc} D' & & D \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S, \end{array}$$

so that $D \in \text{ob } \mathcal{F}(S)$ and $D' \in \text{ob } \mathcal{F}(S')$. Then we define

$$\text{Hom}_f(D', D) := \text{Hom}_{\mathcal{F}(S')}(D', f^* D)$$

and

$$\text{Hom}_{\mathcal{D}}(D', D) := \coprod_{f \in \text{Hom}_{\mathcal{C}}(S', S)} \text{Hom}_f(D', D).$$

As above, we may also realize $\text{Hom}_{\mathcal{D}}(D', D)$ as the set of all pairs (f, ϕ) with $f \in \text{Hom}_{\mathcal{C}}(S', S)$ and $\phi \in \text{Hom}_{\mathcal{F}(S')}(D', f^*D)$. We define a on morphisms by $a(f, \phi) := f$.

To realize \mathcal{D} as a category, we must now define compositions and identity morphisms.

To define compositions of morphisms in \mathcal{D} , suppose we have a diagram

$$\begin{array}{ccccc} D'' & \xrightarrow{\psi} & D' & \xrightarrow{\phi} & D \\ | & & | & & | \\ S'' & \xrightarrow{g} & S' & \xrightarrow{f} & S, \end{array}$$

so that $\psi \in \text{Hom}_{\mathcal{F}(S'')}(D'', g^*D')$ and $\phi \in \text{Hom}_{\mathcal{F}(S')}(D', f^*D)$. Then applying the functor g^* to ϕ , we get a composition

$$D'' \xrightarrow{\psi} g^*D' \xrightarrow{g^*\phi} g^*f^*D \xrightarrow[\sim]{\varphi_{f,g}} (fg)^*D.$$

We define $\phi\psi$ to be the composite morphism $D'' \rightarrow (fg)^*D$ in $\mathcal{F}(S'')$.

To define identity morphisms in \mathcal{D} , suppose D is an object over S , and let $\text{id}_D^{\mathcal{F}(S)}$ denote the identity morphism of D in $\mathcal{F}(S)$. We define id_D in \mathcal{D} to be the element in $\text{Hom}_{\mathcal{F}(S)}(D, \text{id}_S^*D)$ obtained by applying the isomorphism $\varphi_S: \text{id}_{\mathcal{F}(S)} \xrightarrow{\sim} \text{id}_S^*$ to

$$D \xrightarrow{\text{id}_D^{\mathcal{F}(S)}} D.$$

To check that our definitions make \mathcal{D} into a category, we must verify that compositions are associative and that each id_D is a left and right identity. These three conditions are satisfied precisely because of the three corresponding axioms for a presheaf of categories. Our construction thus gives one way to understand why we took these axioms in the definition of a presheaf of categories.

It is now clear that a is functor.

It remains to check that \mathcal{D} is a fibered category — that is, that each diagram

$$\begin{array}{ccc} & D & \\ & | & \\ S' & \xrightarrow{f} & S \end{array}$$

can be completed to a cartesian square. The presheaf \mathcal{F} affords a canonical way to do so: we take f^*D as the object and

$$\text{id}_D^{\mathcal{F}(S')} \in \text{Hom}_{\mathcal{F}(S')}(f^*D, f^*D) = \text{Hom}_f(f^*D, D)$$

as the morphism $f^*D \rightarrow D$. We leave it as an exercise to check that this morphism is cartesian.

Thus, as claimed, a presheaf of categories is a rigidified version of a fibered category, in that there is a distinguished way to fill in each cartesian square.

Example 2.5.1. We say a category is *discrete* if all its morphisms are identity morphisms. Any set X can be regarded as a category in a trivial way: we just take the discrete category whose objects are the elements of X .

Let F be any presheaf of sets on \mathcal{C} . For each $S \in \text{ob } \mathcal{C}$, we can regard $F(S)$ as a discrete category, so realizing F as a (strict) presheaf of categories. Thus F canonically yields a fibered category over \mathcal{C} , which we typically still denote by F .

Example 2.5.2. As a special case of the previous example, let $S \in \text{ob } \mathcal{C}$, and take the presheaf of sets $h_S = \text{Hom}_{\mathcal{C}}(-, S)$. Then h_S as a fibered category is just \mathcal{C}/S (1.5.4), with the functor $\mathcal{C}/S \rightarrow \mathcal{C}$ sending $(T \rightarrow S) \mapsto T$.

2.6. Morphisms of Fibered Categories. Our next goal is to obtain a notion of when a presheaf of categories is a sheaf of categories — that is, when it’s a stack. We will work with fibered categories instead of presheaves of categories, per se. Before defining stacks, we need to develop some more definitions. In this section we define the category of morphisms between two fibered categories over the same base category.

Let \mathcal{D} and \mathcal{E} be fibered categories over \mathcal{C} . Informally, the morphisms $\mathcal{D} \rightarrow \mathcal{E}$ of fibered categories should be functors that interact well with the fibered category structures of \mathcal{D} and \mathcal{E} . If we think of \mathcal{D} and \mathcal{E} as presheaves of categories, then it seems reasonable to demand that such functors be “invisible” to the base category \mathcal{C} . It is also reasonable to demand that such functors preserve pullbacks.

Definition 2.6.1. Given fibered categories $a: \mathcal{D} \rightarrow \mathcal{C}$ and $b: \mathcal{E} \rightarrow \mathcal{C}$, we define the *morphisms of fibered categories* (over \mathcal{C}) from \mathcal{D} to \mathcal{E} , denoted $\text{Hom}_{\square}(\mathcal{D}, \mathcal{E})$, to be the following category. The objects in $\text{Hom}_{\square}(\mathcal{D}, \mathcal{E})$ are functors $\mathcal{D} \rightarrow \mathcal{E}$ that

- (1) make the diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{E} \\ a \downarrow & & \downarrow b \\ \mathcal{C} & \xlongequal{\quad} & \mathcal{C} \end{array}$$

commute (strictly), and

- (2) send cartesian morphisms in \mathcal{D} to cartesian morphisms in \mathcal{E} .

The morphisms in $\text{Hom}_{\square}(\mathcal{D}, \mathcal{E})$ are the natural transformations of the objects in $\text{Hom}_{\square}(\mathcal{D}, \mathcal{E})$ that, upon applying b , induce the identity natural transformation on a .

Thus a morphism in $\text{Hom}_{\square}(\mathcal{D}, \mathcal{E})$ from F_1 to F_2 gives us, for each $D \in \text{ob } \mathcal{D}$, a morphism $\xi_D: F_1(D) \rightarrow F_2(D)$ in $\mathcal{E}(a(D))$; that is, $b(\xi_D) = \text{id}_{a(D)}$.

The objects in $\text{Hom}_{\square}(\mathcal{D}, \mathcal{E})$ can be described somewhat more concretely as follows. Roughly, an object consists of a functor $F_S: \mathcal{D}(S) \rightarrow \mathcal{E}(S)$ for each $S \in \text{ob } \mathcal{C}$, such that for every $f: S' \rightarrow S$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} \mathcal{D}(S) & \xrightarrow{F_S} & \mathcal{E}(S) \\ f^* \downarrow & & \downarrow f^* \\ \mathcal{D}(S') & \xrightarrow{F_{S'}} & \mathcal{E}(S'). \end{array}$$

commutes weakly. That is, we choose pullback functors $\mathcal{D}(S) \rightarrow \mathcal{D}(S')$ and $\mathcal{E}(S) \rightarrow \mathcal{E}(S')$ over f , and we get a (canonical, because F preserves cartesian morphisms) natural isomorphism $F_{S'} f^* \xrightarrow{\sim} f^* F_S$.

Such a description is much in the spirit of presheaves of categories. To give the correct definition of a morphism between presheaves of categories on \mathcal{C} , the natural isomorphisms $F_{S'} f^* \xrightarrow{\sim} f^* F_S$ would have to be given as part of the data of the morphism, and these natural isomorphisms would have to satisfy a 1-cocycle condition for every composition of three morphisms in \mathcal{C} .

One can check that a morphism in $\text{Hom}_{\square}(\mathcal{D}, \mathcal{E})$ between F_1 and F_2 amounts to, for each $S \in \text{ob } \mathcal{C}$, a natural transformation ξ_S on each

$$\mathcal{D}(S) \begin{array}{c} \xrightarrow{F_1} \\ \xrightarrow{F_2} \end{array} \mathcal{E}(S),$$

such that the ξ_S are compatible with pullbacks every time we have a morphism $S' \rightarrow S$ in \mathcal{C} .

The following results, which we leave as exercises, give further evidence that (2.6.1) is the right definition if we want to think of \mathcal{D} and \mathcal{E} as presheaves of categories.

Exercise 2.6.2. Let \mathcal{D} and \mathcal{E} be fibered categories over \mathcal{C} , and let $F: \mathcal{D} \rightarrow \mathcal{E}$ be a morphism of fibered categories.

- (1) Show that F is essentially surjective if and only if $F: \mathcal{D}(S) \rightarrow \mathcal{E}(S)$ is essentially surjective for all $S \in \text{ob } \mathcal{C}$.
- (2) Show that F is fully faithful if and only if $F: \mathcal{D}(S) \rightarrow \mathcal{E}(S)$ is fully faithful for all $S \in \text{ob } \mathcal{C}$.
- (3) If F is an equivalence of categories, show that there exists a quasi-inverse morphism of fibered categories $G: \mathcal{E} \rightarrow \mathcal{D}$ such that $GF \cong \text{id}_{\mathcal{D}}$ and $FG \cong \text{id}_{\mathcal{E}}$ as morphisms of fibered categories (that is, there exist isomorphisms in $\text{Hom}_{\square}(\mathcal{D}, \mathcal{D})$ and $\text{Hom}_{\square}(\mathcal{E}, \mathcal{E})$, respectively).

2.7. Yoneda’s Lemma for Fibered Categories. For presheaves of sets, Yoneda’s lemma (1.3) says that there is a canonical identification $\text{Hom}_{\text{Presheaf}(\mathcal{C})}(h_S, F) = F(S)$ for every $S \in \text{ob } \mathcal{C}$ and every presheaf F on \mathcal{C} . There is an analogous statement for fibered categories over \mathcal{C} .

Let F be a presheaf of sets on \mathcal{C} . Then we may regard F as a fibered category over \mathcal{C} (2.5.1). Explicitly, $\text{ob } F = \coprod_{S \in \text{ob } \mathcal{C}} F(S)$ consists of all pairs (S, x) with $S \in \text{ob } \mathcal{C}$ and $x \in F(S)$. A morphism from (S', x') to (S, x) consists of a morphism $f: S' \rightarrow S$ in \mathcal{C} and a morphism $x' \rightarrow f^*(x)$ in $F(S')$. Since $F(S')$ is a discrete category, we get a (necessarily unique) morphism only if $x' = f^*(x)$, and so

$$\text{Hom}_F((S', x'), (S, x)) = \{ f \in \text{Hom}_{\mathcal{C}}(S', S) \mid f^*(x) = x' \}.$$

It is immediate from the construction that every morphism in F is cartesian.

In particular, if we take the presheaf of sets h_S for some object $S \in \text{ob } \mathcal{C}$, then we get the fibered category \mathcal{C}/S (2.5.2).

We now state the lemma. Let \mathcal{D} be any fibered category over \mathcal{C} . We get a canonical functor $\text{Hom}_{\square}(h_S, \mathcal{D}) \rightarrow \mathcal{D}(S)$ (here we regard h_S as a fibered category) sending the functor $F: h_S \rightarrow \mathcal{D}$ to $F(S \xrightarrow{\text{id}_S} S) \in \mathcal{D}(S)$.

Lemma. For every $S \in \text{ob } \mathcal{C}$ and every fibered category \mathcal{D} ,

$$\text{Hom}_{\square}(h_S, \mathcal{D}) \rightarrow \mathcal{D}(S)$$

is an equivalence of categories.

We leave the formal proof as an exercise, but we explain intuitively why the lemma is true. An object F of $\text{Hom}_{\square}(h_S, \mathcal{D})$ is a functor $\mathcal{C}/S \rightarrow \mathcal{D}$ over \mathcal{C} that preserves cartesian morphisms. Thus F associates to each $T \xrightarrow{f} S$ in \mathcal{C}/S an object $D_T \in \text{ob } \mathcal{D}(T)$ (really we should write D_f , but, by abuse of notation, we suppress

the f), and to each morphism

$$\begin{array}{ccc} T' & \xrightarrow{g} & T \\ f' \downarrow & & \downarrow f \\ S & \xlongequal{\quad} & S \end{array}$$

in \mathcal{C}/S a morphism $D_{T'} \rightarrow D_T$. Since every morphism in \mathcal{C}/S is cartesian, so is $D_{T'} \rightarrow D_T$; that is, F chooses a pullback $D_{T'}$ of D_T over g for each g . Since F must preserve compositions and identity morphisms, all the various pullbacks are compatible. Since $S \xrightarrow{\text{id}_S} S$ is a terminal object in \mathcal{C}/S , each D_T is a pullback of D_S . The functor in the lemma sends the family $(D_T)_{T \in \mathcal{C}/S} \mapsto D_S$, and therefore the functor is essentially surjective because the family is determined (up to isomorphism) by D_S .

The proof that the functor is fully faithful uses (2.2.3) and (2.2.4).

2.8. Stacks. Let \mathcal{U} be a sieve on $S \in \text{ob } \mathcal{C}$. Then, viewed as a fibered category (2.5.1), \mathcal{U} is the full subcategory of \mathcal{C}/S whose objects are precisely those morphisms that appear in the sieve \mathcal{U} . In fact, every full subcategory of \mathcal{C}/S that is a fibered category over \mathcal{C} arises from a sieve in this way, as one can easily check.

Given a fibered category \mathcal{D} , we define the \mathcal{U} -local sections, or \mathcal{U} -local objects, of \mathcal{D} over S to be the category

$$\mathcal{D}(\mathcal{U}) := \text{Hom}_{\mathbb{Q}}(\mathcal{U}, \mathcal{D}),$$

with \mathcal{U} regarded as a fibered category on the right hand side. Objects of $\mathcal{D}(\mathcal{U})$ are sometimes referred to in the literature as *descent data* or as *gluing data*.

As in the case $\mathcal{U} = h_S$ discussed in the previous subsection, the objects of $\mathcal{D}(\mathcal{U})$ can be regarded roughly as certain families (D_T) of objects in \mathcal{D} , with each $D_T \in \text{ob } \mathcal{D}(T)$, indexed by the morphisms appearing in the sieve \mathcal{U} (here we have again suppressed the actual morphism in favor of its source).

For example, take $\mathcal{C} = \text{Open}(X)$ and let \mathcal{U} be the sieve on X generated by some open cover of X . Let \mathcal{D} be the fibered category of rank n vector bundles over \mathcal{C} . Then a \mathcal{U} -local section of \mathcal{D} over X consists of a vector bundle over each open set U in \mathcal{U} plus gluing data on overlaps.

We now come to the definition of stacks and related separation properties of fibered categories. Given a sieve \mathcal{U} on S , the inclusion of presheaves $\mathcal{U} \hookrightarrow h_S$ corresponds to the full embedding of fibered categories

$$\mathcal{U} \hookrightarrow h_S = \mathcal{C}/S.$$

Thus, for any fibered category \mathcal{D} , there is an induced restriction functor

$$\mathcal{D}(h_S) = \text{Hom}_{\mathbb{Q}}(h_S, \mathcal{D}) \rightarrow \text{Hom}_{\mathbb{Q}}(\mathcal{U}, \mathcal{D}) = \mathcal{D}(\mathcal{U}).$$

Definition 2.8.1. The fibered category \mathcal{D} over \mathcal{C} is a *stack* (respectively, *separated fibered category*; respectively, *preseparated fibered category*) if for every $S \in \text{ob } \mathcal{C}$ and every $\mathcal{U} \in \text{Cov}(S)$, the functor

$$\mathcal{D}(h_S) \rightarrow \mathcal{D}(\mathcal{U})$$

is an equivalence of categories (respectively, fully faithful; respectively, faithful).

Thus, speaking informally, \mathcal{D} is a pre-separated fibered category if distinct morphisms of objects defined globally give distinct systems of compatible local morphisms; \mathcal{D} is a separated fibered category if, further, compatible local morphisms

glue to give global morphisms; and \mathcal{D} is a stack if, further, compatible systems of local objects glue to give global objects. This makes precise our assertion that a stack is a sheaf of categories.

There is also an obvious analogy between (2.8.1) and the definition of a sheaf of sets (1.6.2). Note that here we do not get a natural map $\mathcal{D}(S) \rightarrow \mathcal{D}(U)$. After Yoneda’s lemma (2.7), $\mathcal{D}(S)$ and $\mathcal{D}(h_S)$ are equivalent categories, but the natural functor goes from $\mathcal{D}(h_S)$ to $\mathcal{D}(S)$. To get a functor in the opposite direction, we must choose pullbacks.

The above definition is equivalent to the following slightly altered formulation.

Lemma 2.8.2. *The fibered category \mathcal{D} is a stack (respectively, separated fibered category; respectively, pre-separated fibered category) if and only if for every $S \in \text{ob } \mathcal{C}$ and every cofinal subset $\Sigma \subset \text{Cov}(S)$, we have that*

$$\mathcal{D}(U) \rightarrow \mathcal{D}(V)$$

is an equivalence of categories (respectively, fully faithful; respectively, faithful) for every $V \in \Sigma$ and every sieve $U \supset V$. \square

We omit the proof, though we note that it is not completely trivial, in that one must use that the conditions on $\mathcal{D}(h_S) \rightarrow \mathcal{D}(U)$ hold for *all* objects S in the stated definition.

We caution that the definition of stack in (2.8.1) is not the one most commonly encountered in the literature. Most authors impose an additional condition. Namely, we say that the fibered category \mathcal{D} is a *category fibered in groupoids* if every morphism in \mathcal{D} is cartesian. It is equivalent to say that \mathcal{D} is a fibered category in which $\mathcal{D}(S)$ is a groupoid for every $S \in \text{ob } \mathcal{C}$. Then a stack is defined just as above, with the additional constraint that \mathcal{D} be a category fibered in groupoids. A *prestack* is defined to be a category fibered in groupoids that satisfies our definition of a separated fibered category.

2.9. Filtered Colimits and Limits of Categories. In this subsection and the next, we show how to make any fibered category over a site into a stack. The procedure is analogous to the sheafification of presheaves discussed in §1.7. We begin with the fibered category analog of the $+$ operation.

Definition 2.9.1. Given a fibered category \mathcal{D} over the site \mathcal{C} , we define

$$\mathcal{D}^+(S) = \varinjlim_{U \in \text{Cov}(S)} \mathcal{D}(U)$$

for each $S \in \text{ob } \mathcal{C}$.

The definition begs explanation of how to interpret the colimit of categories appearing on the right hand side. We explain by analogy with filtered colimits of groups. Given a filtered inductive system G_i of groups, we can first form the set colimit $G := \varinjlim G_i$. To make G into a group, we must define a multiplication on it. Taking the colimit over the various multiplication maps $G_i \times G_i \rightarrow G_i$, and using that filtered colimits commute with finite limits, we get the desired multiplication map

$$(\varinjlim G_i) \times (\varinjlim G_i) = \varinjlim (G_i \times G_i) \rightarrow \varinjlim G_i.$$

The case of a filtered system of (small) categories is similar. A category consists of a set E_0 of objects and a set E_1 of morphisms, with functions

$$s, t: E_1 \rightarrow E_0$$

sending each morphism to its source and target, respectively. Composition of morphisms is defined by a function $E_1 \times_s \times_t E_1 \rightarrow E_1$. Identities are specified by a function $i: E_0 \rightarrow E_1$. (Of course, to actually get a category, all these data must satisfy certain axioms.)

Now, the inclusions of the covering sieves $\mathcal{W} \subset \mathcal{V} \subset \mathcal{U}$ induce functors

$$\mathcal{D}(\mathcal{U}) \rightarrow \mathcal{D}(\mathcal{V}), \quad \mathcal{D}(\mathcal{V}) \rightarrow \mathcal{D}(\mathcal{W}), \quad \text{and} \quad \mathcal{D}(\mathcal{U}) \rightarrow \mathcal{D}(\mathcal{W}).$$

Since composition of functors is strictly associative, the functor $\mathcal{D}(\mathcal{U}) \rightarrow \mathcal{D}(\mathcal{W})$ factors through $\mathcal{D}(\mathcal{U}) \rightarrow \mathcal{D}(\mathcal{V}) \rightarrow \mathcal{D}(\mathcal{W})$ on the nose (not just up to isomorphism). Thus the objects and morphisms of the various $\mathcal{D}(\mathcal{U})$ form separate inductive systems of sets in the usual sense, and we can take their colimits in the usual sense to get what we define to be the objects and morphisms of the colimit category. As in the group case discussed above, we get a natural composition law on morphisms, with natural identity morphisms.

Just as for colimits of sets, filtered colimits of categories enjoy a certain universal property. Let \mathcal{E}_i be an inductive system of categories indexed by the filtered category \mathcal{J} . Thus for every $i \rightarrow j \rightarrow k$ in \mathcal{J} , the corresponding functor $\mathcal{E}_i \rightarrow \mathcal{E}_k$ factors through the composition $\mathcal{E}_i \rightarrow \mathcal{E}_j \rightarrow \mathcal{E}_k$ on the nose. For any category \mathcal{F} , the categories $\text{Hom}(\mathcal{E}_i, \mathcal{F})$ then form a strict inverse system. As in the colimit case, we can form the category

$$\varprojlim_{i \in \mathcal{J}} \text{Hom}(\mathcal{E}_i, \mathcal{F})$$

by taking the inverse limits of the underlying object sets and morphism sets. Then $\varinjlim_{i \in \mathcal{J}} \mathcal{E}_i$ has the universal property that

$$\text{Hom}\left(\varinjlim_{i \in \mathcal{J}} \mathcal{E}_i, \mathcal{F}\right) \cong \varprojlim_{i \in \mathcal{J}} \text{Hom}(\mathcal{E}_i, \mathcal{F})$$

(these categories are isomorphic, not just equivalent).

2.10. Stackification. We return to our discussion of \mathcal{D}^+ . Given a morphism $f: T \rightarrow S$ in \mathcal{C} and a covering sieve \mathcal{U} on S , the pullback $f^*(\mathcal{U})$ is a covering sieve on T . Suppose $\mathcal{V} \subset f^*(\mathcal{U})$ is another covering sieve on T . Then we get a morphism of presheaves $\mathcal{V} \rightarrow \mathcal{U}$ via $\mathcal{V} \hookrightarrow f^*(\mathcal{U}) \rightarrow \mathcal{U}$, and hence functors

$$\mathcal{D}(\mathcal{U}) \rightarrow \mathcal{D}(\mathcal{V}) \rightarrow \mathcal{D}^+(T).$$

As we vary \mathcal{U} , the functors $\mathcal{D}(\mathcal{U}) \rightarrow \mathcal{D}^+(T)$ are (strictly) compatible. So, by the universal mapping property of the colimit, we get a functor

$$\mathcal{D}^+(S) \xrightarrow{f^*} \mathcal{D}^+(T).$$

Moreover, given a composition $S'' \rightarrow S' \rightarrow S$ in \mathcal{C} , we again use that functor composition is strictly associative to see that $\mathcal{D}^+(S) \rightarrow \mathcal{D}^+(S'')$ factors through $\mathcal{D}^+(S) \rightarrow \mathcal{D}^+(S') \rightarrow \mathcal{D}^+(S'')$ on the nose.

We have proved the following result.

Lemma 2.10.1. *\mathcal{D}^+ is a strict presheaf of categories on \mathcal{C} .*

Thus \mathcal{D}^+ yields a fibered category, which we still denote as \mathcal{D}^+ .

For each $S \in \text{ob } \mathcal{C}$, we have natural maps

$$\begin{array}{ccc} \mathcal{D}(h_S) & \longrightarrow & \mathcal{D}^+(S) \\ \downarrow & & \\ \mathcal{D}(S) & & \end{array}$$

By Yoneda’s lemma (2.7), the vertical arrow is an equivalence, but there is no canonical quasi-inverse. If, for the moment, we endow \mathcal{C} with the trivial topology (1.6.4), so that the maximal sieves are the only covering sieves, then applying the $^+$ operation to \mathcal{D} yields a new strict presheaf of categories

$$\mathcal{D}_{\text{sp}}: S \mapsto \mathcal{D}(h_S).$$

We thus get a diagram of fibered categories

$$(*) \quad \begin{array}{ccc} \mathcal{D}_{\text{sp}} & \longrightarrow & \mathcal{D}^+ \\ \downarrow & & \\ \mathcal{D} & & \end{array}$$

where, by (2.6.2), $\mathcal{D}_{\text{sp}} \rightarrow \mathcal{D}$ is an equivalence of categories, and \mathcal{D}^+ is taken with respect to the original topology on \mathcal{C} . Display (*) is the fibered category generalization of the canonical map $F \rightarrow F^+$ for presheaves of sets.

The following is the main result on the $^+$ operation.

Theorem 2.10.2. *Let \mathcal{D} be a fibered category over the site \mathcal{C} .*

- (1) \mathcal{D}^+ is a pre-separated fibered category.
- (2) If \mathcal{D} is pre-separated, then \mathcal{D}^+ is separated.
- (3) If \mathcal{D} is separated, then \mathcal{D}^+ is a stack.

Thus for any fibered category \mathcal{D} , \mathcal{D}^{+++} is a stack, called the *stackification* of \mathcal{D} . We omit the proof of the theorem.

Example 2.10.3. Take $\mathcal{C} = \text{Open}(X)$, and let \mathcal{D} be the fibered category of (complex, say) vector bundles over the open subsets of X . Then \mathcal{D} is a stack, because we can glue morphisms and objects defined locally.

Example 2.10.4. Again take $\mathcal{C} = \text{Open}(X)$, but now let \mathcal{D}' denote the fibered category of trivial vector bundles over the open subsets of X . We can still glue morphisms, so \mathcal{D}' is separated, but \mathcal{D}' is not a stack: trivial vector bundles defined locally can certainly glue to give nontrivial vector bundles. We have $\mathcal{D}'^+ \cong \mathcal{D}$, with \mathcal{D} as defined in the previous example.

We end the subsection by expanding on some parallels with sheafification of presheaves of sets. We again omit the proofs.

Let \mathcal{D} and \mathcal{E} be fibered categories over \mathcal{C} , with \mathcal{E} a stack. We know that for presheaves of sets, $\text{Hom}(F^+, G) \rightarrow \text{Hom}(F, G)$ is a bijection of sets whenever G is a sheaf (1.7.3). Applying $\text{Hom}_{\mathcal{C}}(-, \mathcal{E})$ to (*), we get a diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(\mathcal{D}_{\text{sp}}, \mathcal{E}) & \longleftarrow & \text{Hom}_{\mathcal{C}}(\mathcal{D}^+, \mathcal{E}) \\ \uparrow & & \\ \text{Hom}_{\mathcal{C}}(\mathcal{D}, \mathcal{E}) & & \end{array}$$

For any fibered categories \mathcal{D}_1 , \mathcal{D}_2 , and \mathcal{E} , one can check that if $\mathcal{D}_1 \rightarrow \mathcal{D}_2$ is an equivalence of fibered categories, then $\mathrm{Hom}_{\square}(\mathcal{D}_2, \mathcal{E}) \rightarrow \mathrm{Hom}_{\square}(\mathcal{D}_1, \mathcal{E})$ is an equivalence of categories. Thus the vertical arrow is an equivalence. The result for sheaves suggests that since \mathcal{E} is a stack, the horizontal arrow is an equivalence too.

Lemma 2.10.5. *If \mathcal{E} is a stack, then*

$$\mathrm{Hom}_{\square}(\mathcal{D}^+, \mathcal{E}) \rightarrow \mathrm{Hom}_{\square}(\mathcal{D}_{\mathrm{sp}}, \mathcal{E})$$

is an equivalence of categories. \square

There is a version of the usual sheaf Hom on a topological space that applies to the fibered category setting. Recall that for presheaves F and G on a space X , we get a presheaf

$$(**) \quad U \mapsto \mathrm{Hom}(F|_U, G|_U)$$

on X . When F and G are sheaves, so is $(**)$ (note that this says that the fibered category of sheaves over $\mathrm{Open}(X)$ is separated).

For fibered categories, fix $S \in \mathrm{ob} \mathcal{C}$ and $\mathcal{U} \in \mathrm{Cov}(S)$, and let $D, D' \in \mathrm{ob} \mathcal{D}(\mathcal{U})$. Thus if $\mathcal{U} = h_S$, then we can regard D and D' as objects in \mathcal{D} over S , and in general we can regard D and D' as families of objects in \mathcal{D} indexed by the morphisms in \mathcal{U} . Given a morphism $f: T \rightarrow S$ in \mathcal{C} , we can form the pullback sieve $f^*(\mathcal{U})$ on T , so obtaining a morphism of fibered categories $f^*(\mathcal{U}) \rightarrow \mathcal{U}$. Composing with $D, D': \mathcal{U} \rightarrow \mathcal{D}$, we get respective morphisms of fibered categories $f^*(D), f^*(D'): f^*(\mathcal{U}) \rightarrow \mathcal{D}$.

Definition 2.10.6. Given S, \mathcal{U}, D , and D' as above, we define $\mathcal{H} = \mathcal{H}om(D, D')$ to be the presheaf of sets on \mathcal{C}/S

$$(T \xrightarrow{f} S) \mapsto \mathrm{Hom}_{\mathcal{D}(f^*(\mathcal{U}))}(f^*(D), f^*(D')).$$

Usually we suppress f and write $\mathcal{H}(T)$ instead.

Exercise 2.10.7. Show that \mathcal{D} is a separated fibered category (respectively, pre-separated fibered category) if and only if for all $S \in \mathrm{ob} \mathcal{C}$ and all $D, D' \in \mathrm{ob} \mathcal{D}(h_S)$, the presheaf \mathcal{H} , formed as above, is a sheaf (respectively, separated presheaf) on \mathcal{C}/S (recall that \mathcal{C}/S inherits a natural Grothendieck topology from that of \mathcal{C} (1.5.5)).

Since \mathcal{C}/S inherits a natural Grothendieck topology, we can apply the $+$ operation to \mathcal{H} . The following lemmas are useful in proving (2.10.2).

Lemma 2.10.8. *For every morphism $f: T \rightarrow S$ in \mathcal{C} , we have*

$$\mathcal{H}^+(T) = \mathrm{Hom}_{\mathcal{D}^+(T)}(f^*(D), f^*(D')).$$

\square

From this lemma, one can deduce the following.

Lemma 2.10.9. *For all $\mathcal{V} \in \mathrm{Cov}(S)$, we have*

$$\mathcal{H}^+(\mathcal{V}) = \mathrm{Hom}_{\mathcal{D}^+(\mathcal{V})}(D, D').$$

\square

We explain how to interpret the right hand side. We have a diagram of natural maps

$$\begin{array}{ccc} \mathcal{D}^+(h_S) & \longrightarrow & \mathcal{D}^+(\mathcal{V}) \\ \downarrow & & \\ \mathcal{D}(\mathcal{U}) & \longrightarrow & \mathcal{D}^+(S), \end{array}$$

where the vertical map is an equivalence. Since \mathcal{D}^+ is a strict presheaf, the vertical arrow has a canonical quasi-inverse. We then compose functors to realize D and D' as objects in $\mathcal{D}^+(\mathcal{V})$.

3. SHEAVES AND STACKS ON (Aff)

3.1. Sheaves on (Aff). We begin with a few remarks concerning sheaves (of sets) on (Aff). Recall from (1.5.6) that for the Zariski topology on $(\text{Aff}) = (\text{Comm})^{\text{opp}}$, a sieve is a covering sieve exactly when it contains a standard covering sieve, and that a standard covering sieve \mathcal{U} on $\text{Spec}(R)$ is one for which there exist elements $f_1, \dots, f_r \in R$ such that $(f_1, \dots, f_r) = R$ and the morphisms $\text{Spec}(R_{f_i}) \rightarrow \text{Spec}(R)$ generate \mathcal{U} .

For any presheaf F on any site \mathcal{C} , F is a sheaf if and only if the canonical map $F \rightarrow F^+$ is an isomorphism. Consequently, F is a sheaf if and only if for every $S \in \text{ob } \mathcal{C}$ there exists a cofinal subset $\Sigma \subset \text{Cov}(S)$ such that $F(S) \xrightarrow{\sim} F(\mathcal{U})$ for all $\mathcal{U} \in \Sigma$.

Thus, in the case of the Zariski topology on (Aff), the presheaf F is a sheaf if and only if for every ring R , we have

$$F(R) := F(\text{Spec}(R)) \xrightarrow{\sim} F(\mathcal{U})$$

for every standard covering sieve \mathcal{U} on $\text{Spec}(R)$.

Now let \mathcal{C} be any category in which fibered products exist, and let \mathcal{U} be the sieve on S generated by the collection of morphisms $\{U_i \rightarrow S\}_{i \in I}$. Then to give an element of $F(\mathcal{U})$ is to give (using Yoneda's lemma) an element u_i in each $F(U_i)$ such that $u_i|_{U_i \times_S U_j} = u_j|_{U_i \times_S U_j}$ for all pairs i, j . That is,

$$F(\mathcal{U}) = \ker \left[\prod_i F(S_i) \begin{array}{c} \xrightarrow{\text{pr}_1^*} \\ \xrightarrow{\text{pr}_2^*} \end{array} \prod_{i,j} F(U_i \times_S U_j) \right].$$

Therefore, combining the previous two paragraphs, we get that the presheaf F on (Aff) is a sheaf for the Zariski topology if and only if for every affine scheme $S = \text{Spec}(R)$ and every set of elements $f_1, \dots, f_r \in R$ such that $(f_1, \dots, f_r) = R$, the diagram of sets

$$F(S) \rightarrow \prod_{i=1}^r F(S_{f_i}) \begin{array}{c} \xrightarrow{\text{pr}_1^*} \\ \xrightarrow{\text{pr}_2^*} \end{array} \prod_{1 \leq i, j \leq r} F(S_{f_i f_j})$$

is exact; here $S_{f_i f_j} = S_{f_i} \times_S S_{f_j} = \text{Spec}(R_{f_i} \otimes_R R_{f_j}) = \text{Spec}(R_{f_i f_j})$.

3.2. Schemes as Sheaves on (Aff). Let X be any scheme. As usual, we get a presheaf $h_X = \text{Hom}_{(\text{Sch})}(-, X)$ on (Sch), and we can restrict to get a presheaf on (Aff) (here we regard (Aff) as a full subcategory of (Sch), not just the formal opposite category to (Comm)). We write $X(R) := X(\text{Spec}(R)) := h_X(\text{Spec}(R))$.

Lemma 3.2.1. *The presheaf X is a sheaf on (Aff) for the Zariski topology.*

Proof. After §3.1, it suffices to show that if $S = \text{Spec}(R)$ is an affine scheme and $f_1, \dots, f_r \in R$ generate the unit ideal, then

$$X(S) \rightarrow \prod_{i=1}^r X(S_{f_i}) \rightrightarrows \prod_{1 \leq i, j \leq r} X(S_{f_i} \times_S S_{f_j})$$

is exact. Thinking of S as a locally ringed space, we have $S_{f_i} \times_S S_{f_j} = S_{f_i} \cap S_{f_j}$. The result now follows from the fact that morphisms of locally ringed spaces are local. \square

We will later see that each scheme X is in fact a sheaf on (Aff) for the fpqc topology, which is stronger than the Zariski topology.

Lemma 3.2.2. *The functor $(\text{Sch}) \rightarrow \text{Sh}((\text{Aff}))$ in (3.2.1) is fully faithful.*

Proof. We skip the proof of injectivity on Hom sets. The essential point is that morphisms on locally ringed spaces are local.

To check surjectivity, let X and Y be schemes and assume we have a natural transformation of sheaves on (Aff)

$$f: Y(-) \rightarrow X(-).$$

Cover Y by open affines U_i . Then the morphisms $U_i \hookrightarrow Y$ give natural transformations $U_i(-) \hookrightarrow Y(-)$, and we can compose with f to get $U_i(-) \rightarrow X(-)$. By Yoneda’s lemma, since the U_i are affine, each $U_i(-) \rightarrow X(-)$ corresponds to an element $f_i \in X(U_i)$; that is, to a morphism of schemes $f_i: U_i \rightarrow X$. Using injectivity on arrows from the previous paragraph, one checks easily that each f_i and f_j agree on $U_i \cap U_j$. We then glue to get a morphism of schemes $f: Y \rightarrow X$. \square

In light of the lemma, from now on we sometimes write X for both the scheme and the corresponding sheaf on (Aff) , provided no confusion can arise.

3.3. The Sheaves that are Schemes. After (3.2.2), we have a sequence of fully faithful embeddings

$$(\text{Aff}) \hookrightarrow (\text{Sch}) \hookrightarrow \text{Sh}((\text{Aff})) \hookrightarrow \text{Presh}((\text{Aff})).$$

It is natural to ask which sheaves on (Aff) arise from schemes. In this subsection, we obtain an answer that essentially translates into the language of sheaves the usual notion that a scheme is an object glued together out of affine schemes.

Our first goal is to obtain a notion of an open immersion of sheaves. We begin with a lemma.

Lemma 3.3.1.

- (1) *Coproducts commute with the embedding $(\text{Sch}) \hookrightarrow \text{Sh}((\text{Aff}))$. That is, given schemes X_i indexed by the set I , we have*

$$\coprod_{i \in I}^{\text{Sh}((\text{Aff}))} X_i(-) \cong \left(\coprod_{i \in I}^{\text{(Sch)}} X_i \right)(-).$$

- (2) *Let X be a scheme and $\{U_i\}_{i \in I}$ a collection of open subschemes of X , so that $U := \bigcup_i U_i$ is open in X . Then $(\coprod_i U_i)(-) \rightarrow U(-)$ is an epimorphism of sheaves, and $U(-) \rightarrow X(-)$ is a monomorphism of sheaves. Hence $U(-)$ is the sheaf image of the map $(\coprod_i U_i)(-) \rightarrow X(-)$.*

The results in the lemma are what one might guess to be true. They suggest that it is reasonable to try to study schemes from the perspective of sheaves.

Proof.

- (1) First recall that the coproduct of schemes is obtained by taking the disjoint union of the underlying topological spaces with the obvious structure sheaf. For any ring R , the various morphisms $X_j \hookrightarrow \coprod_i X_i$ give a natural map

$$(*) \quad \left(\coprod_i^{\text{Presh}((\text{Aff}))} X_i(-) \right)(R) = \prod_i X_i(R) \xrightarrow{f} \left(\coprod_i^{\text{(Sch)}} X_i \right)(R).$$

f is clearly injective, but it is not in general surjective (for example, $\text{Spec}(R)$ may not be connected, and we may be able to map different components into different X_i). It is easy to see, however, that f is locally (for the

Zariski topology on $\text{Spec}(R)$) surjective. So, sheafifying $(*)$ and using that $(\coprod_i X_i)(-)$ is already a sheaf, we get

$$\left(\coprod_{i \in I} \text{Sh}((\text{Aff})) X_i(-) \right)(R) \xrightarrow{\sim} \left(\coprod_{i \in I} (\text{Sch}) X_i \right)(R).$$

- (2) It follows easily from the definition of a morphism of locally ringed spaces that $U(-) \rightarrow X(-)$ is a monomorphism and that

$$\coprod_i \text{Presh}((\text{Aff})) U_i(-) \rightarrow U(-)$$

is locally surjective. We omit the details. \square

We next define an open immersion of sheaves. First let $X = \text{Spec}(R)$ be affine, and take any collection f_i of elements in R . For each i , we get a principal open subset $X_{f_i} \hookrightarrow X$. We define the sheaf

$$(**) \quad U := \text{im} \left[\left(\coprod_i X_{f_i} \right)(-) \rightarrow X(-) \right] = \text{im}_{\text{Sh}((\text{Aff}))} \left[\left(\coprod_i X_{f_i} \right)(-) \rightarrow X(-) \right].$$

By the lemma, U is precisely the sheaf corresponding to the union of the X_{f_i} 's in X . Display $(**)$ thus allows us to obtain all open subschemes of X in an intrinsic way in $\text{Sh}((\text{Aff}))$.

When $X = \text{Spec}(R)$ is affine, we say that the morphism of sheaves $\iota: Y \rightarrow X(-)$ is an *open immersion* if ι factors as

$$Y \xrightarrow{\sim} U \hookrightarrow X(-)$$

for some U obtained as in $(**)$ for some $f_i \in R$. The definition for arbitrary sheaves on (Aff) is as follows.

Definition 3.3.2. The morphism $\beta: F \rightarrow G$ of sheaves on (Aff) is an *open immersion* if for every affine scheme X and every morphism of sheaves $X \rightarrow G$, the morphism $\beta': F \times_G X \rightarrow X$ in the pullback diagram

$$\begin{array}{ccc} F \times_G X & \xrightarrow{\beta'} & X \\ \downarrow & & \downarrow \\ F & \xrightarrow{\beta} & G \end{array}$$

is an open immersion (in the sense just defined).

As motivation for the definition, let \mathcal{C} be any category and G any presheaf on \mathcal{C} . Then, tautologically,

$$G \cong \varinjlim_{h_S \rightarrow G} h_S,$$

or, informally,

$$G \cong \varinjlim_{h_S \rightarrow G} S.$$

There is now an analogy between the definition of an open immersion $F \rightarrow G$ and the colimit topology on the colimit of topological spaces, where a set in the colimit is open if and only if its inverse image in each S is open.

The following exercise justifies the choice of language in the definition.

Exercise 3.3.3. Show that the morphism of schemes $Y \rightarrow X$ is an open immersion (in the usual sense of algebraic geometry) if and only if the corresponding morphism of sheaves is an open immersion of sheaves (in the sense of (3.3.2)).

We now give an answer to the question posed at the beginning of the subsection.

Proposition 3.3.4. *The sheaf F on (Aff) is a scheme if and only if there exists a collection $X_i \rightarrow F$ of open immersions such that*

- (1) *each X_i is an affine scheme, and*
- (2) *the map of sheaves $\coprod_i X_i \rightarrow F$ is an epimorphism.*

Proof. We leave the implication \implies as an easy exercise. To prove \impliedby , suppose we have a collection of open immersions $X_i \rightarrow F$ satisfying (1) and (2). Since $\coprod_i X_i \rightarrow F$ is an epimorphism, it is a strict epimorphism by (1.10.5). That is, F is the cokernel of the diagram

$$\left(\coprod_i X_i\right) \times_F \left(\coprod_i X_i\right) \rightrightarrows \coprod_i X_i.$$

Applying (1.11.1) twice (as in the proof of (1.11.2)), we have

$$\left(\coprod_i X_i\right) \times_F \left(\coprod_i X_i\right) \cong \coprod_{i,j} X_i \times_F X_j.$$

Since the $X_i \rightarrow F$ are open immersions and each X_i is affine, each $X_i \times_F X_j \rightarrow X_i$ obtained from the pullback diagram

$$\begin{array}{ccc} X_i \times_F X_j & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ X_j & \longrightarrow & F \end{array}$$

is an open immersion of sheaves. Thus, each $X_i \times_F X_j$ necessarily corresponds to an open subscheme of X_i . So the X_i and $X_i \times_F X_j$ constitute gluing data in (Sch), and we glue to obtain a scheme X .

By construction, X is the cokernel in (Sch) of the diagram

$$(***) \quad \coprod_{i,j} X_i \times_F X_j \rightrightarrows \coprod_i X_i.$$

To show $X \cong F$ as sheaves, it suffices to show that X remains the cokernel of (***) regarded as a diagram of sheaves on (Aff) .

The gluing procedure constructs X as a scheme covered by open immersions $X_i \rightarrow X$ such that each $X_i \times_F X_j$ is identified with $X_i \cap X_j$ in X . Thus $X_i \times_F X_j \cong X_i \times_X X_j$ as schemes. Tautologically, the embedding $(\text{Sch}) \hookrightarrow \text{Presh}((\text{Sch}))$ preserves limits, and so, restricting to (Aff) , the embedding $(\text{Sch}) \hookrightarrow \text{Sh}((\text{Aff}))$ preserves limits. Thus the diagram

$$\begin{array}{ccc} (X_i \times_X X_j)(-) & \longrightarrow & X_j(-) \\ \downarrow & & \downarrow \\ X_i(-) & \longrightarrow & X(-) \end{array}$$

is a pullback diagram in $\text{Sh}((\text{Aff}))$. Since the X_i cover X as schemes, the morphism of sheaves $\coprod_i X_i \rightarrow X$ is locally surjective, and hence an epimorphism. Thus, just

as for F , X is the cokernel of

$$\coprod_{i,j} X_i \times_F X_j \cong \coprod_{i,j} X_i \times_X X_j \rightrightarrows \coprod_i X_i.$$

□

3.4. Smooth Morphisms and Etale Morphisms. For many purposes, the Zariski topology on (Aff) (and its analogous version on (Sch)) is unsatisfactory: there are too few open sets, or, in the language of §1, too few covering sieves. In this subsection we introduce some of the kinds of morphisms that we'll use to create finer topologies.

Definition 3.4.1. The morphism of schemes $f: Y \rightarrow X$ is *smooth* if f is flat and locally of finite presentation and all geometric fibers are nonsingular algebraic varieties (we allow nonconnected varieties, as well as empty varieties).

We translate the definition into the affine case with $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$, and f corresponding to the ring homomorphism $\phi: A \rightarrow B$. f is flat if and only if B is a flat A -module. f is locally of finite presentation if and only if B is of finite presentation if and only if B has a finite presentation as an A -algebra; that is, B is generated as an A -algebra by finitely many elements b_1, \dots, b_n , such that the kernel of the map

$$\begin{aligned} A[T_1, \dots, T_n] &\rightarrow B \\ T_i &\mapsto b_i \end{aligned}$$

is finitely generated as an ideal. One can show that if B is of finite presentation, then every choice of finitely many generators induces a finite presentation. When A is Noetherian, B is of finite presentation if and only if B is finitely generated. In non-Noetherian situations, finitely presented is a better behaved notion than finitely generated.

A geometric point of X is an element of $X(k)$ for some algebraically closed field k . A geometric fiber is the variety $\text{Spec}(k) \times_X Y$ obtained from the pullback diagram

$$\begin{array}{ccc} \text{Spec}(k) \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \text{Spec}(k) & \longrightarrow & X, \end{array}$$

where $\text{Spec}(k) \rightarrow X$ is a geometric point. In the affine case, a geometric point corresponds to a homomorphism $A \rightarrow k$, with k algebraically closed, and the resulting geometric fiber is the affine scheme $\text{Spec}(k \otimes_A B)$.

Definition 3.4.2. Let $f: Y \rightarrow X$ be a morphism of schemes.

- (1) We say f is smooth of *relative dimension* n if f is smooth and every geometric fiber is purely n -dimensional (by convention, the empty variety has every possible dimension).
- (2) We say f is *étale* if f is smooth of relative dimension 0.

For example, let E/F be a finite field extension, and let $F \hookrightarrow \bar{F}$ be an algebraic closure of F . It is a fact that the geometric fiber $\text{Spec}(E \otimes_F \bar{F})$ is nonsingular if and only if E/F is a separable extension. As an example, suppose F has characteristic p and $E = F(\sqrt[p]{a})$ for some $a \in F$, so that E is purely inseparable. Then $E =$

$F[T]/(T^p - a)$, and $E \otimes_F \bar{F} = \bar{F}[T]/(T^p - a) = \bar{F}[T]/(T - \sqrt[p]{a})^p$. So $E \otimes_F \bar{F}$ has nilpotent elements, whence $\text{Spec}(E \otimes_F \bar{F})$ is singular.

We conclude the subsection with a list of some of the properties enjoyed by smooth and étale morphisms.

Let $f: X \rightarrow S$ be a smooth morphism and $T \rightarrow S$ arbitrary. Then the morphism $f': X \times_S T \rightarrow T$ obtained from the pullback diagram

$$\begin{array}{ccc} X \times_S T & \xrightarrow{f'} & T \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

is also smooth, as each condition in the definition is easily seen to hold. If f is of relative dimension n , then it is again obvious that f' is too. In particular, if f is étale, then so is f' . Thus smooth and étale morphisms are stable under base change.

Let

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

be a composition of morphisms of schemes. Then it is easy to see that if f and g are smooth, then so is fg , and that if, further, f is of relative dimension n and g is of relative dimension m , then fg is of relative dimension $n + m$. Thus smooth and étale morphisms are stable under composition.

Now suppose $f: X \rightarrow S$ and $g: Y \rightarrow S$ are two smooth morphisms. Then it follows from the previous two facts that $X \times_S Y \rightarrow S$ is also smooth. Moreover, if f and g are of respective relative dimensions n and m , then $X \times_S Y \rightarrow S$ is of relative dimension $n + m$.

3.5. Pretopologies. For the moment, we return to the general setting of Grothendieck topologies on a category \mathcal{C} . This material could have been included in §1.

In practice, one often defines a Grothendieck topology on a particular category by declaring certain families $\{S_i \rightarrow S\}_{i \in I}$ of morphisms to be “coverings”: each family generates a sieve, and the resulting topology is the minimal one such that all sieves so obtained are covering sieves. When the families $\{S_i \rightarrow S\}$ satisfy certain properties, we get a nice characterization of all resulting covering sieves.

Definition. Let \mathcal{C} be a category in which all fibered products exist (for example, (Aff) or (Sch)). A *Grothendieck pretopology*, or a *base* for a Grothendieck topology, consists of certain families, called *covering families*, of morphisms in \mathcal{C} , each family of the form $\{S_i \rightarrow S\}_{i \in I}$ for some $S \in \text{ob } \mathcal{C}$, satisfying the following properties.

- (PT1) Any isomorphism $S' \xrightarrow{\sim} S$ is a covering family on S for any object S .
- (PT2) If $\{S_i \rightarrow S\}_{i \in I}$ is a covering family on S , and $T \rightarrow S$ is any morphism, then $\{S_i \times_S T \rightarrow T\}_{i \in I}$ is a covering family on T .
- (PT3) If $\{S_i \rightarrow S\}_{i \in I}$ is a covering family on S , and for each i , $\{S_{ij} \rightarrow S_i\}_{j \in J_i}$ is a covering family on S_i , then the family $\{S_{ij} \rightarrow S\}$ obtained by compositions is a covering family on S .

Properties (PT1)–(PT3) are analogous to properties (GT1)–(GT3) in the definition of a Grothendieck topology (1.5.1).

Given a pretopology on \mathcal{C} , we get a Grothendieck topology on \mathcal{C} by taking the covering sieves to be those sieves that contain some covering family. We leave it as

an exercise to verify that this definition actually gives a topology. This topology is the minimal topology such that the sieves generated by the covering families are covering sieves.

3.6. Some Topologies on (Aff). Let (P) be a property of morphisms of schemes (for example, flat, smooth, or étale). Suppose that (P)

- (1) holds for all isomorphisms;
- (2) is stable under base change (that is, if $f: X \rightarrow S$ has property (P) and $T \rightarrow S$ is any morphism, then $f': X \times_S T \rightarrow T$ has property (P)); and
- (3) is stable under composition (that is, given a composition $Z \xrightarrow{g} Y \xrightarrow{f} X$, if f and g have property (P) , then so does fg).

Then we claim that we get a pretopology, and hence a Grothendieck topology, on (Aff) (or on (Sch)) by taking the covering families to be all finite surjective families of morphisms $\{S_i \rightarrow S\}_{i \in I}$ such that each $S_i \rightarrow S$ satisfies (P) . By finite, we just mean that the index set I is finite, and by surjective, we mean that the union of the images of the S_i is all of S .

Given our assumptions on (P) , all we really need to check to verify that we get a pretopology is that surjective families are stable under base change and composition, the latter of which is clear. Stability under base change follows from the following exercise.

Exercise 3.6.1. For any scheme S , write S_{set} for the underlying set of the topological space associated to S . Show that for any pullback diagram of schemes

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & S, \end{array}$$

the natural map $(X \times_S Y)_{\text{set}} \rightarrow X_{\text{set}} \times_{S_{\text{set}}} Y_{\text{set}}$ is surjective.

We now define several pretopologies on (Aff) in the way described above by taking morphisms $U \rightarrow X$ satisfying the following properties (P) :

- (1) $U \hookrightarrow X$ is a principal open immersion (that is, $U \hookrightarrow X$ factors as $U \xrightarrow{\sim} X_f \hookrightarrow X$ for some principal open affine subscheme X_f);
- (2) $U \hookrightarrow X$ is an open immersion;
- (3) $U \rightarrow X$ is étale;
- (4) $U \rightarrow X$ is smooth;
- (5) $U \rightarrow X$ is flat and finitely presented;
- (6) $U \rightarrow X$ is flat and quasi-compact.

Note that each property in the list implies the one after it. Except for (1), we could just as well define the above pretopologies on (Sch).

Several remarks are in order. First, the quasi-compactness requirement in (6) is redundant: every affine scheme is quasi-compact, so every morphism of affine schemes is quasi-compact. We mention quasi-compactness only because it is necessary to get the right pretopology in (Sch).

Second, it is a fact that if $f: \text{Spec}(B) \rightarrow \text{Spec}(A)$ is flat and finitely presented, then f is an open mapping. When A is Noetherian, this is exercise 7.25 in Atiyah-MacDonald. For the general case, see EGA IV, Theorem 2.4.6.

Consequently, all morphisms in (1)–(5) are open mappings. Again, since affine schemes are quasi-compact, we could therefore drop the finiteness requirement on our covering families without changing the topologies (though we’d get different pretopologies, of course).

Third, the finiteness requirement is necessary in (6). Without it, the family

$$\{\mathrm{Spec}(A_P) \rightarrow \mathrm{Spec}(A)\}_{P \in \mathrm{Spec}(A)}$$

would always be a flat surjective covering family. When $\mathrm{Spec}(A)$ is infinite, there need be no finite surjective subfamily, a possibility we do not wish to allow.

Definition 3.6.2. We say that the above pretopologies generate the following topologies:

- (A) the *Zariski topology*, generated by (1) and (2);
- (B) the *étale topology*, generated by (3) and (4);
- (C) the *fppf topology*, generated by (5); and
- (D) the *fqc topology*, generated by (6).

Here fppf abbreviates the French for faithfully flat and finitely presented, and fqc abbreviates the French for faithfully flat and quasi-compact.

We conclude the subsection by justifying that (1) and (2), and (3) and (4), generate the same topologies. Both cases are obtained from the following argument. Let (P_1) and (P_2) be two properties of morphisms that hold for isomorphisms and are stable under base change and compositions.

Trivially, if $(P_1) \implies (P_2)$, then any covering sieve for the (P_1) -topology is a covering sieve for the (P_2) -topology.

Next, suppose that (P_1) morphisms are open, and that for any (P_2) -morphism $f: Y \rightarrow X$ and any point x in the image of Y , there exists an affine scheme Z and a (P_1) -morphism $g: Z \rightarrow X$ such that g factors through f :

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ g \downarrow & & \downarrow f \\ X & \xlongequal{\quad} & X. \end{array}$$

Then it follows immediately that any covering sieve for the (P_2) -topology is a covering sieve for the (P_1) -topology.

It is now immediate that (1) and (2) generate the same topology. To see that the argument applies to étale and smooth morphisms, we cite the following result.

Lemma 3.6.3. *If $f: X \rightarrow S$ is a smooth surjective morphism of schemes, then there exists a scheme S' and a surjective étale morphism $S' \rightarrow S$ factoring through f .*

Proof. EGA IV, Corollary 17.16.3. □

3.7. Faithfully Flat Algebras. One of the basic features of the topologies defined in the previous subsection is that they admit descent of quasi-coherent sheaves of modules (or, working in (Comm), of modules over rings). The essential case is that of a singleton flat covering family; that is, of a surjective flat morphism $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$. We now study such morphisms from the perspective of rings.

Given an A -algebra B and an A -module M , we write M_B for $M \otimes_A B$.

Proposition 3.7.1. *Let B be a flat A -algebra. The following conditions are equivalent.*

- (1) For all A -modules $M \neq 0$, we have $M_B \neq 0$.
- (2) For all A -modules M , the map $m \mapsto m \otimes 1_B$ gives an injection $M \hookrightarrow M_B$.
- (3) The morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is surjective.
- (4) For every maximal ideal m of A , there exists a maximal ideal n of B such that $n \cap A = m$.

Definition 3.7.2. We say B is a *faithfully flat* A -algebra if B satisfies the equivalent conditions in (3.7.1).

It is a general fact about exact additive functors that condition (1) is equivalent to the functor

$$- \otimes_A B: M \mapsto M_B$$

being faithful, hence the terminology “faithfully flat.”

Proof of 3.7.1. We give a sketch. The proposition appears in Atiyah-MacDonald as exercise 3.16.

Condition (2) says that, in the case $M = A/I$ for some ideal I , we have

$$A/I \hookrightarrow (A/I) \otimes_A B = B/IB.$$

Equivalently, $(IB) \cap A = I$ for all ideals $I \subset A$, from which it is easy to deduce (3) and (4).

Since B is flat, tensoring with B preserves submodules, and in particular submodules generated by a single element. Therefore condition (1) is equivalent to

$$B/IB = (A/I)_B \neq 0 \quad \text{for all ideals } I \neq A,$$

which in turn is equivalent to

$$B/mB = (A/m)_B \neq 0 \quad \text{for all maximal ideals } m \subset A.$$

It is now easy to check the equivalence of (1) with (3) and (4).

It remains to show (1) \implies (2). Let K be the kernel of $M \rightarrow M_B$. Then by flatness, K_B is the kernel of $M_B \rightarrow (M_B)_B = M_{B \otimes_A B}$. So, assuming (1), it suffices to show $K_B = 0$, or that $\phi: M_B \rightarrow M_{B \otimes_A B}$ is an injection. But ϕ is a section of the map $M_{B \otimes_A B} = M \otimes_A B \otimes_A B \rightarrow M \otimes_A B = M_B$ sending $m \otimes b \otimes c \mapsto m \otimes bc$, and so indeed is injective. \square

The strategy used in the last paragraph is typical of faithfully flat descent: one proves the desired result in the situation obtained from faithfully flat base change, and then faithful flatness means the result “descends” to the original situation.

Let B be an A -algebra and M an A -module. The two algebra maps $B \rightarrow B \otimes_A B$ sending $b \mapsto b \otimes 1_B$ and $b \mapsto 1_B \otimes b$ give two corresponding A -module maps $M_B \rightarrow M_{B \otimes_A B}$.

Proposition 3.7.3. *If B is a faithfully flat A -algebra, then for any A -module M , the diagram*

$$0 \rightarrow M \rightarrow M_B \rightrightarrows M_{B \otimes_A B}$$

is exact.

Proof. By faithful flatness, it suffices to tensor the entire diagram with B and show that the resulting diagram

$$0 \rightarrow M_B \xrightarrow{f} M_{B \otimes_A B} \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} M_{B \otimes_A B \otimes_A B}$$

is exact. Here

$$\begin{aligned} f(m \otimes b) &= m \otimes 1 \otimes b, \\ g_1(m \otimes b \otimes c) &= m \otimes b \otimes 1 \otimes c, \\ g_2(m \otimes b \otimes c) &= m \otimes 1 \otimes b \otimes c. \end{aligned}$$

Exactness at M_B is clear from (3.7.1). So let $x = \sum_i m_i \otimes b_i \otimes c_i$ be in the equalizer of g_1 and g_2 , so that

$$(*) \quad \sum_i m_i \otimes b_i \otimes 1 \otimes c_i = \sum_i m_i \otimes 1 \otimes b_i \otimes c_i.$$

Applying the map $M_{B \otimes_A B \otimes_A B} \rightarrow M_{B \otimes_A B}$ sending $m \otimes b \otimes c \otimes d \mapsto m \otimes b \otimes cd$ to $(*)$, we get

$$x = \sum_i m_i \otimes b_i \otimes c_i = \sum_i m_i \otimes 1 \otimes b_i c_i.$$

Similarly, applying the map $M_{B \otimes_A B} \rightarrow M_B$ sending $m \otimes b \otimes c \mapsto m \otimes bc$ to x , and following with f , we get

$$x \mapsto \sum_i m_i \otimes b_i c_i \xrightarrow{f} \sum_i m_i \otimes 1 \otimes b_i c_i = x.$$

Hence $x \in \text{im } f$, as desired. \square

Corollary 3.7.4. *If B is a faithfully flat A -algebra, then for all A -modules M and N , the diagram*

$$0 \rightarrow \text{Hom}_A(M, N) \rightarrow \text{Hom}_B(M_B, N_B) \rightrightarrows \text{Hom}_{B \otimes_A B}(M_{B \otimes_A B}, N_{B \otimes_A B})$$

is exact.

Proof. By the universal property of extension of scalars, we have

$$\text{Hom}_B(M_B, N_B) = \text{Hom}_A(M, N_B)$$

and

$$\text{Hom}_{B \otimes_A B}(M_{B \otimes_A B}, N_{B \otimes_A B}) = \text{Hom}_A(M, N_{B \otimes_A B}).$$

The result now follows from (3.7.3) applied to N . \square

3.8. Faithfully Flat Descent of Modules. For each affine scheme $S = \text{Spec}(A)$, let

$$\mathcal{D}(S) := (A\text{-Mod}).$$

Then \mathcal{D} defines a (non-strict) presheaf of categories on (Aff) : given a morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$, we get a pullback functor $(A\text{-Mod}) \rightarrow (B\text{-Mod})$ sending

$$M \mapsto M \otimes_A B.$$

The various canonical isomorphisms provided by the tensor product constitute the rest of the data necessary to make \mathcal{D} a presheaf of categories, and it is easy to check that these data satisfy the presheaf axioms.

\mathcal{D} thus defines a fibered category, which we denote by (Mod) , over (Aff) . Of course, (Mod) is equivalent to the fibered category of quasi-coherent sheaves over (Aff) .

Theorem 3.8.1. *(Mod) is a stack over (Aff) (for any of the topologies defined in (3.6.2)).*

We obtain the proof as a series of lemmas. It suffices to work just with the fpqc topology, since it's the finest.

Lemma 3.8.2. *(Mod) is a separated fibered category.*

Proof. Let S be an affine scheme, and fix functors $D, D' \in \text{ob}(\text{Mod})(h_S)$. Then we can form the presheaf of sets $\mathcal{H} = \mathcal{H}om(D, D')$ on $(\text{Aff})/S$ as described in §2.10. By (2.10.7), it suffices to show that \mathcal{H} is a sheaf.

By definition of the fpqc topology on (Aff) , the covering sieves generated by all finite flat surjective covering families $\{Y_i \rightarrow X\}_{i=1}^n$ are cofinal amongst all covering sieves. Thus the same is true of the induced topology on $(\text{Aff})/S$. So let $X = \text{Spec}(A)$ be a scheme over S , and let $\{Y_i \rightarrow X\}$ be a finite flat covering family, with $Y_i = \text{Spec}(B_i)$. To show that \mathcal{H} is a sheaf on $(\text{Aff})/S$, it therefore suffices to show, just as described for the Zariski topology on (Aff) in §3.1, that the diagram of sets

$$(*) \quad \mathcal{H}(X) \rightarrow \prod_i \mathcal{H}(Y_i) \rightrightarrows \prod_{i,j} \mathcal{H}(Y_i \times_X Y_j)$$

is exact (here, as is customary when denoting objects in a category of the form \mathcal{C}/S , we have suppressed the morphisms in favor of their sources).

Now for any $g: Z = \text{Spec}(R) \rightarrow S$ in $(\text{Aff})/S$, we have

$$\mathcal{H}(Z) = \text{Hom}_{(\text{Mod})(g^*(h_S))}(g^*(D), g^*(D')) = \text{Hom}_{(\text{Mod})(h_Z)}(g^*(D), g^*(D')).$$

By Yoneda's lemma for fibered categories, $\mathcal{H}(T)$ is canonically identified with

$$\text{Hom}_{(\text{Mod})(Z)}(M, N) = \text{Hom}_{(R\text{-Mod})}(M, N),$$

where the R -modules M and N are the respective images of D and D' under the canonical functor $(\text{Mod})(h_Z) \rightarrow (\text{Mod})(Z)$.

Thus, to show that $(*)$ is exact, it suffices to show

$$(**) \quad \text{Hom}_A(M, N) \rightarrow \prod_i \text{Hom}_{B_i}(M_{B_i}, N_{B_i}) \rightrightarrows \prod_{i,j} \text{Hom}_{B_i \otimes_A B_j}(M_{B_i \otimes_A B_j}, N_{B_i \otimes_A B_j})$$

is exact for any A -modules M and N .

Define $B := \prod_i B_i$, so that B is faithfully flat over A . Then $(**)$ is just the diagram

$$\text{Hom}_A(M, N) \rightarrow \text{Hom}_B(M_B, N_B) \rightrightarrows \text{Hom}_{B \otimes_A B}(M_{B \otimes_A B}, N_{B \otimes_A B}),$$

which is indeed exact by (3.7.4). \square

It remains to show that for any covering sieve \mathcal{U} on the affine scheme $X = \text{Spec}(A)$, the natural functor

$$(\text{Mod})(h_X) \rightarrow (\text{Mod})(\mathcal{U})$$

is essentially surjective. More concretely, this means that every \mathcal{U} -local section arises from an A -module over X .

The essential case is when \mathcal{U} is generated by a faithfully flat covering $Y = \text{Spec}(B) \rightarrow X$. Regard \mathcal{U} as a fibered category, so that \mathcal{U} is a full fibered subcategory of $(\text{Aff})/X$, and fix some \mathcal{U} -local section D . For every object $Z = \text{Spec}(C) \rightarrow X$ in \mathcal{U} , D gives a C -module $D(Z)$ (again, we abuse notation by suppressing the morphism). For all affine X -schemes Z' and Z in \mathcal{U} and every morphism $Z' \rightarrow Z$

over X , we get a morphism $D(Z') \rightarrow D(Z)$ in (Mod) . Since every morphism in \mathcal{U} is cartesian, so is $D(Z') \rightarrow D(Z)$, so that $D(Z')$ is a pullback of $D(Z)$.

In particular, we have a diagram of A -algebra homomorphisms

$$A \rightarrow B \rightrightarrows B \otimes_A B \rightrightarrows B \otimes_A B \otimes_A B,$$

where the top and bottom maps $B \rightarrow B \otimes_A B$ are given by

$$(***) \quad \begin{aligned} b &\mapsto 1 \otimes b, \\ &b \mapsto b \otimes 1, \end{aligned}$$

respectively, and the top, middle, and bottom maps $B \otimes_A B \rightarrow B \otimes_A B \otimes_A B$ are given by

$$\begin{aligned} b \otimes c &\mapsto 1 \otimes b \otimes c, \\ b \otimes c &\mapsto b \otimes 1 \otimes c, \\ b \otimes c &\mapsto b \otimes c \otimes 1, \end{aligned}$$

respectively.

There are corresponding morphisms of affine schemes

$$X \leftarrow Y \leftarrow Y \times_X Y \leftarrow Y \times_X Y \times_X Y.$$

Applying D , we get a diagram in (Mod)

$$(\dagger) \quad N \leftarrow N' \leftarrow N''$$

where

$$\begin{aligned} N &:= D(Y), \\ N' &:= D(Y \times_X Y), \\ N'' &:= D(Y \times_X Y \times_X Y) \end{aligned}$$

and all five morphisms are cartesian. By definition of (Mod) , the $B \otimes_A B$ -module $N \otimes_B (B \otimes_A B)$, where we form the tensor product using the top map in $(***)$, is a pullback of N along the top arrow of $Y \times_X Y \rightrightarrows Y$. Since the same is true of N' , there is a canonical $B \otimes_A B$ -module isomorphism $N \otimes_B (B \otimes_A B) \xrightarrow{\sim} N'$ in (Mod) , and we get a B -module map $N \rightarrow N'$. Applying the same reasoning to each morphism in (\dagger) , we get maps

$$N \rightrightarrows N' \rightrightarrows N''$$

(the two on the left are B -module maps, and the three on the right are $B \otimes_A B$ -module maps).

The above says essentially that to give a \mathcal{U} -local section, it is enough to give a B -module N and an isomorphism ϕ between the two pullbacks of N along $Y \times_X Y \rightrightarrows Y$ such that the three pullbacks of ϕ along $Y \times_X Y \times_X Y \rightrightarrows Y$ satisfy a certain cocycle condition. We will not pursue this point further.

We are almost ready to state our next lemma. Given an A -module M , we get an h_X -local section, and hence a \mathcal{U} -local section, by defining the functor on $(\text{Aff})/X$ sending

$$(\text{Spec}(C) \rightarrow X) \mapsto M \otimes_A C.$$

By Yoneda’s lemma for fibered categories, every h_X -local section is isomorphic to one of this form. We get a functor $F: (\text{Mod})(X) = (A\text{-Mod}) \rightarrow (\text{Mod})(\mathcal{U})$.

Given a \mathcal{U} -local section D , we have seen above that there arises a diagram of B -modules $N \rightrightarrows N' \rightrightarrows N''$. We define the functor $G: (\text{Mod})(\mathcal{U}) \rightarrow (A\text{-Mod})$ sending

$$D \mapsto \ker[N \rightrightarrows N'].$$

Lemma 3.8.3. *The functor F is an equivalence of categories, and G is a quasi-inverse.*

Proof. The proof is explained in SGA 4 $\frac{1}{2}$ I, pages 7–8. We give the thrust of the argument here.

There are obvious natural transformations

$$\alpha: \text{id}_{(A\text{-Mod})} \rightarrow GF \quad \text{and} \quad \beta: FG \rightarrow \text{id}_{(\text{Mod})(\mathcal{U})}.$$

We have already seen that α is an isomorphism in (3.7.3).

To show that β is an isomorphism, it suffices to do so in the situation obtained by any faithfully flat base change $Z \rightarrow X$. In particular, we can take the base change $Y \rightarrow X$, obtaining

$$\begin{array}{ccc} Y \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

Now the top row admits a section, namely the diagonal morphism Δ . Since id_Y factors through $p: Y \times_X Y \rightarrow Y$, p generates the maximal sieve h_Y . By Yoneda's lemma for fibered categories, any functor $D \in \text{ob}(\text{Mod})(h_Y)$ is determined up to isomorphism by $D(S \xrightarrow{\text{id}_Y} S)$, which completes the proof. \square

The proof of (3.8.1) now follows formally from the faithfully flat case, similar to the end of the proof of (3.8.2). We skip the details.

3.9. Further Results on Descent. We can extend the result of the previous subsection to show that certain properties of modules descend, in addition to the modules themselves.

Lemma 3.9.1. *Let B be a faithfully flat A -algebra, and let (P) be a property of modules (for example, finitely generated). Then M has property (P) as an A -module if and only if M_B has property (P) as a B -module, where (P) is any of the following properties:*

- (1) *finitely generated;*
- (2) *finitely presented;*
- (3) *flat;*
- (4) *finitely generated and projective; or*
- (5) *finitely generated and projective of rank n .*

Proof. The implication \implies is easy for each property, and we skip it.

We prove the reverse implication \impliedby . For (1), write M as a directed colimit of all finitely generated submodules $N \subset M$,

$$M = \varinjlim_{\substack{N \subset M \\ \text{finitely} \\ \text{generated}}} N.$$

Since tensoring commutes with directed colimits, we have

$$M_B = \varinjlim_{N \subset M} N_B$$

Since M_B is finitely generated, we can find a finitely generated N such that $N_B = M_B$. Hence, by faithful flatness, $N = M$.

For (2), assume that M_B is finitely presented. Then, by (1), M is finitely generated. So there exists an exact sequence

$$0 \rightarrow K \rightarrow A^n \rightarrow M \rightarrow 0.$$

Tensoring with B , we get an exact sequence

$$0 \rightarrow K_B \rightarrow B^n \rightarrow M_B \rightarrow 0.$$

Since M_B is finitely presented, any surjection $N \twoheadrightarrow M_B$, with N finitely generated, has finite kernel (Matsumura, Theorem 2.6). In particular, K_B is finitely generated, and by (1) again, so is K .

We leave (3) as an easy exercise.

For (4), we use that M is finitely generated and projective if and only if M is finitely presented and flat (Bourbaki, *Commutative Algebra*).

(5) is clear from (4). □

The lemma provides us with new stacks on (Aff) (for any of our topologies), each a substack of (Mod). For example, fix an integer $n > 0$, and let $\mathcal{D}_n(\text{Spec}(A))$ be the category of finitely generated projective A -modules of rank n ; that is, the category of rank n vector bundles on $\text{Spec}(A)$. By (5), \mathcal{D}_n defines a stack. We will see later that \mathcal{D}_n actually defines an *algebraic* stack.

The presheaf sending $S = \text{Spec}(A)$ to the category of schemes over S defines the fibered category of schemes over affine schemes, which we denote by (Sch)/(Aff). Our goal for the rest of the subsection is to show that (Sch)/(Aff) is a separated fibered category (for any of our topologies).

Lemma 3.9.2. *Let $g: T \rightarrow S$ be a flat morphism of schemes. Then for any quasi-compact morphism of schemes $f: X \rightarrow S$, we have*

$$\overline{g^{-1}(f(X))} = g^{-1}(\overline{f(X)}).$$

Proof. SGA 1 VIII, Theorem 4.1. □

Lemma 3.9.3. *Let $g: T \rightarrow S$ be a flat quasi-compact morphism of schemes. Then the subspace topology on $g(T)$ agrees with the quotient topology on $g(T)$ arising from $T \twoheadrightarrow g(T)$.*

Proof. SGA 1 VIII, Corollary 4.2. □

As a special case, if g is faithfully flat and quasi-compact, then the topology on S is necessarily the quotient topology.

Proposition 3.9.4. *Let $g: T \rightarrow S$ be a faithfully flat quasi-compact morphism of schemes. Let $R = T \times_S T$, so that we have a diagram of schemes*

$$R \begin{array}{c} \xrightarrow{p} \\ \rightrightarrows \\ \xrightarrow{q} \end{array} T \xrightarrow{g} S.$$

Then for any scheme X , the diagram

$$\text{Hom}_{(\text{Sch})}(S, X) \rightarrow \text{Hom}_{(\text{Sch})}(T, X) \rightrightarrows \text{Hom}_{(\text{Sch})}(R, X)$$

is exact.

Proof. Let $f: T \rightarrow X$ be such that fp and fq give the same morphism $u: R \rightarrow X$:

$$R \begin{array}{c} \xrightarrow{p} \\ \rightrightarrows \\ \xrightarrow{q} \end{array} T \xrightarrow{f} X.$$

Since $R_{\text{set}} \rightarrow T_{\text{set}} \times_{S_{\text{set}}} T_{\text{set}}$, f is constant on the fibers of g , and we get a unique set map $f': S_{\text{set}} \rightarrow X_{\text{set}}$. Since S has the quotient topology, f' is a continuous map of topological spaces.

Let $v = gp = gq$ denote the morphism $R \rightarrow S$. We get a diagram of sheaves on S

$$(*) \quad 0 \rightarrow \mathcal{O}_S \rightarrow g_*\mathcal{O}_T \rightrightarrows v_*\mathcal{O}_R.$$

When S and T are affine, $(*)$ is exact by (3.7.3) (taking $M = A$, where $S = \text{Spec}(A)$). Exactness of $(*)$ in the general case is obtained easily from the affine case, using quasi-compactness of g .

Using f' , push $(*)$ forward to X :

$$(**) \quad 0 \rightarrow f'_*\mathcal{O}_S \rightarrow f_*\mathcal{O}_T \rightrightarrows u_*\mathcal{O}_R.$$

Since pushing forward is an exact functor, $(**)$ remains exact. Therefore the map of sheaves $f^\#: \mathcal{O}_X \rightarrow f_*\mathcal{O}_T$ factors through $f'_*\mathcal{O}_S$, and we get the desired morphism of schemes $(f', f^\#): S \rightarrow X$. \square

We obtain the following as a formal consequence, justifying the claim made in §3.2.

Corollary 3.9.5. *Let X be any scheme. Then X determines a sheaf on (Aff) for the fpqc topology (and hence for any of our topologies).* \square

We also obtain our desired result on $(\text{Sch})/(\text{Aff})$.

Corollary 3.9.6. *$(\text{Sch})/(\text{Aff})$ is a separated fibered category over (Aff) for the fpqc topology (and hence for any of our topologies). Moreover, $(\text{Sch})/(\text{Aff})$ is a stack on (Aff) for the Zariski topology.*

Proof. It follows formally from (3.9.5) that $(\text{Sch})/(\text{Aff})$ is a separated fibered category. See SGA 1 VIII, Theorem 5.2.

The claim that $(\text{Sch})/(\text{Aff})$ is a stack for the Zariski topology follows from a standard argument on gluing locally ringed spaces. \square

4. ALGEBRAIC STACKS: MOTIVATION AND EXAMPLES

Our goal in the remainder of these notes is an exposition of some of the basic theory of algebraic stacks. The precise definition of an algebraic stack is somewhat involved, so in this section we discuss a number of examples to help us get a feel for the subject.

There is a sequence of fully faithful embeddings

$$(*) \quad (\text{Aff}) \hookrightarrow (\text{Sch}) \hookrightarrow (\text{Alg Sp}) \hookrightarrow \text{Sh}^{\text{fpqc}}((\text{Aff})) \\ \hookrightarrow \text{Sh}^{\text{fppf}}((\text{Aff})) \hookrightarrow \text{Sh}^{\text{ét}}((\text{Aff})) \hookrightarrow \text{Sh}^{\text{Zar}}((\text{Aff})) \hookrightarrow \text{Presh}((\text{Aff})).$$

Here (Alg Sp) is the category of algebraic spaces (for sake of time, we will not give the definition), and, for example, $\text{Sh}^{\text{fpqc}}((\text{Aff}))$ is the category of sheaves on (Aff) for the fpqc topology.

We have seen that fibered categories are the categorical generalization of pre-sheaves of sets on (Aff). Similarly, stacks for the fpqc, fppf, étale, and Zariski topologies are the categorical generalizations of sheaves of sets for the fpqc, fppf, étale, and Zariski topologies, respectively. In some sense, algebraic stacks are the categorical generalization of algebraic spaces.

We remark that, from the perspective of (*), there is some ambiguity as to which spaces (or objects) are the natural ones to study. The objects in each category in (*) can be obtained as colimits of affine schemes, and we can distinguish these categories by restricting the types of colimits under consideration. But it is not obvious which colimits should be preferred.

4.1. Preliminary Examples. We begin by defining several fibered categories on (Aff), each of which is, in fact, an algebraic stack. We choose to be vague on the details in many places for the time being. Fuller explanations will come later.

- (1) *Vector bundles of rank n.* Let $n \geq 1$, and let VB_n be the fibered category associated to the presheaf sending S to the category of rank n vector bundles over S .
- (2) *Schemes.* Given a scheme X , we get the presheaf of sets $\text{Hom}_{(\text{Sch})}(-, X)$, which we can interpret as a fibered category.

We get the following exercise from (1) and (2).

Exercise. Show that, for any scheme X , the category of morphisms of fibered categories $VB_n(X)$ is equivalent to the category of rank n vector bundles over X .

Thus we can recover vector bundles over all schemes just from vector bundles over affine schemes.

We continue with our list of examples.

- (3) *Algebraic spaces.* Just as for schemes, the algebraic space X gives a presheaf of sets on (Aff), and hence a fibered category.
- (4) *Curves of genus g.* We get a fibered category associated to the presheaf sending S to the category of curves of genus g over S .
- (5) *Curves of genus g with line bundle.* The same as (4), only we use the presheaf sending S to the category of curves equipped with a line bundle over S .
- (6) *Curves of genus g with rank n vector bundle.* The same as (4), only we use the presheaf sending S to the category of curves equipped with a rank n vector bundle over S .

- (7) *Finite flat group schemes.* Fix $n \geq 1$. We get a fibered category associated to the presheaf sending S to the category of finite flat group schemes of order n over S .
- (8) *Quotient by a group scheme.* Fix a commutative ring R , and let X be a scheme (or an algebraic space) over R . Let G be a smooth, separated, and finitely presented group scheme over R , and let $G \times X \rightarrow X$ be an action of G on X over R . Then one can form the quotient stack $G \backslash X$.

We discuss VB_n in more detail. Though we did not explicitly describe the morphisms we allow between vector bundles in $VB_n(S)$, the most obvious possibility is to allow all vector bundle homomorphisms (in the usual sense). Indeed, this choice defines a stack on (Aff). We shall see, however, that the traditional definition of an algebraic stack \mathcal{D} requires that each category $\mathcal{D}(S)$ be a groupoid. Thus, to get an algebraic stack, we should only consider isomorphisms between vector bundles in $VB_n(S)$, and similarly for each of the fibered categories defined above.

To get an intuitive feel for VB_n , consider first the case of schemes. One way to get a rough picture of the scheme X is to consider all geometric points $X(k)$ for an arbitrary algebraically closed field k . For example, if $X = \mathbb{A}^n$, then $X(k) = k^n$. If $X = \mathbb{P}^n$, then $X(k) = \mathbb{P}^n(k)$.

We pursue the same approach for VB_n (though it will turn out that algebraic closure plays no role). Given a field k , $VB_n(k)$ is the category whose objects are n -dimensional k -vector spaces, and whose morphisms are the isomorphisms between these vector spaces. Since all n -dimensional vector spaces over k are isomorphic, $VB_n(k)$ is equivalent to a category with a single object, say k^n for definiteness, whose morphisms are all isomorphisms $k^n \xrightarrow{\sim} k^n$; that is, whose morphisms are $GL_n(k)$. This stack is an algebraic version of the classifying space of GL_n .

The example of VB_n motivates a first crude — and, indeed, incorrect! — notion of an algebraic stack. Roughly, we require that both the objects and their automorphism groups in the stack be “algebraic” in some sense. For morphisms, the automorphism groups should be algebraic groups or group schemes. Indeed, we saw GL_n appear in the case of VB_n . For objects, the situation is more subtle: we wish to work only up to equivalence of categories, so the objects aren’t even well-defined. We will return to this point later.

4.2. Curves. Our next examples of an algebraic stacks will involve curves of genus 0 and 1. We first need some general background on curves before specializing to the genus 0 and 1 cases.

Definition 4.2.1. Let S be a scheme, and fix an integer $g \geq 0$.

- (1) A *curve of genus g* over S is a scheme C and a morphism $\pi: C \rightarrow S$ such that
- (i) π is smooth of relative dimension 1;
 - (ii) π is proper; and
 - (iii) for every geometric point $\bar{s} \rightarrow S$, the geometric fiber $C_{\bar{s}}$ obtained from the pullback diagram

$$\begin{array}{ccc} C_{\bar{s}} & \longrightarrow & C \\ \downarrow & & \downarrow \\ \bar{s} & \longrightarrow & S \end{array}$$

is an irreducible curve of genus g over \bar{s} .

- (2) We define $\mathcal{M}_g(S)$ to be the category whose objects are curves of genus g over S and whose morphisms are the isomorphisms of these curves as schemes over S .

Let $\pi: C \rightarrow S$ be a curve of genus g over S . Condition (1ii) requires that C be separated over S . Condition (1iii) rules out empty fibers, so the map π must be surjective.

Since π is smooth and surjective, π admits a section locally in the étale topology; that is, there exists a scheme S' and an étale surjective map $S' \rightarrow S$ that factors as $S' \rightarrow C \xrightarrow{\pi} S$ (3.6.3). To explain the terminology, here we view S' as an open cover of S in the étale topology, and the map $S' \rightarrow C$ as a section.

We digress for the rest of the subsection to consider several examples of curves. We shall see that π need not admit a section Zariski locally. Thus the importance of using the étale topology on (Aff) already becomes apparent. In each case we take $g = 0$. We refer to a curve of genus 0 over S as a \mathbb{P}^1 -bundle over S .

Example 4.2.2. The simplest example is the trivial \mathbb{P}^1 -bundle

$$\mathbb{P}_S^1 = \mathbb{P}_{\mathbb{Z}}^1 \times_{\text{Spec}(\mathbb{Z})} S \rightarrow S.$$

Example 4.2.3. We next consider a nontrivial \mathbb{P}^1 -bundle that is locally trivial in the Zariski topology. We work over \mathbb{C} . Let $G = GL_3(\mathbb{C})$ and consider the subgroups $B \subset P \subset G$, where

$$B = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\} \quad \text{and} \quad P = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\}.$$

P is the stabilizer of a line in \mathbb{C}^3 , so G/P is identified with $\{\text{lines in } \mathbb{C}^3\} = \mathbb{P}_{\mathbb{C}}^2$. Similarly, G/B is identified with the set of all pairs (ℓ, p) where ℓ is a line in \mathbb{C}^3 and p is a plane in \mathbb{C}^3 containing ℓ . The natural map $\pi: G/B \rightarrow G/P$ sends $(\ell \subset p) \mapsto \ell$ and has fibers isomorphic to $\mathbb{P}_{\mathbb{C}}^1$. We leave it as an exercise to find sections for π Zariski locally and to trivialize π Zariski locally.

Example 4.2.4. We next take $S = \text{Spec}(\mathbb{R})$. The projective curve $C \subset \mathbb{P}_{\mathbb{R}}^2$ defined by $x^2 + y^2 + z^2 = 0$ becomes isomorphic to \mathbb{P}^1 over \mathbb{C} , but C is not isomorphic to \mathbb{P}^1 over \mathbb{R} . Indeed, $C \rightarrow \text{Spec}(\mathbb{R})$ admits no section, since the equation $x^2 + y^2 + z^2 = 0$ has no real solutions in projective space.

Example 4.2.5. For our final example, take $R = \mathbb{C}[T, T^{-1}, U, U^{-1}]$ and let $S = \text{Spec}(R)$. Then $S = \mathbb{G}_m^2$ over \mathbb{C} . We take C to be the closed subscheme of $\mathbb{P}^2 \times \mathbb{G}_m^2$ defined by the equation

$$x^2 - Ty^2 = Uz^2,$$

where $x, y,$ and z are homogenous coordinates on \mathbb{P}^2 .

The geometric fiber over the point $(a, b) \in \mathbb{G}_m^2$ is the curve in $\mathbb{P}_{\mathbb{C}}^2$ defined by $x^2 - ay^2 = bz^2$. If we define $f = x^2 - Ty^2 - Uz^2$, then we see that C is smooth because f and its partial derivatives $f_x, f_y,$ and f_z have no common zeroes on $\mathbb{P}^2 \times \mathbb{G}_m^2$. Since we don't allow T or U to take the value 0, none of the geometric fibers is a degenerate conic. Thus all fibers over \mathbb{C} are isomorphic to $\mathbb{P}_{\mathbb{C}}^1$. So indeed, $\pi: C \rightarrow \mathbb{G}_m^2$ is a \mathbb{P}^1 -bundle.

Since $x^2 - Ty^2 = Uz^2$ has no nontrivial solutions in the function field $\mathbb{C}(T, U)$, there is no section of π over the generic point of \mathbb{G}_m^2 . Thus there is no nonempty Zariski open subset of \mathbb{G}_m^2 over which π has a section.

Now consider the étale map $\beta: \mathbb{G}_m^2 \rightarrow \mathbb{G}_m^2$ sending $(T', U') \mapsto (T'^2, U')$. Putting $T = T'^2$ and $U = U'$, we have that C pulls back along β to the curve C' in $\mathbb{P}^2 \times \mathbb{G}_m^2$ defined by $x^2 - T'^2 y^2 = U' z^2$. That is, C' is defined by $(x + Ty)(x - Ty) = U' z^2$, from which it is clear that C' admits a section $\mathbb{G}_m^2 \rightarrow C'$. So indeed, C admits a section étale locally, as claimed.

4.3. Curves with Section. Let $\pi: C \rightarrow S$ be a curve of genus g , and now assume that π admits a section $c: S \rightarrow C$. Given a geometric point $\bar{s} \rightarrow S$, we get a geometric point $c(\bar{s}) \in C_{\bar{s}}(\bar{s})$:

$$\begin{array}{ccccc} S & \xrightarrow{c} & C & \xrightarrow{\pi} & S \\ \uparrow & & \uparrow & & \uparrow \\ \bar{s} & \xrightarrow{c(\bar{s})} & C_{\bar{s}} & \longrightarrow & \bar{s}. \end{array}$$

Thinking of the geometric point as a divisor of degree 1, we get a line bundle $\mathcal{O}_{C_{\bar{s}}}(c(\bar{s}))$ of degree 1 on $C_{\bar{s}}$. We claim that all line bundles so obtained on the geometric fibers arise as the pullbacks of a single line bundle on C . Thus we obtain a certain canonical line bundle on C .

As a warm-up, consider the case of C a complete nonsingular curve over $\text{Spec}(k)$, with k an algebraically closed field. Given a point $c \in C$, we get the line bundle $\mathcal{O}_C(c)$ of degree 1 obtained from the divisor c of degree 1. Now let c vary in C . Intuitively, we expect that there exists a line bundle \mathcal{L} on $C \times C$ such that the pullback of \mathcal{L} to $C \times \{c\}$ is isomorphic to $\mathcal{O}_C(c)$.

To find \mathcal{L} , it suffices to find a divisor D on the surface $C \times C$ such that

$$D \cap (C \times \{c\}) = \{(c, c)\}.$$

The obvious choice is to take $D = \Delta(C)$, where $\Delta: C \rightarrow C \times C$ is the diagonal morphism. Indeed, since C is proper over $\text{Spec}(k)$, Δ is a closed immersion.

In working over an arbitrary base scheme S , we first quote the following result.

Lemma. *Let $Y \rightarrow S$ and $X \rightarrow S$ be schemes over S , smooth of relative dimensions n and m , respectively. Let $i: Y \hookrightarrow X$ be a closed immersion over S , and consider the exact sequence of sheaves on X*

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow i_* \mathcal{O}_Y \rightarrow 0,$$

where \mathcal{J} is the quasi-coherent sheaf of ideals defining Y . Then for all $y \in Y$, there exists a Zariski open neighborhood U of y in X such that $\mathcal{J}|_U$ is generated by a regular sequence of length $m - n$.

Proof. EGA IV, (17.12.1). □

In the case $m - n = 1$, the lemma says that \mathcal{J} is generated Zariski locally on Y by 1 nonzerodivisor. That is, \mathcal{J} is locally free of rank 1 as an \mathcal{O}_X -module on some open set containing Y . Since \mathcal{J} equals \mathcal{O}_X on $X - Y$, \mathcal{J} is certainly free of rank 1 on $X - Y$. Hence \mathcal{J} is a line bundle.

To obtain the analog of \mathcal{L} in the general case, we take $Y := C$ and $X := C \times_S C$. Then C and $C \times_S C$ are smooth of relative dimensions 1 and 2, respectively, over S , and the diagonal morphism $\Delta: C \rightarrow C \times_S C$ is a closed immersion. So the lemma applies, and

$$\mathcal{J} := \ker[\mathcal{O}_{C \times_S C} \rightarrow \Delta_* \mathcal{O}_C]$$

is a line bundle on $C \times_S C$. We take \mathcal{L} to be the dual bundle

$$\mathcal{L} := \mathcal{J}^* = \text{Hom}_{\mathcal{O}_{C \times_S C}}(\mathcal{J}, \mathcal{O}_{C \times_S C}).$$

To obtain the analog of $\mathcal{O}_C(c)$, we apply the lemma to $Y := S$ and $X := C$. Indeed, S and C are smooth of relative dimensions 0 and 1, respectively, over S . Further, it is easy to check that the diagram

$$\begin{array}{ccc} S & \xrightarrow{c} & C \\ c \downarrow & & \downarrow (\text{id}_C, c\pi) \\ C & \xrightarrow{\Delta} & C \times_S C \end{array}$$

is a cartesian square. (Here $C \xrightarrow{(\text{id}_C, c\pi)} C \times_S C$ is the analog of $C \times \{c\} \rightarrow C \times C$ in the case of C a complete nonsingular curve over $\text{Spec}(k)$.) Since Δ is a closed immersion, so therefore is c . Thus

$$\mathcal{J}_c := \ker[\mathcal{O}_C \rightarrow c_*\mathcal{O}_S]$$

is a line bundle on C . We define

$$\mathcal{O}_C(c) := \mathcal{J}_c^*.$$

Alternatively, $\mathcal{O}_C(c)$ is just the pullback of \mathcal{L} along $(\text{id}_C, c\pi)$.

Now let $\bar{s} \rightarrow S$ be a geometric point, and consider the usual pullback diagram

$$\begin{array}{ccc} C_{\bar{s}} & \longrightarrow & C \\ \downarrow & & \downarrow \\ \bar{s} & \longrightarrow & S. \end{array}$$

One checks easily that the pullback of $\mathcal{O}_C(c)$ to $C_{\bar{s}}$ agrees with $\mathcal{O}_{C_{\bar{s}}}(c(\bar{s}))$, proving the claim made at the beginning of the subsection.

4.4. Cohomology and Base Change. Suppose we have a curve $\pi: C \rightarrow S$ and a line bundle \mathcal{L} on C . In later subsections, we’ll need to make use of the derived functors $R^i\pi_*$ applied to \mathcal{L} . In this subsection, we cover the needed background on cohomology and base change.

A good reference is Mumford’s *Abelian Varieties*, chapter II, §5. Mumford restricts to the locally Noetherian case. To eliminate Noetherian hypotheses, we appeal to the following results from EGA. Suppose that X is a scheme over $\text{Spec}(A)$. First note that

$$A = \varinjlim A_0,$$

where the colimit is taken over all subrings $A_0 \subset A$ that are finitely generated as \mathbb{Z} -algebras. Of course, each such A_0 is Noetherian.

Lemma 4.4.1.

- (1) [EGA, (8.9.1)] *The following are equivalent.*
 - (a) *X is finitely presented over $\text{Spec}(A)$ (that is, X is locally finitely presented, quasi-compact, and quasi-separated).*

- (b) *There exist a finitely generated \mathbb{Z} -algebra $A_0 \subset A$ and a scheme X_0 of finite type over $\mathrm{Spec}(A_0)$ such that $X \cong X_0 \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(A)$:*

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A_0). \end{array}$$

Moreover, in the situation of (a) and (b), if \mathcal{F} is a locally finitely presented quasi-coherent \mathcal{O}_X -module, then there exist A_0 and X_0 as in (b) and a coherent \mathcal{O}_{X_0} -module \mathcal{F}_0 such that \mathcal{F} is the pullback of \mathcal{F}_0 to X .

- (2) [EGA IV, (8.10.5)] *Moreover, in the situation of (1a) and (1b), if X is proper over $\mathrm{Spec}(A)$, then A_0 and X_0 can in addition be chosen such that X_0 is proper over $\mathrm{Spec}(A_0)$.*
- (3) [EGA IV, (11.2.6)] *Moreover, in the situation of (1a) and (1b), if \mathcal{F} is A -flat, then \mathcal{F}_0 can in addition be chosen to be A_0 -flat. \square*

The following proposition is the main result on cohomology and base change. It is clear from the lemma that the locally Noetherian case of the proposition implies the proposition in general, so we just state the general form.

Proposition 4.4.2. *Let $S = \mathrm{Spec}(A)$ be any affine scheme, and let $\pi: X \rightarrow S$ be a scheme finitely presented and proper over S . Suppose \mathcal{F} is a quasi-coherent \mathcal{O}_X -module that is locally finitely presented on X and flat over S (for example, any vector bundle on X if π is flat). Then, as a complex, $R\pi_*\mathcal{F}$ is quasi-isomorphic to a finite complex*

$$K^0 \rightarrow K^1 \rightarrow \dots \rightarrow K^n$$

of finitely generated projective A -modules.

Moreover, the complex K^\bullet is compatible with base change in the following sense. For any scheme S' and cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \pi' \downarrow & & \downarrow \pi \\ S' & \longrightarrow & S, \end{array}$$

let \mathcal{F}' denote the pullback of \mathcal{F} to X' . Then $R\pi'_\mathcal{F}'$ is quasi-isomorphic to the pullback of K^\bullet to S' . \square*

The proposition furnishes an immediate corollary. Given a point $s \in S$, we write X_s for the fiber $X \times_S \mathrm{Spec}(\kappa(s))$ and $\mathcal{F}|_{X_s}$ for the pullback of \mathcal{F} to X_s . We define the function χ on the underlying topological space of S by letting $\chi(s)$ equal the Euler-Poincaré characteristic of $H^\bullet(X_s, \mathcal{F}|_{X_s})$, that is, the alternating sum of the $\kappa(s)$ -dimensions of the cohomology groups.

Corollary 4.4.3. *Let $X \rightarrow S$ and the \mathcal{O}_X -module \mathcal{F} be such that the conclusion of (4.4.2) holds. Then, in the notation of (4.4.2), χ is equal at each $s \in S$ to the alternating sum of the ranks of the stalks K_s^0, \dots, K_s^n . In particular, χ is locally constant on S . \square*

Example 4.4.4. Let $S = \mathrm{Spec}(A)$ and take $X = \mathbb{P}_S^1$. Then $\mathbb{P}_S^1 \rightarrow S$ satisfies the hypotheses of (4.4.2). When $A = \mathbb{Z}$, $\mathbb{P}_{\mathbb{Z}}^1$ is covered by open affines U and V , each isomorphic to $\mathbb{A}_{\mathbb{Z}}^1$, such that the intersection $U \cap V$ is isomorphic to \mathbb{G}_m (intuitively,

we think of $U = \mathbb{P}^1 - \{0\}$ and $V = \mathbb{P}^1 - \{\infty\}$. Changing base, we get the same covering of \mathbb{P}_S^1 for an arbitrary ring A . So, all the data of \mathbb{P}_S^1 is contained in the data of the ring maps

$$A[T] \hookrightarrow A[T, T^{-1}] \quad \text{and} \quad A[T^{-1}] \hookrightarrow A[T, T^{-1}].$$

To give a quasi-coherent $\mathcal{O}_{\mathbb{P}_S^1}$ -module \mathcal{F} is to give an $A[T]$ -module M , an $A[T^{-1}]$ -module N , an $A[T, T^{-1}]$ -module P , and maps

$$\phi: M \rightarrow P \quad \text{and} \quad \psi: N \rightarrow P$$

such that

$$M \otimes_{A[T]} A[T, T^{-1}] \xrightarrow{\sim} P \quad \text{and} \quad N \otimes_{A[T^{-1}]} A[T, T^{-1}] \xrightarrow{\sim} P.$$

\mathcal{F} is flat if and only if M , N , and P are flat over $A[T]$, $A[T^{-1}]$, and $A[T, T^{-1}]$, respectively.

Define maps $\alpha, \beta: M \times N \rightarrow P$ by

$$\alpha: (m, n) \mapsto \phi(m) \quad \text{and} \quad \beta: (m, n) \mapsto \psi(n).$$

In terms of M , N , and P , the cohomology $H^\bullet(\mathbb{P}_S^1, \mathcal{F})$ is given by

$$\begin{aligned} H^0(\mathbb{P}_S^1, \mathcal{F}) &= \ker(\alpha - \beta) \\ H^1(\mathbb{P}_S^1, \mathcal{F}) &= \text{cok}(\alpha - \beta) \\ H^i(\mathbb{P}_S^1, \mathcal{F}) &= 0 \text{ for } i \geq 2. \end{aligned}$$

We now restrict to the case that A is Noetherian. It is then a standard result that H^0 and H^1 are finitely generated A -modules. By choosing representatives in P of finitely many generators in H^1 , we get a map $A^n \rightarrow P$ for some n . We get a pullback diagram of A -modules

$$\begin{array}{ccc} (M \times N) \times_P A^n & \longrightarrow & A^n \\ \downarrow & & \downarrow \\ M \times N & \xrightarrow{\alpha - \beta} & P, \end{array}$$

in which we can view the horizontal rows as complexes. The vertical arrows then give a quasi-isomorphism of complexes. We leave it as an exercise to show that $(M \times N) \times_P A^n$ is flat and finitely generated, so that it is a vector bundle (Bourbaki, *Commutative Algebra*). Thus the top row is a finite complex of vector bundles representing $H^\bullet(\mathbb{P}_S^1, \mathcal{F})$ — that is, a complex of the form predicted by (4.4.2).

4.5. Line Bundles and Projective Space Bundles. Given a line bundle \mathcal{L} on a scheme X , one often uses \mathcal{L} to obtain morphisms to projective spaces.

We first review the case of a scheme over $\text{Spec}(k)$, say

$$\pi: X \rightarrow \text{Spec}(k),$$

with k an algebraically closed field. Suppose \mathcal{L} is a line bundle on X generated by global sections $s_0, \dots, s_n \in \mathcal{L}(X)$; that is, the corresponding morphism of sheaves $\mathcal{O}_X^{n+1} \rightarrow \mathcal{L}$ is an epimorphism. Then we get a morphism $X \rightarrow \mathbb{P}_k^n$ that, on k -points, is the map $x \mapsto (s_0(x), \dots, s_n(x))$. Conversely, every morphism $X \rightarrow \mathbb{P}_k^n$ is determined up to isomorphism by a line bundle \mathcal{L} and an epimorphism $\mathcal{O}_X^{n+1} \rightarrow \mathcal{L}$.

More canonically, given an $n + 1$ -dimensional k -vector space V and an epimorphism of sheaves

$$\pi^*V = V \otimes_k \mathcal{O}_X \rightarrow \mathcal{L},$$

we get a morphism

$$X \rightarrow \mathbb{P}(V),$$

where $\mathbb{P}(V)$ is the space of hyperplanes in V (or, equivalently, of 1-dimensional quotients of V).

We now replace $\text{Spec}(k)$ with an arbitrary base scheme S . Suppose we have a rank $n + 1$ vector bundle \mathcal{E} over S . Then we get the projective space bundle $\mathbb{P}(\mathcal{E})$ over S . Intuitively, one can think of $\mathbb{P}(\mathcal{E})$ as the scheme of 1-dimensional quotient vector bundles of \mathcal{E} , or as a \mathbb{P}^n -bundle over S .

$\mathbb{P}(\mathcal{E})$ is characterized as follows. If $\pi: X \rightarrow S$ is an arbitrary S -scheme, then to give an S -morphism $X \rightarrow \mathbb{P}(\mathcal{E})$ is to give (up to an obvious notion of isomorphism) a line bundle \mathcal{L} on X and an epimorphism of sheaves $\pi^*\mathcal{E} \rightarrow \mathcal{L}$.

For later use, we note that if $\bar{s} \rightarrow S$ is a geometric point, then

$$\mathbb{P}(\mathcal{E})_{\bar{s}} = \mathbb{P}(\mathcal{E}_{\bar{s}}) \cong \mathbb{P}_{\bar{s}}^n,$$

where $\mathcal{E}_{\bar{s}} = \mathcal{E} \otimes \kappa(\bar{s})$ is the pullback of \mathcal{E} to \bar{s} .

In the remainder of the subsection, we apply the theory of this subsection and §4.4 to the case of $S = \text{Spec}(A)$ an affine scheme, $\pi: C \rightarrow S$ a curve of genus g , and \mathcal{L} a line bundle on C . We wish to find a condition on \mathcal{L} so that \mathcal{L} gives a map from C to a certain projective space.

By (4.4.2), $R\pi_*\mathcal{L}$ is represented by a finite complex of vector bundles $K^0 \rightarrow \dots \rightarrow K^n$. We have the following lemma, whose proof we omit.

Lemma 4.5.1. *Let*

$$(*) \quad K^0 \rightarrow \dots \rightarrow K^n$$

be a complex of vector bundles on a scheme S . Suppose that for every geometric point $\bar{s} \rightarrow S$, the i th cohomology of the pullback complex

$$K_{\bar{s}}^0 \rightarrow \dots \rightarrow K_{\bar{s}}^n$$

vanishes for all $i > 0$. Then

- (1) *the 0th cohomology $\ker[K^0 \rightarrow K^1]$ of (*) is a vector bundle on S , whose rank at the point $s \in S$ equals*

$$\dim_{\kappa(\bar{s})} H^0(K_{\bar{s}}^\bullet) = \dim_{\kappa(\bar{s})} \ker[K_{\bar{s}}^0 \rightarrow K_{\bar{s}}^1],$$

where $\bar{s} \rightarrow S$ is any geometric point with image s ; and

- (2) *all higher cohomology of (*) vanishes. □*

To put ourselves in the situation of the lemma, assume that for every geometric point $\bar{s} \rightarrow S$, the cohomology $H^i(C_{\bar{s}}, \mathcal{L}|_{C_{\bar{s}}}) = 0$ for all $i > 0$. Then

$$\mathcal{E} := \pi_*\mathcal{L}$$

is a vector bundle on S . Pulling back \mathcal{E} , we get an adjunction map

$$\alpha: \pi^*\mathcal{E} \rightarrow \mathcal{L}.$$

By the discussion at the beginning of this subsection, α corresponds to a morphism of schemes $C \rightarrow \mathbb{P}(\mathcal{E})$ over S exactly when α is an epimorphism of sheaves. Using Nakayama's lemma, we see α is an epimorphism if and only if every geometric point $\bar{c} \rightarrow C$ gives a surjection

$$(**) \quad (\pi^*\mathcal{E})_{\bar{c}} \twoheadrightarrow \mathcal{L}_{\bar{c}}.$$

So let $\bar{c} \rightarrow C$ be a geometric point. Composing with $C \xrightarrow{\pi} S$, we can regard \bar{c} as a geometric point of S . To avoid confusing notation, we write $\bar{s} \rightarrow S$ for the geometric point \bar{c} :

$$\begin{array}{ccc} \bar{c} & \longrightarrow & C \\ \parallel & & \downarrow \pi \\ \bar{s} & \longrightarrow & S. \end{array}$$

From the diagram, we get a natural map $\iota: \bar{c} \rightarrow C_{\bar{s}}$, say with image $c \in C_{\bar{s}}$.

Let $\mathcal{L}|_{C_{\bar{s}}}(-c)$ denote the kernel of the adjunction map $\mathcal{L} \rightarrow \iota_*\iota^*(\mathcal{L}|_{C_{\bar{s}}}) = \iota_*\mathcal{L}_{\bar{c}}$. It is easy to check $\mathcal{L}|_{C_{\bar{s}}}(-c) \cong \mathcal{L}|_{C_{\bar{s}}} \otimes \mathcal{O}_{C_{\bar{s}}}(-c)$, where $\mathcal{O}_{C_{\bar{s}}}(-c)$ is the line bundle of degree -1 obtained from the divisor $-c$. So we get an exact sequence

$$0 \rightarrow \mathcal{L}|_{C_{\bar{s}}} \otimes \mathcal{O}_{C_{\bar{s}}}(-c) \rightarrow \mathcal{L}|_{C_{\bar{s}}} \rightarrow \iota_*\mathcal{L}_{\bar{c}} \rightarrow 0$$

of sheaves on $C_{\bar{s}}$, and hence a long exact sequence on cohomology

$$\cdots \rightarrow H^0(C_{\bar{s}}, \mathcal{L}|_{C_{\bar{s}}}) \rightarrow H^0(C_{\bar{s}}, \iota_*\mathcal{L}_{\bar{c}}) \rightarrow H^1(C_{\bar{s}}, \mathcal{L}|_{C_{\bar{s}}} \otimes \mathcal{O}_{C_{\bar{s}}}(-c)) \rightarrow \cdots.$$

But now

$$H^0(C_{\bar{s}}, \mathcal{L}|_{C_{\bar{s}}}) = (\pi_*\mathcal{L})_{\bar{s}} = (\pi^*\mathcal{E})_{\bar{c}}$$

and

$$H^0(C_{\bar{s}}, \iota_*\mathcal{L}_{\bar{c}}) = \mathcal{L}_{\bar{c}},$$

so we get the desired surjection (***) if

$$H^1(C_{\bar{s}}, \mathcal{L}|_{C_{\bar{s}}} \otimes \mathcal{O}_{C_{\bar{s}}}(-c)) = 0.$$

Let ω denote the canonical bundle on $C_{\bar{s}}$. Then by Serre duality,

$$H^1(C_{\bar{s}}, \mathcal{L}|_{C_{\bar{s}}} \otimes \mathcal{O}_{C_{\bar{s}}}(-c)) \cong H^0(C_{\bar{s}}, \omega \otimes \mathcal{L}|_{C_{\bar{s}}}^* \otimes \mathcal{O}_{C_{\bar{s}}}(c)).$$

Indeed, the latter will vanish if

$$(***) \quad \deg \mathcal{L}|_{C_{\bar{s}}} \geq 2g.$$

We take the desired condition on \mathcal{L} to be that (***) holds for all geometric points $\bar{s} \rightarrow S$.

Moreover, let $\bar{s} \rightarrow S$ be a geometric point, and suppose we have strict inequality

$$\deg \mathcal{L}|_{C_{\bar{s}}} > 2g.$$

Write $\pi_{\bar{s}}$ for the map $C_{\bar{s}} \rightarrow \bar{s}$. Then the map $C_{\bar{s}} \rightarrow \mathbb{P}(\mathcal{E})_{\bar{s}} = \mathbb{P}(\mathcal{E}_{\bar{s}})$ obtained from base change is the map obtained from the epimorphism of sheaves

$$\pi_{\bar{s}}^*(\mathcal{E}_{\bar{s}}) = (\pi^*\mathcal{E})|_{C_{\bar{s}}} \rightarrow \mathcal{L}|_{C_{\bar{s}}}.$$

So, by our assumption on $\deg \mathcal{L}|_{C_{\bar{s}}}$, the map $C_{\bar{s}} \rightarrow \mathbb{P}(\mathcal{E})_{\bar{s}}$ is a closed immersion.

We record our results in the following proposition.

Proposition 4.5.2. *Let $\pi: C \rightarrow S$ be a curve of genus g over the affine scheme S , and let \mathcal{L} be a line bundle on C . Assume that for every geometric point $\bar{s} \rightarrow S$, we have $H^i(C_{\bar{s}}, \mathcal{L}|_{C_{\bar{s}}}) = 0$ for all $i > 0$. Then*

- (1) *the sheaf $\mathcal{E} := \pi_*\mathcal{L}$ is a vector bundle on S ;*
- (2) *if $\deg \mathcal{L}|_{C_{\bar{s}}} \geq 2g$ for every geometric point \bar{s} , then the adjunction map $\pi^*\mathcal{E} \rightarrow \mathcal{L}$ corresponds to a morphism $C \rightarrow \mathbb{P}(\mathcal{E})$ over S ; and*
- (3) *moreover, if $\deg \mathcal{L}|_{C_{\bar{s}}} > 2g$, then the morphism $C_{\bar{s}} \rightarrow \mathbb{P}(\mathcal{E})_{\bar{s}}$ is a closed immersion. \square*

4.6. Curves of Genus 0. In this subsection, we apply the generalities on curves developed in §§4.2–4.5 to the case of $\pi: C \rightarrow S$ a curve of genus 0, with $S = \text{Spec}(A)$ affine. As in §4.2, we refer to a curve of genus 0 as a \mathbb{P}^1 -bundle.

To begin, suppose that π admits a section $c: S \rightarrow C$. We saw in §4.3 that c must be a closed immersion, and we defined the line bundle $\mathcal{O}_C(c)$ of degree 1 to be the dual of the sheaf of ideals defining the closed subscheme $c(S)$. The line bundle $\mathcal{O}_C(c)$ has the property that, for any geometric fiber $C_{\bar{s}}$, the pullback $\mathcal{O}_C(c)|_{C_{\bar{s}}}$ is the line bundle of degree 1 obtained from a certain point, regarded as a divisor, in $C_{\bar{s}}$.

We apply the results of §4.4 and §4.5 to the line bundle $\mathcal{O}_C(c)$. Consider first the case that A is a field k . Since $\deg \mathcal{O}_C(c) = 1$, we get from the Riemann-Roch Theorem that

$$\dim_k H^i(\mathbb{P}_k^1, \mathcal{O}_C(c)) = \begin{cases} 2 & i = 0 \\ 0 & i > 0. \end{cases}$$

Returning to the general case of arbitrary A , it follows from (4.5.2) that

$$\mathcal{E} := \pi_* \mathcal{O}_C(c)$$

is a vector bundle of rank 2 on S . Since for every geometric point $\bar{s} \rightarrow S$ we have

$$(*) \quad \deg \mathcal{O}_C(c)|_{C_{\bar{s}}} = 1 > 0 = 2g,$$

we get a map from C to the \mathbb{P}^1 -bundle $\mathbb{P}(\mathcal{E})$. We claim this map is an isomorphism.

Consider again the case that A is a field k . Then by (*), $C \rightarrow \mathbb{P}(\mathcal{E}) \cong \mathbb{P}_k^1$ is a closed immersion. It is a standard result that any closed immersion of curves over k is an isomorphism, so we have $C \xrightarrow{\sim} \mathbb{P}(\mathcal{E})$. For the general case, we cite the following result.

Lemma 4.6.1. [*EGA IV, (17.9.5)*] *Let X and Y be schemes locally of finite presentation over a scheme S , and assume further that X is flat over S . Then an S -morphism*

$$f: X \rightarrow Y$$

is an isomorphism (respectively, open immersion) if and only if for every point $s \in S$, the map on fibers

$$f_s: X_s \rightarrow Y_s$$

obtained from the base change $\text{Spec}(\kappa(s)) \rightarrow S$ is an isomorphism (respectively, open immersion). \square

Thus the lemma and the case $A = k$ imply that the map $C \rightarrow \mathbb{P}(\mathcal{E})$ is an isomorphism in general.

To conclude the subsection, let $\pi: C \rightarrow S$ be an arbitrary curve of genus 0 over an affine scheme S . We have already seen that π admits sections étale locally (3.6.3). So, étale locally, C is isomorphic to the \mathbb{P}^1 -bundle associated to a rank 2 vector bundle on S . Zariski localizing further, so that we trivialize this vector bundle, we get that C is isomorphic to the trivial \mathbb{P}^1 -bundle $\mathbb{P}_S^1 \rightarrow S$. We have proved the following proposition.

Proposition 4.6.2. *Any curve $C \rightarrow S$ of genus 0 over the affine scheme S is isomorphic étale locally to the trivial \mathbb{P}^1 -bundle $\mathbb{P}_S^1 \rightarrow S$.* \square

It is immediate that the proposition in fact holds for any scheme S , affine or otherwise.

The proposition justifies our terminology “ \mathbb{P}^1 -bundle” for a curve of genus 0, provided we use the étale topology. We have seen that local triviality need not hold in the Zariski topology (4.2.4).

4.7. Curves of Genus 1 and Line Bundles. Our next example of an algebraic stack will be \mathcal{M}_1 , the moduli stack of curves of genus 1. We begin with some basic facts about line bundles \mathcal{L} on a curve E of genus 1 over a field k .

(1) (Riemann-Roch Theorem) We have

$$\dim_k H^0(E, \mathcal{L}) - \dim_k H^1(E, \mathcal{L}) = \deg \mathcal{L}.$$

(2) The canonical line bundle ω on E has degree 0.

(3) If $\deg \mathcal{L} > 0$, then $H^1(E, \mathcal{L}) = 0$.

Our first task is to define the degree of a line bundle \mathcal{L} on a curve E of genus 1 over an arbitrary base scheme S . In §4.4 we defined the function $\chi: S \rightarrow \mathbb{Z}$ sending each point $s \in S$ to the Euler-Poincaré characteristic of $H^\bullet(E_s, \mathcal{L}|_{E_s})$. By (4.4.3), χ is locally constant.

Definition 4.7.1. We say the line bundle \mathcal{L} on E has *degree* n if the associated function $\chi: S \rightarrow \mathbb{Z}$ is the constant function $s \mapsto n$.

To study \mathcal{M}_1 , we will make use of the following related fibered category.

Definition 4.7.2. We define the *moduli stack \mathcal{M} of curves of genus 1 with degree 3 line bundle* to be the fibered category obtained from the following presheaf of categories on (Aff). Let S be an affine scheme.

- An object in the category $\mathcal{M}(S)$ is a pair (π, \mathcal{L}) consisting of a curve $\pi: E \rightarrow S$ of genus 1 and a line bundle \mathcal{L} of degree 3 on E . By abuse of notation, we typically write (E, \mathcal{L}) for the pair (π, \mathcal{L}) .
- A morphism $(E', \mathcal{L}') \rightarrow (E, \mathcal{L})$ in $\mathcal{M}(S)$ consists of an isomorphism of schemes $f: E' \xrightarrow{\sim} E$ over S and an isomorphism of $\mathcal{O}_{E'}$ -modules $\mathcal{L}' \xrightarrow{\sim} f^*\mathcal{L}$.
- Given a morphism $f: S' \rightarrow S$ of affine schemes, the pullback functor $\mathcal{M}(S) \rightarrow \mathcal{M}(S')$ is given by $(E, \mathcal{L}) \mapsto (E \times_S S', f^*\mathcal{L})$.

Let $\pi: E \rightarrow S$ be a curve of genus 1 over the affine scheme S , and let \mathcal{L} be a line bundle of degree 3 on E . We wish to apply the theory of §4.4 and §4.5 to \mathcal{L} . Since \mathcal{L} has degree 3, the pullback $\mathcal{L}|_{E_s}$ is a degree 3 line bundle for each $s \in S$. Therefore

$$\dim_{\kappa(s)} H^i(E_s, \mathcal{L}|_{E_s}) = \begin{cases} 3 & i = 0 \\ 0 & i > 0 \end{cases}$$

for each $s \in S$. Moreover, for any geometric point $\bar{s} \rightarrow S$, we have

$$\deg \mathcal{L}|_{E_{\bar{s}}} = 3 > 2 = 2g.$$

Hence (4.5.2) applies, so that

$$\mathcal{E} := \pi_*\mathcal{L}$$

is a vector bundle of rank 3 on S ; we get a morphism

$$E \rightarrow \mathbb{P}(\mathcal{E})$$

over S ; and the map on fibers $E_s \rightarrow \mathbb{P}(\mathcal{E})_s \cong \mathbb{P}_{\kappa(s)}^2$ is a closed immersion for each point $s \in S$. Since \mathcal{L} has degree 3, E_s embeds as a cubic in $\mathbb{P}_{\kappa(s)}^2$.

We claim that the map $E \rightarrow \mathbb{P}(\mathcal{E})$ is a closed immersion. The proof is an application of (4.7.5) below. We obtain (4.7.5), in turn, from the next two results from EGA. Recall that a morphism of schemes $f: X \rightarrow Y$ is *finite* if for every open affine $U = \text{Spec}(A) \subset Y$, the inverse image $f^{-1}(U)$ is affine, say equal to $\text{Spec}(B)$, with B a finite A -module. The morphism f is *quasi-finite* if for every point $y \in Y$, the underlying set of the fiber X_y is finite.

Proposition 4.7.3. [EGA IV, (8.11.1)] *Let $f: X \rightarrow Y$ be a proper, locally finitely presented, and quasi-finite morphism of schemes. Then f is finite.* \square

Using Nakayama's lemma, one can deduce the following.

Proposition 4.7.4. [EGA IV, (8.11.5)] *Let $f: X \rightarrow Y$ be a finitely presented morphism of schemes. Then the following are equivalent.*

- (1) *The morphism f is a closed immersion.*
- (2) *The morphism f is proper, and for every point $y \in Y$, the map $X_y \rightarrow \text{Spec}(\kappa(y))$ is a closed immersion.*
- (3) *The morphism f is proper, and for every point $y \in Y$, we have $X_y = \emptyset$ or $X_y \xrightarrow{\sim} \text{Spec}(\kappa(y))$.* \square

Corollary 4.7.5. *Let*

$$X \xrightarrow{f} Y \xrightarrow{\rho} S$$

be a diagram of schemes, and let π denote the composite $\rho \circ f$. Suppose that π is proper and finitely presented, and that ρ is separated and locally of finite type. Assume that for all points $s \in S$, the map on fibers $f_s: X_s \rightarrow Y_s$ is a closed immersion. Then f is a closed immersion.

Proof. Since π is proper and ρ is separated, f is proper [EGA II, (5.4.3)]. Since π is (locally) finitely presented and ρ is locally of finite type, f is locally finitely presented [EGA IV, (1.4.3)]. Since a proper morphism is quasi-compact, we conclude that f is proper and finitely presented.

Let $y \in Y$ be any point. By (4.7.4), it suffices to show $X_y \rightarrow \text{Spec}(\kappa(y))$ is a closed immersion. Say y has image $s \in S$. We get the following diagram.

$$\begin{array}{ccccc} X & \longleftarrow & X_s & \longleftarrow & X_y \\ f \downarrow & & f_s \downarrow & & \downarrow \\ Y & \longleftarrow & Y_s & \longleftarrow & \text{Spec}(\kappa(y)) \\ \rho \downarrow & & \downarrow & & \\ S & \longleftarrow & \text{Spec}(\kappa(s)) & & \end{array}$$

Every square in the diagram is cartesian. So, since the vertical arrow $f_s: X_s \rightarrow Y_s$ is a closed immersion by hypothesis, we conclude $X_y \rightarrow \text{Spec}(\kappa(y))$ is a closed immersion as well. \square

Corollary 4.7.6. *In the notation from earlier in the subsection, the map $E \rightarrow \mathbb{P}(\mathcal{E})$ is a closed immersion.* \square

Since $E \rightarrow \mathbb{P}(\mathcal{E})$ is a closed immersion, we get an exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})} \rightarrow \mathcal{O}_E \rightarrow 0$$

of sheaves on $\mathbb{P}(\mathcal{E})$, where \mathcal{J} is the sheaf of ideals defining E . Since E is smooth of relative dimension 1 over S , and $\mathbb{P}(\mathcal{E})$ is smooth of relative dimension 2 over S , we get from (4.3) that \mathcal{J} is a line bundle.

Now consider the dual line bundle \mathcal{J}^* . For each $s \in S$, $\mathcal{J}^*|_{\mathbb{P}(\mathcal{E})_s}$ is the line bundle associated to the divisor E_s in $\mathbb{P}(\mathcal{E})_s \cong \mathbb{P}_{\kappa(s)}^2$. Since E_s embeds as a cubic, $\mathcal{J}^*|_{\mathbb{P}(\mathcal{E})_s}$ has degree 3. Hence

$$\mathcal{J}^*|_{\mathbb{P}(\mathcal{E})_s} \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})_s}(3).$$

On $\mathbb{P}(\mathcal{E})$, we have the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and one might then hope that \mathcal{J}^* is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(3)$. Unfortunately, \mathcal{J}^* and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(3)$ need not be isomorphic when working over a general base scheme S . We claim, however, that the line bundle

$$\mathcal{N} := \mathcal{J} \otimes_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(3)$$

on $\mathbb{P}(\mathcal{E})$ arises as the pullback of a line bundle on S . More precisely, let ρ denote the structure map $\rho: \mathbb{P}(\mathcal{E}) \rightarrow S$. We claim that $\rho_*\mathcal{N}$ is a line bundle on S , and that the adjunction map

$$\rho^*\rho_*\mathcal{N} \rightarrow \mathcal{N}$$

is an isomorphism.

The proof of the claim is similar to other arguments we’ve seen with cohomology and base change, especially the discussion leading up to (4.5.2), using the fact that $\mathcal{N}|_{\mathbb{P}(\mathcal{E})_s} \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})_s}$. We omit the details.

4.8. Curves of Genus 1 and Trivial Bundles. Let $\pi: E \rightarrow S$ be a curve of genus 1 over the affine scheme $S = \text{Spec}(A)$, and let \mathcal{L} be a line bundle of degree 3 on E . We summarize the results and notation of the previous subsection:

- $\mathcal{E} := \pi_*\mathcal{L}$ is a vector bundle of rank 3 on S with corresponding projective space bundle $\rho: \mathbb{P}(\mathcal{E}) \rightarrow S$;
- the adjunction map $\pi^*\mathcal{E} \rightarrow \mathcal{L}$ gives a closed immersion $E \rightarrow \mathbb{P}(\mathcal{E})$ over S ;
- $\mathcal{J} := \ker[\mathcal{O}_{\mathbb{P}(\mathcal{E})} \rightarrow \mathcal{O}_E]$ is a line bundle on $\mathbb{P}(\mathcal{E})$;
- $\mathcal{N} := \mathcal{J} \otimes_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(3)$ is a line bundle on $\mathbb{P}(\mathcal{E})$;
- $\rho_*\mathcal{N}$ is a line bundle on S , and the adjunction map $\rho^*\rho_*\mathcal{N} \rightarrow \mathcal{N}$ is an isomorphism.

Zariski locally on S , both \mathcal{E} and $\rho_*\mathcal{N}$ are trivial bundles. If both are actually trivial, then we can add a choice of trivializations to the datum (E, \mathcal{L}) .

Definition 4.8.1. Let S be an affine scheme. We define $\widehat{\mathcal{M}}(S)$ to be the following category. An object in $\widehat{\mathcal{M}}(S)$ is a quadruple $(\pi, \mathcal{L}, \alpha, \beta)$ consisting of, in the above notation,

- a curve $\pi: E \rightarrow S$ of genus 1;
- a line bundle \mathcal{L} of degree 3 on E ;
- an isomorphism $\alpha: \mathcal{E} \xrightarrow{\sim} \mathcal{O}_S^3$; and
- an isomorphism $\beta: \rho_*\mathcal{N} \xrightarrow{\sim} \mathcal{O}_S$.

We typically write $(E, \mathcal{L}, \alpha, \beta)$ for the quadruple $(\pi, \mathcal{L}, \alpha, \beta)$. A morphism

$$(E, \mathcal{L}, \alpha, \beta) \rightarrow (E', \mathcal{L}', \alpha', \beta')$$

is a morphism $(E, \mathcal{L}) \rightarrow (E', \mathcal{L}')$ in \mathcal{M} (4.7.2) that is compatible with the α ’s and β ’s in the obvious sense. We make $\widehat{\mathcal{M}}$ into a presheaf on (Aff) , hence a fibered category, by taking the obvious pullback functors, as in (4.7.2).

Note that there is a natural morphism of fibered categories $\widehat{\mathcal{M}} \rightarrow \mathcal{M}$ sending $(E, \mathcal{L}, \alpha, \beta) \mapsto (E, \mathcal{L})$.

Let $(E, \mathcal{L}, \alpha, \beta) \in \text{ob } \widehat{\mathcal{M}}(S)$. Though it appears somewhat unnatural to have fixed isomorphisms α and β , we will see that there is a simple description of $\widehat{\mathcal{M}}(S)$ in terms of cubic polynomials over the ring A .

From the isomorphism $\alpha: \mathcal{E} \xrightarrow{\sim} \mathcal{O}_S^3$, we get an identification $\mathbb{P}(\mathcal{E}) = \mathbb{P}_S^2$. Applying ρ^* to the isomorphism

$$\rho_* \mathcal{N} \xrightarrow{\beta} \mathcal{O}_S,$$

and using the canonical isomorphism $\rho^* \rho_* \mathcal{N} \xrightarrow{\sim} \mathcal{N}$, we get an identification $\mathcal{N} = \mathcal{O}_{\mathbb{P}_S^2}$. Hence we get an identification $\mathcal{J}^* = \mathcal{O}_{\mathbb{P}_S^2}(3)$. Now in the exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{\mathbb{P}_S^2} \rightarrow \mathcal{O}_E \rightarrow 0,$$

the inclusion $\mathcal{J} \hookrightarrow \mathcal{O}_{\mathbb{P}_S^2}$ is, by definition, a global section of \mathcal{J}^* . Thus the datum $(E, \mathcal{L}, \alpha, \beta)$ yields a canonical global section of $\mathcal{O}_{\mathbb{P}_S^2}(3)$ — that is, a homogenous cubic G in the polynomial ring $A[X, Y, Z]$. Moreover, let $V(G)$ denote the closed subscheme in \mathbb{P}_S^2 determined by the ideal $(G) \subset A[X, Y, Z]$. Then $V(G)$ is precisely the scheme-theoretic image of E in \mathbb{P}_S^2 .

For each affine scheme $S = \text{Spec}(A)$, let

$$(\mathbb{A}_{\mathbb{Z}}^{10} - \mathbb{A}_{\mathbb{Z}}^0)(S) = \left\{ \sum_{i+j+k=3} a_{ijk} X^i Y^j Z^k \in A[X, Y, Z] \mid (a_{ijk}) = A \right\}$$

denote the set of cubics over A . We have thus defined a map on objects $\widehat{\mathcal{M}}(S) \rightarrow (\mathbb{A}_{\mathbb{Z}}^{10} - \mathbb{A}_{\mathbb{Z}}^0)(S)$. Since each curve E is smooth over its base, the image of $\widehat{\mathcal{M}}(S)$ is contained in the set

$$\mathbb{A}_{\text{sm}}^{10}(S) := \{ G \in (\mathbb{A}_{\mathbb{Z}}^{10} - \mathbb{A}_{\mathbb{Z}}^0)(S) \mid V(G) \text{ is smooth over } S \}$$

of smooth cubics over A .

Regarding the set $\mathbb{A}_{\text{sm}}^{10}(S)$ as a discrete category, we claim that the map $\widehat{\mathcal{M}}(S) \rightarrow \mathbb{A}_{\text{sm}}^{10}(S)$ is a functor. Indeed, it suffices to show that isomorphic quadruples $(E, \mathcal{L}, \alpha, \beta)$ and $(E', \mathcal{L}', \alpha', \beta')$ give the same cubic G . This is clear from the above algorithm for G .

Proposition 4.8.2. *The functor*

$$\widehat{\mathcal{M}}(S) \rightarrow \mathbb{A}_{\text{sm}}^{10}(S)$$

is an equivalence of categories.

Proof. We construct a quasi-inverse $\mathbb{A}_{\text{sm}}^{10}(S) \rightarrow \widehat{\mathcal{M}}(S)$. Given a smooth cubic G , we take

$$E := V(G).$$

Let i denote the closed immersion $i: E \hookrightarrow \mathbb{P}_S^2$, ρ the structure map $\rho: \mathbb{P}_S^2 \rightarrow S$, and π the composite

$$\pi: E \xrightarrow{i} \mathbb{P}_S^2 \xrightarrow{\rho} S.$$

Then E is smooth and proper (because E is closed in projective space) over S . When A is an algebraically closed field k , it is a standard result that a nonsingular plane curve of degree d has genus $\frac{1}{2}(d-1)(d-2)$. Since E is defined by a cubic polynomial, E embeds as a degree 3 subvariety. Hence, for any A , E is a curve of genus 1.

We take the line bundle \mathcal{L} on E to be

$$\mathcal{L} := i^* \mathcal{O}_{\mathbb{P}_S^2}(1).$$

Since E has degree 3 in \mathbb{P}_S^2 when A is a field, \mathcal{L} is a degree 3 line bundle (for any A). On \mathbb{P}_S^2 , $\mathcal{O}_{\mathbb{P}_S^2}$ is generated by the global sections X, Y , and Z , so we get a canonical epimorphism of sheaves

$$\mathcal{O}_{\mathbb{P}_S^2}^3 \twoheadrightarrow \mathcal{O}_{\mathbb{P}_S^2}(1).$$

Applying i^* gives $\mathcal{O}_E^3 \rightarrow \mathcal{L}$. Hence, by adjunction, we get

$$\mathcal{O}_S^3 \rightarrow \pi_* \mathcal{L}.$$

We take this last map to be α^{-1} . Note that, by (4.5.2), $\pi_* \mathcal{L}$ is a vector bundle on S .

We claim that α^{-1} is an isomorphism. Using Nakayama’s lemma, it suffices to prove the claim when A is a field k . Then $\mathcal{O}_S^3 = \mathcal{O}_k^3 = k^3$, and $\pi_* \mathcal{L}$ is a 3-dimensional vector space over k . The map $\alpha^{-1}: k^3 \rightarrow \pi_* \mathcal{L}$ sends the standard basis elements e_1, e_2 , and e_3 of k^3 to the images of X, Y , and Z , respectively, in $\pi_* \mathcal{L}$. Therefore α^{-1} fails to be an isomorphism if and only if α^{-1} fails to be injective if and only if the images of X, Y , and Z in $\pi_* \mathcal{L}$ are linearly dependent over k . So suppose that such a nontrivial k -linear relation

$$aX + bY + cZ = 0$$

exists in $\pi_* \mathcal{L}$. That is, the polynomial $aX + bY + cZ$ vanishes on the points of E , so that E is contained in the line L determined by $aX + bY + cZ$. Since E and L are nonsingular, we must have $E \cong L$. But E has genus 1 and L has genus 0, a contradiction.

It remains to define β . As usual, we define the ideal sheaf \mathcal{J} on \mathbb{P}_S^2 via the exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{\mathbb{P}_S^2} \rightarrow \mathcal{O}_E \rightarrow 0.$$

Now the global section G in $\mathcal{O}_{\mathbb{P}_S^2}(3)$ gives a map $\mathcal{O}_{\mathbb{P}_S^2} \rightarrow \mathcal{O}_{\mathbb{P}_S^2}(3)$ sending $1 \mapsto G$. The dual map $\mathcal{O}_{\mathbb{P}_S^2}(-3) \rightarrow \mathcal{O}_{\mathbb{P}_S^2}$ is given by multiplication by G and induces an epimorphism $\mathcal{O}_{\mathbb{P}_S^2}(-3) \twoheadrightarrow \mathcal{J}$. Since, by (4.3), \mathcal{J} is generated locally by a nonzerodivisor, the multiplication-by- G map is injective. Thus $\mathcal{O}_{\mathbb{P}_S^2}(-3) \xrightarrow{\sim} \mathcal{J}$, and we get a canonical isomorphism

$$(*) \quad \mathcal{N} := \mathcal{J} \otimes_{\mathcal{O}_{\mathbb{P}_S^2}} \mathcal{O}_{\mathbb{P}_S^2}(3) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}_S^2}.$$

Taking the inverse of the isomorphism $(*)$ and using adjunction, we get a canonical map

$$\mathcal{O}_S \rightarrow \rho_* \mathcal{N}.$$

We take this last map to be β^{-1} . To check that β^{-1} is an isomorphism, it suffices to assume that A is a field, and one then uses the now familiar techniques of cohomology and base change. We omit the details.

In summary, we have now constructed our candidate quasi-inverse $\mathbb{A}_{\text{sm}}^{10}(S) \rightarrow \widehat{\mathcal{M}}(S)$. The proof that the functor is a quasi-inverse is not hard, and we again omit the details. \square

The force of the proposition, as will become apparent in §5, is that $\mathbb{A}_{\text{sm}}^{10}$ is representable. We omit the proof.

Proposition 4.8.3. *The presheaf $S \mapsto \mathbb{A}_{\text{sm}}^{10}(S)$ is represented by an open subscheme $\mathbb{A}_{\text{sm}}^{10}$ of $\mathbb{A}_{\mathbb{Z}}^{10}$. \square*

Once we have the definition of an algebraic stack, it will follow without difficulty from the natural functor $\widehat{\mathcal{M}} \rightarrow \mathcal{M}$ that \mathcal{M} is an algebraic stack.

5. ALGEBRAIC STACKS: GENERAL THEORY

5.1. **Fibered Products of Categories.** In this subsection and the next, we discuss the notions of fibered products of categories and of fibered products of fibered categories. The material involves only basic categorical notions, and could have been included in §2. We’ll need fibered products of fibered categories to give the definition of an algebraic stack.

We first define fibered products of categories. There are two versions of the definition. Let

$$(*) \quad \begin{array}{ccc} & \mathcal{E} & \\ & \downarrow G & \\ \mathcal{D} & \xrightarrow{F} & \mathcal{C} \end{array}$$

be a diagram of categories and covariant functors.

Definition 5.1.1. Given a diagram $(*)$, the *strict fibered product* $\mathcal{D} \times_{\mathcal{C}}^{\text{str}} \mathcal{E}$ is the following category. An object in $\mathcal{D} \times_{\mathcal{C}}^{\text{str}} \mathcal{E}$ is a pair (D, E) of objects $D \in \text{ob } \mathcal{D}$ and $E \in \text{ob } \mathcal{E}$ such that $FD = GE$ in \mathcal{C} . A morphism $(D', E') \rightarrow (D, E)$ in $\mathcal{D} \times_{\mathcal{C}}^{\text{str}} \mathcal{E}$ is a pair (f, g) of morphisms $f: D' \rightarrow D$ and $g: E' \rightarrow E$ such that $Ff = Gg$ in \mathcal{C} .

When \mathcal{C} , \mathcal{D} , and \mathcal{E} are small, one can express $\mathcal{D} \times_{\mathcal{C}}^{\text{str}} \mathcal{E}$ as follows. Write \mathcal{C}_0 and \mathcal{C}_1 for the object set and morphism set, respectively, of \mathcal{C} . Define \mathcal{D}_0 , \mathcal{D}_1 , \mathcal{E}_0 , and \mathcal{E}_1 similarly for \mathcal{D} and \mathcal{E} . The functors F and G give functions between sets

$$\begin{array}{ccc} \mathcal{E}_0 & & \mathcal{E}_1 \\ \downarrow G & \text{and} & \downarrow G \\ \mathcal{D}_0 \xrightarrow{F} \mathcal{C}_0 & & \mathcal{D}_1 \xrightarrow{F} \mathcal{C}_1. \end{array}$$

Then the object set of $\mathcal{D} \times_{\mathcal{C}}^{\text{str}} \mathcal{E}$ is the usual fibered product of sets $\mathcal{D}_0 \times_{\mathcal{C}_0} \mathcal{E}_0$, and the morphism set is $\mathcal{D}_1 \times_{\mathcal{C}_1} \mathcal{E}_1$.

There is an analogy between strict fibered products of categories and strict presheaves of categories. Just as for strict presheaves of categories, the definition of a strict fibered product is too rigid to be very useful. The second version of the definition is more flexible.

As motivation, we consider an example from topology. Let

$$\begin{array}{ccc} & Y & \\ & \downarrow G & \\ X & \xrightarrow{F} & S \end{array}$$

be a diagram of continuous maps of topological spaces. Then one can form the fibered product space $X \times_S Y$ as usual. However, in homotopy theory, it is often advantageous to consider the homotopy fibered product

$$\left\{ (x, y, p) \mid \begin{array}{l} x \in X, y \in Y, \text{ and } p: [0, 1] \rightarrow S \text{ is a path} \\ \text{with } p(0) = F(x) \text{ and } p(1) = G(y) \end{array} \right\}.$$

The general definition mimics this construction.

Definition 5.1.2. Given a diagram $(*)$, the *fibred product* $\mathcal{D} \times_{\mathcal{C}} \mathcal{E}$ is the following category. An object in $\mathcal{D} \times_{\mathcal{C}} \mathcal{E}$ is a triple (D, E, φ) consisting of an object $D \in \text{ob } \mathcal{D}$, an object $E \in \text{ob } \mathcal{E}$, and an isomorphism $\varphi: FD \xrightarrow{\sim} GE$ in \mathcal{C} . A morphism $(D', E', \varphi') \rightarrow (D, E, \varphi)$ is a pair (f, g) of morphisms $f: D' \rightarrow D$ and $g: E' \rightarrow E$ such that the diagram

$$\begin{array}{ccc} FD' & \xrightarrow{Ff} & FD \\ \varphi' \downarrow & & \downarrow \varphi \\ GE' & \xrightarrow{Gg} & GE \end{array}$$

commutes in \mathcal{C} .

We consider several examples.

Example 5.1.3. Let

$$\begin{array}{ccc} & & Y \\ & & \downarrow G \\ X & \xrightarrow{F} & \mathcal{D} \end{array}$$

be a diagram of categories, with X and Y both small and discrete. Then $X \times_{\mathcal{D}} Y$ is a small discrete category, with underlying object set

$$\{(x, y, \varphi) \mid x \in \text{ob } X, y \in \text{ob } Y, \text{ and } \varphi: Fx \xrightarrow{\sim} Gy\}$$

Example 5.1.4. Given categories \mathcal{D} and \mathcal{E} , the *product* $\mathcal{D} \times \mathcal{E}$ is the category of pairs (D, E) of objects $D \in \text{ob } \mathcal{D}$ and $E \in \text{ob } \mathcal{E}$. A morphism $(D', E') \rightarrow (D, E)$ in $\mathcal{D} \times \mathcal{E}$ is just a pair (f, g) of morphisms $f: D' \rightarrow D$ and $g: E' \rightarrow E$.

Let \mathcal{C} be the trivial category — that is, a category with a single object $*$ and a single morphism id_* . Then $\mathcal{D} \times_{\mathcal{C}} \mathcal{E}$ and $\mathcal{D} \times_{\mathcal{C}}^{\text{str}} \mathcal{E}$ are both isomorphic (not just equivalent) to $\mathcal{D} \times \mathcal{E}$.

Example 5.1.5. Let

$$\mathcal{E} \begin{array}{c} \xrightarrow{F} \\ \rightrightarrows \\ \xrightarrow{G} \end{array} \mathcal{D}$$

be a diagram of categories. We define the *homotopy kernel*

$$\text{hoker} \left[\mathcal{E} \begin{array}{c} \xrightarrow{F} \\ \rightrightarrows \\ \xrightarrow{G} \end{array} \mathcal{D} \right]$$

to be the category of pairs (E, φ) consisting of an object $E \in \text{ob } \mathcal{E}$ and an isomorphism $\varphi: FE \xrightarrow{\sim} GE$. A morphism $(E', \varphi') \rightarrow (E, \varphi)$ in $\text{hoker}[\mathcal{E} \rightrightarrows \mathcal{D}]$ is a morphism $f: E' \rightarrow E$ in \mathcal{E} that makes the diagram

$$\begin{array}{ccc} FE' & \xrightarrow{Ff} & FE \\ \varphi' \downarrow & & \downarrow \varphi \\ GE' & \xrightarrow{Gf} & GE \end{array}$$

commute in \mathcal{D} .

Now consider the fibered product $\mathcal{E} \times_{\mathcal{D} \times \mathcal{D}} \mathcal{D}$ obtained in the diagram

$$\begin{array}{ccc} \mathcal{E} \times_{\mathcal{D} \times \mathcal{D}} \mathcal{D} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \Delta \\ \mathcal{E} & \xrightarrow{F \times G} & \mathcal{D} \times \mathcal{D}, \end{array}$$

where $\Delta: \mathcal{D} \rightarrow \mathcal{D} \times \mathcal{D}$ is the diagonal functor. We leave it as an exercise to check that $\mathcal{E} \times_{\mathcal{D} \times \mathcal{D}} \mathcal{D}$ is equivalent to $\text{hoker}[\mathcal{E} \rightrightarrows \mathcal{D}]$.

Example 5.1.6. Let X be small and discrete, \mathcal{D} a groupoid, and F a functor

$$X \xrightarrow{F} \mathcal{D}.$$

By (5.1.3), $X \times_{\mathcal{D}} X$ is small. Let s and t denote the projection functors $X \times_{\mathcal{D}} X \rightarrow X$. Then we can interpret the diagram

$$X \times_{\mathcal{D}} X \begin{array}{c} \xrightarrow{s} \\ \rightrightarrows \\ \xrightarrow{t} \end{array} X$$

as a category \mathcal{D}' in a natural way: \mathcal{D}' has object set $\text{ob } X$ and morphism set $\text{ob } X \times_{\mathcal{D}} X$, with s and t sending each morphism to its source and target, respectively. Compositions are defined via the functor $(X \times_{\mathcal{D}} X) \times_X (X \times_{\mathcal{D}} X) \xrightarrow{s \times t} X \times_{\mathcal{D}} X$, and identities via the diagonal functor $X \xrightarrow{\Delta} X \times_{\mathcal{D}} X$.

Our construction gives a natural functor $F': \mathcal{D}' \rightarrow \mathcal{D}$. When F is essentially surjective, it is easy to check that F' is an equivalence.

Example 5.1.7. Let G be a group. Recall from (1.6.6) that $G \setminus *$ is the groupoid consisting of a single object $*$ with automorphism group G .

Let H and K be subgroups of G . There is a natural diagram

$$\begin{array}{ccc} & & K \setminus * \\ & & \downarrow \\ H \setminus * & \longrightarrow & G \setminus *. \end{array}$$

It is easy to check that the fibered product $(H \setminus *) \times_{G \setminus *} (K \setminus *)$ is isomorphic to the groupoid $(K \times H) \setminus G$ (1.6.6), where $K \times H$ acts on G as $(k, h) \cdot g = kgh^{-1}$.

We'll use the following exercise in the next subsection.

Exercise 5.1.8. Let

$$\begin{array}{ccccc} \mathcal{D}' & \longrightarrow & \mathcal{C}' & \longleftarrow & \mathcal{E}' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{C} & \longleftarrow & \mathcal{E} \end{array}$$

be a commutative (in the strong sense) diagram of categories and functors, and suppose that the vertical arrows are equivalences. Show that the natural functor

$$\mathcal{D}' \times_{\mathcal{C}'} \mathcal{E}' \rightarrow \mathcal{D} \times_{\mathcal{C}} \mathcal{E}$$

is an equivalence.

The analogous statement of the exercise for strict fibered categories is false. Thus the exercise supports the claim that (5.1.2) is the better notion of a fibered product of categories.

5.2. Fibered Products of Fibered Categories. Let

$$d: \mathcal{D} \rightarrow \mathcal{C}, \quad e: \mathcal{E} \rightarrow \mathcal{C}, \quad \text{and} \quad f: \mathcal{F} \rightarrow \mathcal{C}$$

be fibered categories over \mathcal{C} . Let

$$\alpha \in \text{ob Hom}_{\square}(\mathcal{E}, \mathcal{D}) \quad \text{and} \quad \beta \in \text{ob Hom}_{\square}(\mathcal{F}, \mathcal{D}),$$

so that we have a diagram

$$\begin{array}{ccc} & & \mathcal{F} \\ & & \downarrow \beta \\ \mathcal{E} & \xrightarrow{\alpha} & \mathcal{D} \end{array}$$

of fibered categories over \mathcal{C} . One can construct the fibered product $\mathcal{E} \times_{\mathcal{D}} \mathcal{F}$ as in the previous subsection, but $\mathcal{E} \times_{\mathcal{D}} \mathcal{F}$ will fail to be a fibered category over \mathcal{C} in all but the simplest cases. We thus wish to define a fibered category version of the fibered product $\mathcal{E} \overset{\text{fc}}{\times}_{\mathcal{D}} \mathcal{F}$.

Heuristically, for each $S \in \text{ob } \mathcal{C}$, we want

$$(\mathcal{E} \overset{\text{fc}}{\times}_{\mathcal{D}} \mathcal{F})(S) = \mathcal{E}(S) \times_{\mathcal{D}(S)} \mathcal{F}(S).$$

To get pullbacks functors in $\mathcal{E} \overset{\text{fc}}{\times}_{\mathcal{D}} \mathcal{F}$, we should use those in \mathcal{E} , \mathcal{F} , and \mathcal{D} . That is, given a morphism $g: S' \rightarrow S$ in \mathcal{C} and an object (E, F, φ) in $(\mathcal{E} \overset{\text{fc}}{\times}_{\mathcal{D}} \mathcal{F})(S)$, we should take, informally, $g^*(E, F, \varphi) := (g^*E, g^*F, g^*\varphi)$. Of course, the choices of pullbacks are noncanonical, and one must take care in interpreting $(g^*E, g^*F, g^*\varphi)$. The formal definition sidesteps such complications.

Definition 5.2.1. Let \mathcal{C} , \mathcal{D} , \mathcal{E} , and \mathcal{F} be as above. The *fibered product of fibered categories* $\mathcal{E} \overset{\text{fc}}{\times}_{\mathcal{D}} \mathcal{F}$ is the following category. An object in $\mathcal{E} \overset{\text{fc}}{\times}_{\mathcal{D}} \mathcal{F}$ is a triple (E, F, φ) consisting of an object $E \in \text{ob } \mathcal{E}$, an object $D \in \text{ob } \mathcal{D}$, both lying over the same object $S \in \text{ob } \mathcal{C}$, and an isomorphism $\varphi: \alpha E \xrightarrow{\sim} \beta F$ in $\mathcal{D}(S)$ (so that φ lies over id_S). A morphism $(E', F', \varphi') \rightarrow (E, F, \varphi)$ is a pair (ϵ, η) of morphisms $\epsilon: E' \rightarrow E$ and $\eta: F' \rightarrow F$, both lying over the same morphism in \mathcal{C} , such that the diagram

$$\begin{array}{ccc} \alpha E' & \xrightarrow{\alpha \epsilon} & \alpha E \\ \varphi' \downarrow & & \downarrow \varphi \\ \beta F' & \xrightarrow{\beta \eta} & \beta F \end{array}$$

commutes in \mathcal{D} .

Thus $\mathcal{E} \overset{\text{fc}}{\times}_{\mathcal{D}} \mathcal{F}$ is a certain (not full, usually) subcategory of $\mathcal{E} \times_{\mathcal{D}} \mathcal{F}$.

We say a diagram of fibered categories

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\alpha'} & \mathcal{F} \\ \beta' \downarrow & & \downarrow \beta \\ \mathcal{E} & \xrightarrow{\alpha} & \mathcal{D} \end{array}$$

over \mathcal{C} is *cartesian* if there is an isomorphism φ between the two compositions $\alpha\beta'$, $\beta\alpha': \mathcal{G} \rightarrow \mathcal{D}$ in $\text{Hom}_{\square}(\mathcal{G}, \mathcal{D})$ such that the induced map

$$\mathcal{G} \xrightarrow{(\beta', \alpha', \varphi)} \mathcal{E} \overset{\text{fc}}{\times}_{\mathcal{D}} \mathcal{F}$$

is an equivalence of categories.

Example 5.2.2. If \mathcal{E} and \mathcal{F} are any fibered categories over the same base \mathcal{C} , then we can always form the *product of fibered categories*

$$\mathcal{E} \times^{\text{fc}} \mathcal{F} := \mathcal{E} \times_{\mathcal{C}}^{\text{fc}} \mathcal{F},$$

where we view \mathcal{C} as a fibered category over itself via the identity functor. An object of $\mathcal{E} \times^{\text{fc}} \mathcal{F}$ over $S \in \text{ob } \mathcal{C}$ is a pair $(E, F) \in \text{ob}(E(S) \times \mathcal{F}(S))$, and a morphism $(E', F') \rightarrow (E, F)$ is a pair (ϵ, η) of morphisms $\epsilon: E' \rightarrow E$ and $\eta: F' \rightarrow F$, both lying over the same morphism in \mathcal{C} .

The basic properties of fibered products of fibered categories are as follows.

Proposition 5.2.3. *Let \mathcal{C} , \mathcal{D} , \mathcal{E} , and \mathcal{F} be as at the beginning of the subsection, and let $\mathcal{G} = \mathcal{E} \times_{\mathcal{D}}^{\text{fc}} \mathcal{F}$. Let $p: \mathcal{G} \rightarrow \mathcal{E}$ and $q: \mathcal{G} \rightarrow \mathcal{F}$ denote the canonical projections.*

- (1) \mathcal{G} is naturally a fibered category over \mathcal{C} .
- (2) The functors p and q are morphisms of fibered categories.
- (3) \mathcal{G} has the following universal mapping property: for any fibered category \mathcal{H} over \mathcal{C} , there is an isomorphism of categories

$$\text{Hom}_{\square}(\mathcal{H}, \mathcal{G}) \xrightarrow{\sim} \text{Hom}_{\square}(\mathcal{H}, \mathcal{E}) \times_{\text{Hom}_{\square}(\mathcal{H}, \mathcal{D})} \text{Hom}_{\square}(\mathcal{H}, \mathcal{F}).$$

- (4) If \mathcal{C} is a site, and if \mathcal{D} , \mathcal{E} , and \mathcal{F} are stacks, then \mathcal{G} is a stack.

Proof. We leave (1), (2), and (3) as straightforward exercises. We prove (4). To check that \mathcal{G} is separated, let $S \in \text{ob } \mathcal{C}$ and let $G, G' \in \text{ob } \mathcal{G}$. By (2.10.7) and Yoneda’s lemma for fibered categories, it suffices to show that the presheaf of sets $\mathcal{H}om(G', G)$ on \mathcal{C}/S defined by

$$(T \xrightarrow{\pi} S) \mapsto \text{Hom}_{\mathcal{G}(S)}(\pi^* G', \pi^* G)$$

is a sheaf.

Say $G = (E, F, \varphi)$ and $G' = (E', F', \varphi')$. It is easy to check that

$$\mathcal{H}om(G', G) = \mathcal{H}om(E', E) \times_{\mathcal{H}om(\alpha_{E'}, \beta_F)} \mathcal{H}om(F', F)$$

as presheaves on \mathcal{C}/S . Since \mathcal{D} , \mathcal{E} , and \mathcal{F} are separated, the right hand side is a limit of sheaves, hence is itself a sheaf by (1.8.4).

To complete the proof that \mathcal{G} is a stack, let $S \in \text{ob } \mathcal{C}$ and $\mathcal{U} \in \text{Cov}(S)$. Recall that for any fibered categories \mathcal{X} and \mathcal{Y} , we have $\mathcal{Y}(\mathcal{X}) = \text{Hom}_{\square}(\mathcal{X}, \mathcal{Y})$. We get a diagram

$$\begin{array}{ccc} \mathcal{G}(h_S) & \longrightarrow & \mathcal{G}(\mathcal{U}) \\ \downarrow & & \\ \mathcal{G}(S) & & \end{array}$$

where the vertical arrow is the usual Yoneda equivalence. We must show that the horizontal arrow is an equivalence. That is, using the universal mapping property (3), we must show that the natural functor

$$(*) \quad \mathcal{E}(h_S) \times_{\mathcal{D}(h_S)} \mathcal{F}(h_S) \rightarrow \mathcal{E}(\mathcal{U}) \times_{\mathcal{D}(\mathcal{U})} \mathcal{F}(\mathcal{U})$$

is an equivalence. Since \mathcal{D} , \mathcal{E} , and \mathcal{F} are stacks, the functors

$$\mathcal{D}(h_S) \rightarrow \mathcal{D}(\mathcal{U}), \quad \mathcal{E}(h_S) \rightarrow \mathcal{E}(\mathcal{U}), \quad \text{and} \quad \mathcal{F}(h_S) \rightarrow \mathcal{F}(\mathcal{U})$$

are all equivalences. It follows from (5.1.8) that (*) is an equivalence too. \square

5.3. Representable Fibered Categories and Representable Morphisms.

We now define representable fibered categories and representable morphisms of fibered categories. We'll need these definitions for that of an algebraic stack. Recall that any presheaf of sets on (Aff) gives a fibered category over (Aff) in a natural way (2.5.1). In particular, any algebraic space S gives the representable presheaf of sets $h_S = \text{Hom}(-, S)$ on (Aff) , from which we obtain the fibered category $(\text{Aff})/S \subset (\text{Alg Sp})/S$ of affine schemes over S (2.5.2).

From now on, we use the same symbol S to denote both the algebraic space S and the category $(\text{Aff})/S$, provided our meaning is clear from context. Let \mathcal{D} be a fibered category over (Aff) . When S is an affine scheme, we reserve the notation $\mathcal{D}(S)$ for the subcategory of \mathcal{D} of objects over the scheme $S \in \text{ob}(\text{Aff})$ and morphisms over id_S . However, by Yoneda's lemma for fibered categories (2.7), the categories $\mathcal{D}(S)$ and

$$\mathcal{D}((\text{Aff})/S) = \text{Hom}_{\overline{(\text{Aff})}}((\text{Aff})/S, \mathcal{D})$$

are equivalent. So no real confusion seems possible!

Definition 5.3.1.

- (1) A fibered category \mathcal{D} over (Aff) is *strictly representable* if \mathcal{D} is the fibered category obtained from an algebraic space.
- (2) A fibered category \mathcal{D} over (Aff) is *representable* if \mathcal{D} is equivalent, as a fibered category, to a strictly representable fibered category.
- (3) A morphism of fibered categories $\mathcal{E} \rightarrow \mathcal{D}$ over (Aff) is *representable* if for every affine scheme S and every morphism of fibered categories $S \rightarrow \mathcal{D}$, the fibered product $\mathcal{E} \times_{\mathcal{D}}^{\text{fc}} S$ is a representable fibered category.

To be quite explicit, we mean in (2) that there exists a strictly representable fibered category \mathcal{E} and a morphism of fibered categories α in $\text{Hom}_{\overline{(\text{Aff})}}(\mathcal{D}, \mathcal{E})$ such that α is an equivalence.

The next lemma follows from the properties of algebraic spaces. We omit its proof.

Lemma 5.3.2. *Let $\mathcal{E} \rightarrow \mathcal{D}$ be a representable morphism of fibered categories. Then for every algebraic space X and morphism $X \rightarrow \mathcal{D}$, the fibered product $\mathcal{E} \times_{\mathcal{D}}^{\text{fc}} X$ is representable. \square*

Let X and Y be algebraic spaces. Using Yoneda's lemma, there is a bijective correspondence between morphisms of algebraic spaces $X \rightarrow Y$ and morphisms of fibered categories $X \rightarrow Y$ over (Aff) . So, if (P) is a property of morphisms of algebraic spaces, we define a morphism of fibered categories $X \rightarrow Y$ to have property (P) exactly when the corresponding morphism of algebraic spaces has property (P) . Provided (P) satisfies certain restrictions, we can go further to say when any representable morphism of fibered categories has property (P) .

Definition 5.3.3. Let (P) be a property of morphisms of algebraic spaces that satisfies the following.

- (a) (P) is stable under base change; that is, in any cartesian diagram of algebraic spaces

$$\begin{array}{ccc} Y \times_X X' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X, \end{array}$$

if the morphism $Y \rightarrow X$ has property (P) , then so does $Y \times_X X' \rightarrow X'$.

- (b) (P) is étale-local on the base; that is, let $Y \rightarrow X$ be any morphism, and suppose that there exists a surjective family $\{X_i \rightarrow X\}$ of étale morphisms such that each projection $X_i \times_X Y \rightarrow X_i$ has property (P) . Then $Y \rightarrow X$ has property (P) .

Now let $\alpha: \mathcal{D} \rightarrow \mathcal{E}$ be a representable morphism of fibered categories over (Aff) . We say that α satisfies property (P) if for every affine scheme S and cartesian diagram

$$\begin{array}{ccc} S \times_{\mathcal{E}}^{\text{fc}} \mathcal{D} & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{\alpha} & \mathcal{E}, \end{array}$$

the morphism $S \times_{\mathcal{E}}^{\text{fc}} \mathcal{D} \rightarrow S$ satisfies property (P) .

In anticipation of the definition of an algebraic stack, note that the properties of smoothness and surjectivity both satisfy (a) and (b). So it makes sense for a representable morphism of fibered categories to be smooth or surjective.

5.4. Algebraic Stacks.

We finally come to the definition of an algebraic stack.

Definition 5.4.1. An *algebraic stack* is a fibered category \mathcal{X} over (Aff) such that

- (1) \mathcal{X} is a category fibered in groupoids — that is, for every affine scheme S , the category $\mathcal{X}(S)$ is a groupoid;
- (2) \mathcal{X} is a stack for the étale topology on (Aff) ;
- (3) the diagonal morphism $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{E}}^{\text{fc}} \mathcal{X}$ is representable; and
- (4) there exists an algebraic space X and a smooth surjective morphism of fibered categories, called a *presentation*, $X \rightarrow \mathcal{X}$.

There is no assumption in (4) on the morphism $X \rightarrow \mathcal{X}$ being representable, so it is not clear that it makes sense for such a morphism to be smooth and surjective. When \mathcal{X} satisfies condition (3), however, the following lemma says that any morphism from a representable fibered category into \mathcal{X} is automatically representable.

Lemma 5.4.2. *Let \mathcal{X} be a fibered category over (Aff) such that the diagonal morphism*

$$\Delta: \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{E}}^{\text{fc}} \mathcal{X}$$

is representable. Then for any algebraic space X and any morphism of fibered categories $\alpha: X \rightarrow \mathcal{X}$, α is representable.

Proof. Let S be an affine scheme and $\beta: S \rightarrow \mathcal{X}$ a morphism of fibered categories. Let $\mathcal{D} = X \times_{\mathcal{X}}^{\text{fc}} S$. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & X \times_{\mathcal{X}}^{\text{fc}} S \\ \downarrow & & \downarrow \alpha \times \beta \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times_{\mathcal{X}}^{\text{fc}} \mathcal{X}, \end{array}$$

where $\mathcal{E} := \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{X}}^{\text{fc}} \mathcal{X}}^{\text{fc}} (X \times_{\mathcal{X}}^{\text{fc}} S)$. Since Δ is representable, so is \mathcal{E} . On the other hand, it is easy to check that \mathcal{E} and \mathcal{D} are equivalent. The lemma follows. \square

We continue our discussion of condition (3) in the definition of an algebraic stack. Let \mathcal{D} be a fibered category over a base category \mathcal{C} . We define $\text{sk}_{\mathcal{C}} \mathcal{D}$, the *fibered category skeleton* of \mathcal{D} , to be the fibered category obtained from the presheaf of categories on \mathcal{C}

$$S \mapsto \{\text{Isomorphism classes of objects in } \mathcal{D}(S)\}.$$

Now take $\mathcal{C} = (\text{Aff})$. Then \mathcal{D} is representable if and only if $\text{sk}_{(\text{Aff})} \mathcal{D}$ is a representable presheaf of sets — that is, each category $(\text{sk}_{(\text{Aff})} \mathcal{D})(S)$ is small and discrete, so that $\text{sk}_{(\text{Aff})} \mathcal{D}$ can be regarded as a presheaf of sets, and this presheaf of sets is representable.

Consider the diagonal morphism $\Delta: \mathcal{D} \rightarrow \mathcal{D} \times_{\mathcal{D}}^{\text{fc}} \mathcal{D}$. By definition, Δ is representable if and only if for every affine scheme S and morphism $S \rightarrow \mathcal{D} \times_{\mathcal{D}}^{\text{fc}} \mathcal{D}$, the fibered category $S \times_{\mathcal{D} \times_{\mathcal{D}}^{\text{fc}} \mathcal{D}}^{\text{fc}} \mathcal{D}$ is representable. Fix an algebraic space X and an equivalence of fibered categories

$$\alpha: X \xrightarrow{\sim} S \times_{\mathcal{D} \times_{\mathcal{D}}^{\text{fc}} \mathcal{D}}^{\text{fc}} \mathcal{D}.$$

Then $\text{id}_X \in X(X)$ gives a distinguished object in

$$\left(S \times_{\mathcal{D} \times_{\mathcal{D}}^{\text{fc}} \mathcal{D}}^{\text{fc}} \mathcal{D} \right)(X) = S(X) \times_{\mathcal{D}(X) \times \mathcal{D}(X)} \mathcal{D}(X),$$

hence a distinguished object $g: X \rightarrow S$ in $S(X)$.

Now, the morphism $S \rightarrow \mathcal{D} \times_{\mathcal{D}}^{\text{fc}} \mathcal{D}$ is equivalent, by Yoneda's lemma for fibered categories, to the choice of objects D_1 and D_2 in $\mathcal{D}(S)$. We leave it as an exercise to check that $X \xrightarrow{g} S$ represents the presheaf of sets on $(\text{Aff})/S$

$$(T \xrightarrow{\pi} S) \mapsto \text{Isom}_{\mathcal{D}(T)}(\pi^* D_1, \pi^* D_2).$$

We denote this presheaf $\mathcal{I}som(D_1, D_2)$. It is a subpresheaf of $\mathcal{H}om(D_1, D_2)$ (2.10.6), and equals $\mathcal{H}om(D_1, D_2)$ when \mathcal{D} is a category fibered in groupoids.

The content of condition (3) is thus that the $\mathcal{I}som$ presheaves are always representable. This makes precise the notion that the morphisms in an algebraic stack should be of an algebraic nature.

It does not appear completely natural to demand that an algebraic stack be a category fibered in groupoids, so it seems reasonable to pursue a version of the definition absent this requirement. We shall not do so here, but we remark that such a formulation should probably demand that all $\mathcal{H}om$ sheaves be representable.

Condition (4) in the definition of an algebraic stack makes precise the notion that the objects in an algebraic stack should be of an algebraic nature.

5.5. Constructions. We describe two common techniques to produce new algebraic stacks from old ones.

Proposition 5.5.1. *Let*

$$\begin{array}{ccc} & \mathfrak{Y} & \\ & \downarrow & \\ \mathfrak{X} & \longrightarrow & \mathfrak{Z} \end{array}$$

be a diagram of algebraic stacks. Then the fibered product $\mathfrak{X} \times_{\mathfrak{Z}}^{\text{fc}} \mathfrak{Y}$ is an algebraic stack.

Proof. We check the four conditions in the definition (5.4.1). Condition (1) is clear from the definition of fibered product of fibered categories (5.2.1). Condition (2) is done in part (4) of (5.2.3). We leave (3) as an easy exercise. For (4), choose algebraic spaces $X, Y,$ and Z and presentations

$$X \rightarrow \mathfrak{X}, \quad Y \rightarrow \mathfrak{Y}, \quad \text{and} \quad Z \rightarrow \mathfrak{Z}.$$

We get a diagram

$$\begin{array}{ccccc} (X \times_Z Y) \times_Z (Y \times_Z Z) & \longrightarrow & (X \times_Z Z) \times_Z (Y \times_Z Z) & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X \times_Z Y & \longrightarrow & X \times_Z Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X \times Y & \longrightarrow & X \times Y & \longrightarrow & Z \times Z, \end{array}$$

where all products and fibered products are taken in the fibered category sense. One checks easily that every square in the diagram is cartesian. Since the top right vertical arrow $Z \rightarrow \mathfrak{Z}$ is smooth and surjective, so is the top middle vertical arrow. Since the bottom left horizontal arrow $X \times_Z Y \rightarrow X \times_Z Y$ is smooth and surjective, so is the top left horizontal arrow. Therefore

$$(X \times_Z Z) \times_Z^{\text{fc}} (Y \times_Z Z) \rightarrow X \times_Z Y$$

is a composition of smooth surjective morphisms, hence is itself smooth and surjective. Since $(X \times_Z Z) \times_Z^{\text{fc}} (Y \times_Z Z)$ is representable, we’re done. \square

The following lemma will be used in the next proposition.

Lemma 5.5.2. *Let*

$$\begin{array}{ccc} \mathfrak{Y}' & \xrightarrow{f'} & \mathfrak{X}' \\ g' \downarrow & & \downarrow g \\ \mathfrak{Y} & \xrightarrow{f} & \mathfrak{X} \end{array}$$

be a cartesian diagram of fibered categories. Suppose that f is representable, smooth, and surjective, and that g' is representable. Then g is representable.

Proof. Without loss of generality, we may assume that \mathfrak{X} is an affine scheme. Then, since f is representable, \mathfrak{Y} is (equivalent to) an algebraic space, smooth and surjective over \mathfrak{X} . Moreover, since \mathfrak{Y} is an algebraic space, there exists a scheme Y and an étale surjective morphism $Y \rightarrow \mathfrak{Y}$. So, again without loss of generality, we may

further reduce to the case of $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ a smooth surjective morphism of schemes, with \mathcal{X} affine.

Consider the special case that f admits a section σ . Then base change by $\mathcal{X}' \xrightarrow{g} \mathcal{X}$ gives the diagram

$$\begin{array}{ccccc} \mathcal{X}' & \xrightarrow{\sigma'} & \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \\ g \downarrow & & \downarrow g' & & \downarrow g \\ \mathcal{X} & \xrightarrow{\sigma} & \mathcal{Y} & \xrightarrow{f} & \mathcal{X}. \end{array}$$

Since g' is representable, so is g .

In the general case, f admits sections étale locally (3.6.3). Since representability is a local condition in the étale topology, the lemma follows. \square

Proposition 5.5.3. *Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of fibered categories over (Aff) , with \mathcal{X} a category fibered in groupoids. If \mathcal{Y} is an algebraic stack and f is representable, then \mathcal{X} is an algebraic stack.*

Proof. To find a presentation for \mathcal{X} , choose a presentation $Y \rightarrow \mathcal{Y}$ of \mathcal{Y} , and consider the diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

Since f is representable, so is $\mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} Y$. Since $Y \rightarrow \mathcal{Y}$ is smooth and surjective, so is $\mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} Y \rightarrow \mathcal{X}$. Hence $\mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} Y \rightarrow \mathcal{X}$ is a presentation.

We next show that the diagonal morphism $\Delta_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} \mathcal{X}$ is representable. Let $\Delta_{\mathcal{Y}}: \mathcal{Y} \rightarrow \mathcal{Y} \times_{\mathcal{Y}}^{\text{fc}} \mathcal{Y}$ denote the diagonal morphism of \mathcal{Y} , and form the fibered product $\mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} \mathcal{X}$. We get a diagram

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} \mathcal{X} \\ & & \downarrow & & \downarrow f \times f \\ & & \mathcal{Y} & \xrightarrow{\Delta_{\mathcal{Y}}} & \mathcal{Y} \times_{\mathcal{Y}}^{\text{fc}} \mathcal{Y}, \end{array}$$

where the square is cartesian and the composite $\mathcal{X} \rightarrow \mathcal{Y}$ is just f . By hypothesis, $\Delta_{\mathcal{Y}}$ is representable. Therefore $\mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} \mathcal{X}$ is representable. Since a composition of representable morphisms is representable, it suffices to show that $\mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}}^{\text{fc}} \mathcal{X}$ is representable.

Now let $Y \rightarrow \mathcal{Y}$ be a presentation, and set $\mathcal{X}_Y := \mathcal{X} \overset{\text{fc}}{\times}_{\mathcal{Y}} Y$. Then, by the first part of the proof, \mathcal{X}_Y is representable. Consider the diagram

$$\begin{array}{ccc} \mathcal{X}_Y & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X}_Y \overset{\text{fc}}{\times}_Y \mathcal{X}_Y & \longrightarrow & \mathcal{X} \overset{\text{fc}}{\times}_{\mathcal{Y}} \mathcal{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{Y}. \end{array}$$

Both squares are cartesian. Since \mathcal{X}_Y is representable, so is $\mathcal{X}_Y \overset{\text{fc}}{\times}_Y \mathcal{X}_Y$. Therefore the top left vertical arrow $\mathcal{X}_Y \rightarrow \mathcal{X}_Y \overset{\text{fc}}{\times}_Y \mathcal{X}_Y$ is a representable morphism. Since $Y \rightarrow \mathcal{Y}$ is representable, surjective, and smooth, so is the middle horizontal arrow $\mathcal{X}_Y \overset{\text{fc}}{\times}_Y \mathcal{X}_Y \rightarrow \mathcal{X} \overset{\text{fc}}{\times}_{\mathcal{Y}} \mathcal{X}$. Hence $\mathcal{X} \rightarrow \mathcal{X} \overset{\text{fc}}{\times}_{\mathcal{Y}} \mathcal{X}$ is representable by (5.5.2).

We lastly show that \mathcal{X} is a stack for the étale topology. Since $\mathcal{X} \rightarrow \mathcal{X} \overset{\text{fc}}{\times} \mathcal{X}$ is representable, we know from the discussion in §5.4 that all *Isom* presheaves are representable (by algebraic spaces). Therefore the *Isom* presheaves are all sheaves for the étale topology (or even for the fpqc topology), and \mathcal{X} is separated.

It remains to show that for all affine schemes S and all $\mathcal{U} \in \text{Cov}(S)$, the natural map $\mathcal{X}(h_S) \rightarrow \mathcal{X}(\mathcal{U})$ is essentially surjective. In other words, given a morphism of fibered categories $\alpha: \mathcal{U} \rightarrow \mathcal{X}$, we must show that there exists a morphism $h_S \rightarrow \mathcal{X}$ such that the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\alpha} & \mathcal{X} \\ \downarrow & & \parallel \\ h_S & \longrightarrow & \mathcal{X} \end{array}$$

commutes weakly.

Since \mathcal{Y} is a stack, there exists a morphism $h_S \rightarrow \mathcal{Y}$ that makes the square

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\alpha} & \mathcal{X} \\ \downarrow & & \downarrow f \\ h_S & \longrightarrow & \mathcal{Y} \end{array}$$

commute weakly. Thus we get a morphism

$$\mathcal{U} \rightarrow h_S \overset{\text{fc}}{\times}_{\mathcal{Y}} \mathcal{X}.$$

Since f is representable, the fibered product $h_S \overset{\text{fc}}{\times}_{\mathcal{Y}} \mathcal{X}$ is too, hence is an (algebraic) stack. Thus there exists a weak lifting $h_S \rightarrow h_S \overset{\text{fc}}{\times}_{\mathcal{Y}} \mathcal{X}$, and the composite

$$h_S \rightarrow h_S \overset{\text{fc}}{\times}_{\mathcal{Y}} \mathcal{X} \rightarrow \mathcal{X}$$

gives the desired morphism. □

5.6. Elliptic Curves. As an application of the previous subsection, we return to the discussion of curves of genus 1 at the end of §4. We begin with the definition of an elliptic curve.

Definition 5.6.1.

- (1) An *elliptic curve* over a base scheme S is a pair (π, e) consisting of a curve $\pi: E \rightarrow S$ of genus 1 and a section $e: S \rightarrow E$ of π . We typically write (E, e) for the pair (π, e) .
- (2) Given an affine scheme S , we define $(\text{Ell})(S)$ to be the category whose objects are elliptic curves over S , and whose morphisms are isomorphisms of schemes over S compatible with the given sections. We make (Ell) into a presheaf over (Aff) , hence a fibered category, by taking the obvious pullback functors, similar to (4.7.2).

Our goal for the subsection is the following.

Theorem 5.6.2. *The fibered category (Ell) is an algebraic stack.*

Let (E, e) be an elliptic curve over the scheme S . We know from §4.3 that e is a closed immersion, and that

$$\mathcal{J}_e := \ker[\mathcal{O}_E \rightarrow e_*\mathcal{O}_S]$$

is a line bundle on E . We define

$$\mathcal{O}_E(e) := \mathcal{J}_e^*.$$

One verifies immediately that $\mathcal{O}_E(e)$ is a line bundle of degree 1 (4.7.1) on E .

Recall that we defined \mathcal{M} (4.7.2) to be the fibered category over (Aff) of pairs (E, \mathcal{L}) consisting of a curve $\pi: E \rightarrow S$ of genus 1 and a line bundle \mathcal{L} on E of degree 3. There is a natural morphism of fibered categories

$$\alpha: (\text{Ell}) \rightarrow \mathcal{M},$$

defined as follows. Given an elliptic curve (E, e) , the line bundle

$$\mathcal{O}_E(3e) := \mathcal{O}_E(e)^{\otimes 3}$$

has degree 3 on E . We define α by sending $(E, e) \mapsto (E, \mathcal{O}_E(3e))$.

For our present purposes, we shall take it on faith that \mathcal{M} is an algebraic stack. To show that (Ell) is an algebraic stack, it then suffices, by (5.5.3), to show that α is representable.

We define the auxiliary fibered category $\tilde{\mathcal{M}}$, the “tautological curve of genus 1 over \mathcal{M} ,” to be the category of triples (E, \mathcal{L}, e) consisting of an elliptic curve (E, e) and a line bundle \mathcal{L} on E of degree 3. There are natural morphisms of fibered categories

$$(\text{Ell}) \xrightarrow{\sigma} \tilde{\mathcal{M}} \xrightarrow{\tau} \mathcal{M}$$

given by

$$\sigma: (E, e) \mapsto (E, \mathcal{O}_E(3e), e) \quad \text{and} \quad \tau: (E, \mathcal{L}, e) \mapsto (E, \mathcal{L}).$$

Evidently $\alpha = \tau\sigma$.

We claim that both σ and τ are representable, whence so is α . For τ , let S be an affine scheme, and consider a test diagram

$$\begin{array}{ccc} & S & \\ & \downarrow & \\ \tilde{\mathcal{M}} & \xrightarrow{\tau} & \mathcal{M}. \end{array}$$

By Yoneda’s lemma for fibered categories, the morphism $S \rightarrow \mathcal{M}$ is equivalent to the choice of an object (E, \mathcal{L}) in $\mathcal{M}(S)$. We leave it as a straightforward exercise to verify that E represents the fibered product $\tilde{\mathcal{M}} \times_{\mathcal{M}}^{\text{fc}} S$.

To show that α is representable, it now remains to show that σ is representable. The proof will occupy the rest of the subsection. We first have the following general exercise.

Exercise 5.6.3. Let \mathcal{D} and \mathcal{E} be fibered categories over \mathcal{C} . Let

$$\pi: \mathcal{D} \rightarrow \mathcal{E} \quad \text{and} \quad s: \mathcal{E} \rightarrow \mathcal{D}$$

be morphisms of fibered categories, and let $c: \pi s \xrightarrow{\sim} \text{id}_{\mathcal{E}}$ be a fixed isomorphism of functors in $\text{Hom}_{\mathcal{E}(\mathcal{E})}(\pi s, \text{id}_{\mathcal{E}})$. Let $c\pi$ denote the induced isomorphism $\pi s\pi \xrightarrow{\sim} \pi$. We get a weakly commuting diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{s} & \mathcal{D} \\ s \downarrow & & \downarrow (s\pi, \text{id}_{\mathcal{D}}, c\pi) \\ \mathcal{D} & \xrightarrow{\Delta} & \mathcal{D} \times_{\mathcal{E}}^{\text{fc}} \mathcal{D}. \end{array}$$

Show that the induced map $\mathcal{E} \rightarrow \mathcal{D} \times_{\mathcal{D} \times_{\mathcal{E}}^{\text{fc}} \mathcal{D}}^{\text{fc}} \mathcal{D}$ is an equivalence.

To apply the exercise, let $\rho: \tilde{\mathcal{M}} \rightarrow (\text{Ell})$ be the projection functor

$$(E, \mathcal{L}, e) \mapsto (E, e).$$

Then $\rho\sigma = \text{id}_{(\text{Ell})}$. So, to show that σ is representable, it suffices to show that the diagonal morphism $\Delta: \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}} \times_{(\text{Ell})}^{\text{fc}} \tilde{\mathcal{M}}$ is representable.

Recall from (4.2.1) that \mathcal{M}_1 is the fibered category of curves of genus 1 over (Aff) . There are natural projection morphisms

$$\mathcal{M} \rightarrow \mathcal{M}_1 \quad \text{and} \quad (\text{Ell}) \rightarrow \mathcal{M}_1$$

sending

$$(E, \mathcal{L}) \mapsto E \quad \text{and} \quad (E, e) \mapsto E,$$

respectively. One verifies immediately that the diagram

$$\begin{array}{ccc} \tilde{\mathcal{M}} & \xrightarrow{\rho} & (\text{Ell}) \\ \tau \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{M}_1 \end{array}$$

is cartesian. So, to show that the diagonal map associated to $\tilde{\mathcal{M}} \xrightarrow{\rho} (\text{Ell})$ is representable, it suffices to show that the diagonal map associated to $\mathcal{M} \rightarrow \mathcal{M}_1$ is representable.

We remark that the morphism $\mathcal{M} \rightarrow \mathcal{M}_1$ is not itself representable, but we will now see that its diagonal is. The proof is obtained from the following three lemmas. Throughout, let $S = \text{Spec } A$, and let $\pi: X \rightarrow S$ be a proper, finitely presented, and flat morphism of schemes, so that we may apply the proposition on cohomology and base change (4.4.2).

Lemma 5.6.4. *Let \mathcal{E} be a vector bundle on X . Given a morphism of affine schemes $T \rightarrow S$, let $X_T = X \times_S T$, and let \mathcal{E}_T be the pullback of \mathcal{E} to X_T . Then the presheaf of sets on $(\text{Aff})/S$ defined by*

$$(*) \quad (T \rightarrow S) \mapsto H^0(X_T, \mathcal{E}_T)$$

is represented by a finitely presented A -algebra.

Proof. By (4.4.2), there exists a finite complex $K^0 \rightarrow K^1 \rightarrow \cdots \rightarrow K^n$ of finitely generated projective A -modules representing $R\pi_*\mathcal{E}$. Define

$$M := \text{cok}[(K^1)^* \rightarrow (K^0)^*].$$

We claim that

$$\text{Spec}(\text{Sym}_A M)$$

represents the presheaf $(*)$. The proof consists of unwinding the definitions, and we skip it. \square

Lemma 5.6.5. *Let \mathcal{E} and \mathcal{F} be vector bundles on X . Then the presheaf of sets $\underline{\text{Hom}}(\mathcal{E}, \mathcal{F})$ on $(\text{Aff})/S$ defined by*

$$(T \rightarrow S) \mapsto \text{Hom}_{\mathcal{O}_{X_T}}(\mathcal{E}_T, \mathcal{F}_T)$$

is represented by a finitely presented A -algebra.

Proof. Apply the previous lemma to the vector bundle $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ on X . \square

Lemma 5.6.6. *Let \mathcal{E} and \mathcal{F} be rank n vector bundles on X for some $n \geq 1$. Then the presheaf of sets $\underline{\text{Isom}}(\mathcal{E}, \mathcal{F})$ on $(\text{Aff})/S$ defined by*

$$(T \rightarrow S) \mapsto \text{Isom}_{\mathcal{O}_{X_T}}(\mathcal{E}_T, \mathcal{F}_T)$$

is represented by a finitely presented A -algebra.

Proof. Write S for the representable presheaf $h_{\text{id}_S}: (T \xrightarrow{f} S) \mapsto \{f\}$. For any presheaf F on $(\text{Aff})/S$, to give a morphism of presheaves $S \rightarrow F$ is to give a map of sets

$$S(S \xrightarrow{\text{id}_S} S) = \{\text{id}_S\} \rightarrow F(S \xrightarrow{\text{id}_S} S),$$

which in turn is just to choose an element of $F(S \xrightarrow{\text{id}_S} S)$.

Now consider the diagram of presheaves and natural maps

$$\begin{array}{ccc} \underline{\text{Isom}}(\mathcal{E}, \mathcal{F}) & \longrightarrow & S \\ \downarrow & & \downarrow (\text{id}_{\mathcal{E}}, \text{id}_{\mathcal{F}}) \\ \underline{\text{Hom}}(\mathcal{E}, \mathcal{F}) \times \underline{\text{Hom}}(\mathcal{F}, \mathcal{E}) & \longrightarrow & \underline{\text{Hom}}(\mathcal{E}, \mathcal{E}) \times \underline{\text{Hom}}(\mathcal{F}, \mathcal{F}). \end{array}$$

It is immediate that the diagram is cartesian. Thus, using the previous lemma, $\underline{\text{Isom}}(\mathcal{E}, \mathcal{F})$ is a fibered product of presheaves represented by finitely presented A -algebras, hence is itself represented by a finitely presented A -algebra. \square

To complete the proof that $\mathcal{M} \rightarrow \mathcal{M} \overset{\text{fc}}{\times}_{\mathcal{M}_1} \mathcal{M}$ is representable, we must, by definition, show that for every affine scheme S and morphism $S \rightarrow \mathcal{M} \overset{\text{fc}}{\times}_{\mathcal{M}_1} \mathcal{M}$, the fibered product $\mathcal{F} := \mathcal{M} \overset{\text{fc}}{\times}_{\mathcal{M} \overset{\text{fc}}{\times}_{\mathcal{M}_1} \mathcal{M}} S$ is representable. By Yoneda's lemma, the morphism $S \rightarrow \mathcal{M} \overset{\text{fc}}{\times}_{\mathcal{M}_1} \mathcal{M}$ is equivalent to the choice of objects (E, \mathcal{L}) and (E', \mathcal{L}') in $\mathcal{M}(S)$ and an isomorphism $\varphi: E \rightarrow E'$ of S -schemes. Pulling back \mathcal{L}' , we get line bundles \mathcal{L} and $\varphi^*\mathcal{L}'$ on E . We leave it as an exercise to show that \mathcal{F} is equivalent to the fibered category obtained from $\underline{\text{Isom}}(\mathcal{L}, \varphi^*\mathcal{L}')$. This completes the proof of (5.6.2).

5.7. Quotients by GL_n . In this subsection, we discuss quotient stacks obtained from general linear groups acting on algebraic spaces.

In our discussion of curves of genus 1 in §4.8, we defined the fibered category $\widehat{\mathcal{M}}$ and saw that it is represented by an open subscheme $\mathbb{A}_{\text{sm}}^{10}$ of $\mathbb{A}_{\mathbb{Z}}^{10}$. We will see below that the natural morphism $\widehat{\mathcal{M}} \rightarrow \mathcal{M}$ gives an equivalence between the fibered category \mathcal{M} and the quotient stack $(\mathbb{G}_m \times GL_3) \backslash \mathbb{A}_{\text{sm}}^{10}$. So one of our goals will be to understand quotients obtained from products of general linear groups.

For simplicity, let $G = GL_n$. Then G is a group scheme defined over \mathbb{Z} . For any affine scheme $S = \text{Spec}(A)$,

$$GL_n(S) = GL_n(A) = \{ M \in \text{Mat}_{n \times n}(A) \mid \det M \in A^\times \}.$$

In this subsection, we write $*$ for $\text{Spec}(\mathbb{Z})$. The point is that $\text{Spec}(\mathbb{Z})$ is a terminal object in (Aff), and here we think of it as a point, in analogy with (Top).

Definition 5.7.1. Let S be an affine scheme. We define $(G \backslash *) (S) = (GL_n \backslash *) (S)$ to be the category of rank n vector bundles over S and vector bundle isomorphisms over S . We use the usual pullback functors to make $G \backslash *$ into a presheaf of categories, hence a fibered category, on (Aff).

The notation $G \backslash *$ in the definition is incompatible with that of (1.6.6). We shall not use the notation of (1.6.6) in this subsection.

Proposition 5.7.2. *The fibered category $G \backslash *$ is an algebraic stack.*

Proof. We check the four conditions in the definition (5.4.1). By definition, $G \backslash *$ is a category fibered in groupoids. Since a vector bundle for the étale topology is automatically a vector bundle for the Zariski topology, we get that $G \backslash *$ is a stack for the étale topology.

To check that the diagonal $G \backslash * \rightarrow (G \backslash *) \overset{\text{fc}}{\times} (G \backslash *)$ is representable, it suffices to check that all $\mathcal{I} \text{som}$ presheaves are representable. Fix an affine scheme S , and let \mathcal{E} and \mathcal{F} be rank n vector bundles over S . Then $\mathcal{I} \text{som}(\mathcal{E}, \mathcal{F})$ sends

$$(T \xrightarrow{\pi} S) \mapsto \text{Isom}(\pi^* \mathcal{E}, \pi^* \mathcal{F}),$$

and

$$\text{Isom}(\pi^* \mathcal{E}, \pi^* \mathcal{F}) \subset \text{Hom}(\pi^* \mathcal{E}, \pi^* \mathcal{F}) = \pi^* \text{Hom}(\mathcal{E}, \mathcal{F}).$$

But $\text{Hom}(\mathcal{E}, \mathcal{F})$ is a vector bundle of rank n^2 , and one can check that isomorphisms sit inside as an open subscheme.

As a presentation for $G \backslash *$, consider the natural map

$$\mathcal{O}_*^n : * = (\text{Aff}) / \text{Spec}(\mathbb{Z}) = (\text{Aff}) \rightarrow G \backslash *$$

sending each affine scheme S to the trivial vector bundle \mathcal{O}_S^n . Given an affine scheme S , a morphism of fibered categories $S \rightarrow G \backslash *$ is equivalent, by Yoneda’s lemma, to choosing a rank n vector bundle \mathcal{E} on S . We get a cartesian diagram

$$\begin{array}{ccc} P_{\mathcal{E}} & \longrightarrow & S \\ \downarrow & & \downarrow \varepsilon \\ * & \xrightarrow{\mathcal{O}_*^n} & G \backslash *, \end{array}$$

where $P_{\mathcal{E}} := * \overset{\text{fc}}{\times}_{G \backslash *} S$.

To describe $P_{\mathcal{E}}$, note that the diagram

$$\begin{array}{ccc} P_{\mathcal{E}} & \longrightarrow & * \overset{\text{fc}}{\times} S \\ \downarrow & & \downarrow \mathcal{O}_*^n \times \mathcal{E} \\ G \backslash * & \xrightarrow{\Delta} & (G \backslash *) \overset{\text{fc}}{\times} (G \backslash *) \end{array}$$

is cartesian. Therefore $P_{\mathcal{E}}$ is represented by the sheaf on $(\text{Aff})/S$

$$\text{Isom}(\mathcal{O}_S^n, \mathcal{E}): (T \xrightarrow{\pi} S) \mapsto \text{Isom}_{\mathcal{O}_T}(\mathcal{O}_T^n, \pi^* \mathcal{E}).$$

Thus $P_{\mathcal{E}}$ admits a natural G -action over S , and $P_{\mathcal{E}}$ is the principal G -bundle over S associated to \mathcal{E} .

Zariski locally on S , we have $\mathcal{E} \cong \mathcal{O}_S^n$, whence $P_{\mathcal{E}} \cong S \times G$ is the trivial G -bundle. So, gluing, we see that $P_{\mathcal{E}}$ is a scheme in general. Moreover, since G is smooth and surjective over $*$, we get that $S \times G \rightarrow S$ is smooth and surjective for any S . Since smoothness and surjectivity are local on the base, it follows that $P_{\mathcal{E}} \rightarrow S$ is smooth and surjective. Thus the morphism $* \rightarrow G \backslash *$ is a presentation. \square

In keeping with the notation of the proof, given a rank n vector bundle \mathcal{E} on the affine scheme S , we write $P_{\mathcal{E}}$ for the associated principal G -bundle. The proof also justifies the following definition.

Definition 5.7.3. Let \mathcal{D} be a category fibered in groupoids over (Aff) . A *vector bundle of rank n* on \mathcal{D} is a morphism of fibered categories $\mathcal{D} \rightarrow G \backslash *$.

Thus we think of $G \backslash *$ as the “classifying stack” BG , and the morphism

$$* \xrightarrow{\mathcal{O}_*^n} G \backslash *$$

as the “universal G -bundle.”

Now let X be any algebraic space on which G acts. We generalize (5.7.1) as follows.

Definition 5.7.4. We define $G \backslash X$ to be the fibered category obtained from the following presheaf on (Aff) . For each affine scheme S , we let $(G \backslash X)(S)$ be the groupoid of pairs (\mathcal{E}, x) consisting of a rank n vector bundle \mathcal{E} on S and a G -equivariant morphism $x: P_{\mathcal{E}} \rightarrow X$. We take the obvious morphisms and pullback functors.

As a special case, fix an affine scheme S , and consider a pair (\mathcal{E}, x) with $\mathcal{E} = \mathcal{O}_S^n$. Then $P_{\mathcal{E}} = G \times S$. A G -equivariant map $x: G \times S \rightarrow X$ determines a map $x_1: S \rightarrow X$ by restricting x to $1_G \times S$. Conversely, any map $x_1: S \rightarrow X$ determines a unique G -equivariant map $G \times S \rightarrow X$, as is easy to see by viewing G , S , and X as presheaves.

Let $g \in G(S)$. Then (\mathcal{O}_S^n, x_1) and (\mathcal{O}_S^n, gx_1) are isomorphic objects in $G \backslash X$. By only considering trivial vector bundles, we’d have a separated fibered category. Allowing arbitrary vector bundles amounts to stackifying.

Proposition 5.7.5. *The fibered category $G \backslash X$ is an algebraic stack.*

Proof. We give a sketch. Given any algebraic G -space Y and a G -equivariant morphism $X \rightarrow Y$, we get a morphism of fibered categories $G \backslash X \rightarrow G \backslash Y$. In particular, we always have a morphism $X \rightarrow *$, where G acts trivially on $*$. Thus

we get a morphism $f: G \backslash X \rightarrow G \backslash *$. By (5.7.2), $G \backslash *$ is an algebraic stack. So, by (5.5.3), it suffices to show that f is representable.

Speaking loosely, the fibers of f are just X , so we certainly expect f to be representable. In fact, the fibers are certain twists of X .

Let S be an affine scheme and \mathcal{E} a rank n vector bundle on S . We get a diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & S \\ \downarrow & & \downarrow \mathcal{E} \\ G \backslash X & \xrightarrow{f} & G \backslash *, \end{array}$$

where \mathcal{F} is the fibered product $(G \backslash X) \times_{G \backslash *}^{\text{fc}} S$. In the special case $\mathcal{E} = \mathcal{O}_S^n$, one can show that \mathcal{F} is represented by the algebraic S -space (or scheme, if X is a scheme) $X \times S \rightarrow S$. But \mathcal{E} is trivial Zariski locally on S , so, gluing, we see that \mathcal{F} is always represented by an algebraic space (or a scheme, if X is a scheme). In general, $\mathcal{F} = G \backslash (X \times P_{\mathcal{E}})$. \square

So far, we’ve always taken $G = GL_n$. Now suppose G is a product of general linear groups. Then we define $G \backslash *$ and $G \backslash X$ exactly as above, only with 1 vector bundle for each factor in the product. We omit the details.

We next come to the quotient recognition problem. How do we identify an algebraic stack \mathcal{X} as being of the form $G \backslash X$ for some X ? We take the following as necessary conditions.

- (1) \mathcal{X} is a category fibered in groupoids over (Aff).
- (2) \mathcal{X} is a stack in the Zariski topology (here the Zariski topology suffices, since we’re interested in vector bundles).
- (3) \mathcal{X} comes equipped with a morphism $\mathcal{E}: \mathcal{X} \rightarrow G \backslash *$ such that the fibered product \mathcal{F} in the diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & * \\ \downarrow & & \downarrow \mathcal{O}_*^n \\ \mathcal{X} & \xrightarrow{\mathcal{E}} & G \backslash * \end{array}$$

is represented by an algebraic space X . Note that there is a natural G -action on \mathcal{F} , hence on X .

It turns out that these conditions are sufficient as well. We omit the proof.

Proposition 5.7.6. *Let \mathcal{X} be a fibered category such that the above three conditions hold. Then \mathcal{X} is equivalent to $G \backslash X$.* \square

Example 5.7.7. Recall that \mathcal{M} is the fibered category of pairs (E, \mathcal{L}) consisting of a curve $\pi: E \rightarrow S$ of genus 1 and a line bundle \mathcal{L} of degree 3 on E (4.7.2). We saw in §4.7 that $\pi_* \mathcal{L}$ is a rank 3 vector bundle on S . In addition, we defined a certain line bundle \mathcal{N} on the projective space bundle $\mathbb{P}(\pi_* \mathcal{L})$, and we saw that the pushforward \mathcal{N}_S of \mathcal{N} to S is a line bundle on S . Let $G = \mathbb{G}_m \times GL_3$. We get a morphism $\mathcal{M} \rightarrow G \backslash *$ sending

$$(E, \mathcal{L}) \mapsto (\mathcal{N}_S, \pi_* \mathcal{L}).$$

Recall that we defined the related fibered category $\widehat{\mathcal{M}}$ in (4.8.1), and we stated that $\widehat{\mathcal{M}}$ is represented by a certain scheme $\mathbb{A}_{\text{sm}}^{10}$ (4.8.2 and 4.8.3). There is a natural

morphism $\widehat{\mathcal{M}} \rightarrow \mathcal{M}$. It is an easy exercise to verify that the diagram

$$\begin{array}{ccc} \widehat{\mathcal{M}} & \longrightarrow & * \\ \downarrow & & \downarrow \mathcal{O}_* \times \mathcal{O}_*^3 \\ \mathcal{M} & \longrightarrow & G \backslash * \end{array}$$

is cartesian. Thus \mathcal{M} is equivalent to $G \backslash \mathbb{A}_{\text{sm}}^{10}$, as claimed.

Example 5.7.8. There is a natural action of GL_n on the scheme of $n \times n$ matrices $M_n \cong \mathbb{A}^{n^2}$. An object in the quotient stack $GL_n \backslash M_n$ over the affine scheme S consists of a rank n vector bundle \mathcal{E} on S equipped with an endomorphism $\mathcal{E} \rightarrow \mathcal{E}$.

Example 5.7.9. We construct the stack \mathcal{X} given by sending each affine scheme S to the groupoid of “finite flat” schemes of order n over S . To be more precise, say $S = \text{Spec}(A)$. Then an object of $\mathcal{X}(S)$ is a scheme $X \rightarrow S$, affine over S and so of the form $\text{Spec}(B)$, such that B is a projective A -module of rank n .

The essential case is when $B = A^n$ as an A -module. To place an A -algebra structure on B , the multiplication law is given by a map $A^n \otimes_A A^n \rightarrow A^n$, and the identity element by a map $A \rightarrow A^n$. The space of all such data is contained in \mathbb{A}^{n^3+n} . The relations required by the commutative, associative, identity, and distributive laws give a closed subscheme X of \mathbb{A}^{n^3+n} . Thus \mathcal{X} is given by $GL_n \backslash X$; dividing allows us to take any rank n vector bundle, not just A^n .

Example 5.7.10. Similarly, we can define the stack of finite flat group schemes of order n over affine schemes. This time, the various necessary maps and relations give a closed subscheme $X' \subset \mathbb{A}^{2(n^3+n)}$, and we realize the stack as $GL_n \backslash X'$.