

PRODUCTS OF CONSECUTIVES THAT ARE SQUARES.

A mathematical vignette

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0. This investigation is designed to allow students to obtain a feel for how numbers interact together and also to illustrate the various dimensions of algebraic usage. The prerequisites are slight – just some familiarity with integer multiplication and squares and the rudiments of Grade 9 algebra. This is presented in the spirit of providing a topic that may be helpful in promoting student understanding, but in no way as a recipe for how it may be presented. The actual discussion can proceed in a number of different ways, particularly if some students asks a question, has a difficulty or makes an insightful comment. Depending on the class, it may be a matter of open-ended investigation, or the teacher may have to explicitly state a problem and then offer general guidance on how it might be pursued. There is nothing sacred about the order of presentation; some teachers may find it preferable to deal first with products of two consecutive integers.

1. A result that lends itself to investigation by secondary students is the proposition that the product of four consecutive positive integers is never square. It can be approached in a number of ways, and depending on the group, questions arise about mathematical practice that can be explored.

The result can be approached by either stating the proposition as a statement to be proved, or by asking students to write out a number of such products to see what they observe. The first few instances are

$$1 \times 2 \times 3 \times 4 = 24;$$

$$2 \times 3 \times 4 \times 5 = 120;$$

$$3 \times 4 \times 5 \times 6 = 360;$$

$$4 \times 5 \times 6 \times 7 = 840;$$

$$5 \times 6 \times 7 \times 8 = 1680.$$

Where the students go from here depends on the ability of at least some of them to recognize properties and patterns. Here the observation might be that the products are each one less than the respective squares of 5, 11, 19, 29, 41. (They might recognize only the first two and have to conjecture the others.) If we throw in $0 \times 1 \times 2 \times 3 = 0$, we get a product that is one less than the square of 1.

Recognizing that there is a pattern, we might try to see if there is a general formula for the product

$$n \times (n + 1) \times (n + 2) \times (n + 3).$$

How students arrive at a suitable formula will depend on their mathematical insight and experience. A knowledgeable student will recognize that the differences for the

sequence 1, 5, 11, 19, 29, 41 form an arithmetic progression and interpolate a quadratic function. The typical high school class will not have access to this (although this situation might be a good place to introduce it).

One insightful observation is that $1 = 2^2 - 3$, $5 = 3^2 - 4$, $11 = 4^2 - 5$, $19 = 5^2 - 6$, $29 = 6^2 - 7$, $41 = 7^2 - 8$. This leads to the conjecture that

$$n \times (n + 1) \times (n + 2) \times (n + 3) = [(n + 2)^2 - (n + 3)] - 1 = [n^2 + 3n + 1]^2 - 1.$$

This is readily checked. Here we see two roles of algebra, the first as a notation to describe a general pattern, and the second as a proof technique for a general result with infinitely many instances.

At this point, students who have been given the proposition will jump to the conclusion that, because the product is one less than a perfect square, it cannot itself be a square. Otherwise, the teacher may have to ask whether it is possible for the product itself to be a square.

This focusses now on the question as to whether two positive squares can differ by 1. A popular approach is for the students to observe from the sequence $\{1, 4, 9, 16, \dots\}$ that two *consecutive* squares cannot differ by 1. How can one make this precise? Again algebra can be used: $(n + 1)^2 - n^2 = 2n + 1 > 1$. Fine; we have the result for consecutive positive squares. But what about *any* two positive squares? This may be splitting hairs, but it is worth having students attempt to give a succinct explicit argument.

An alternative argument starts with the equation $x^2 - y^2 = 1$ to be solved for integers x and y with $x > y$, and exploits the factorization of a difference of squares along with the solution of a simple pair of simultaneous equation, both early topics for a beginning algebra student. The equation can be rewritten

$$(x - y)(x + y) = 1,$$

the integer 1 as a product of two positive integers. The only possibility is $x + y = x - y = 1$ which forces $(x, y) = (1, 0)$. Thus, there is no solution with both x and y positive.

Returning to the fourfold product, we can draw out information by a strategic rearrangement of terms. This is a common enough process in school algebra that students should be tuned into this possibility. We can write the general product as

$$\begin{aligned} n(n + 1)(n + 2)(n + 3) &= [n(n + 3)][(n + 1)(n + 2)] = [n^2 + 3n][n^2 + 3n + 2] \\ &= [(n^2 + 3n + 1) - 1][(n^2 + 3n + 1) + 1] \\ &= [n^2 + 3n + 1]^2 - 1. \end{aligned}$$

One possible motivation for this bit of legerdemain might be looking at small numerical cases and noting that $1 \times 2 \times 3 \times 4 = 4 \times 6$, $2 \times 3 \times 4 \times 5 = 10 \times 12$, and generalizing.

3. Staying with the fourfold products, we see that there is an interesting connection with Pascal's triangle. For each integer greater than or equal to 4, there

exists a second integer m for which

$$\binom{m}{2} = 3\binom{n}{4}.$$

This could be discovered empirically by examining the entries of Pascal's triangle, or else it could be stated and the students asked to prove it by solving an equation for m . In fact, it turns out that

$$m = \binom{n-1}{2}.$$

Now we have that

$$\begin{aligned} n(n-1)(n-2)(n-3) + 1 &= 24\binom{n}{4} + 1 = 8\binom{m}{2} \\ &= 4m(m-1) + 1 = (2m-1)^2 = [(n-1)(n-2) - 1]^2 \\ &= (n^2 - 3n + 1)^2. \end{aligned}$$

If we replace n by $n+3$ in this equation, then we find that

$$(n+3)(n+2)(n+1)n + 1 = (n^2 + 3n + 1)^2,$$

as before.

3. Having disposed of this situation, students could be asked to investigate a number of questions: (a) can the product of two consecutive positive integers ever be a perfect square? (b) can the product of three consecutive positive integers ever be a perfect square? (c) is there a value of k such that some product of k consecutive positive integers is a perfect square? if so, what is the smallest such k ?

Here are the points that might be drawn out for (a). Two consecutive positive integers are coprime (*i.e.* have greatest common divisor 1), so that if their product is square, then each of them must be square. One way of getting at this is to consider the prime factor decomposition and note that squares are characterized by the fact that all their prime divisors do so to an even exponent.

Alternatively, we can note that $n(n+1)$ lies between two consecutive squares, and so cannot be square. This is straightforward, since

$$n^2 < n(n+1) = n^2 + n < (n+1)^2.$$

Another approach takes us into territory we have already visited. Any number x is a square if and only if $4x$ is also a square. Since

$$4n(n+1) = (2n+1)^2 - 1,$$

and two positive squares cannot differ by 1, then $4n(n+1)$ and with it $n(n+1)$ cannot be square.

Because $n(n+1)(n+2)$ is a polynomial of odd degree in n , there is no obvious way to approach the possibility of its being square using elementary algebra. However, we can observe that any odd prime can divide at most one of the three factors, and so must do so to an even degree. Furthermore, when n is odd, the prime 2 can

divide only the middle factor, so that the three factors are pairwise coprime. Thus, if the triple product is square in this case, each of its three factors must also be square, an impossibility.

However, if n is even, then 2 divides both n and $n+2$ and divides exactly one of these to the first power. The bottom line here is that both $n+1$ and $n(n+2)$ are squares. Using the fact that $n(n+2) = (n+1)^2 - 1$, we again find that the triple product cannot be a positive square.

As for (c), this is a deeper problem to be explored by students, to see what properties a square which is the product of k consecutive positive integers must have.