THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

March 11, 2007

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

It is not necessary to do all the problems. Complete solutions to fewer problems are preferred to partial solutions to many.

1. A $m \times n$ rectangular array of distinct real numbers has the property that the numbers in each row increase from left to right. The entries in each column, individually, are rearranged so that the numbers in each column increase from top to bottom. Prove that it the final array, the numbers in each row will increase from left to right.

2. Determine distinct positive integers a, b, c, d, e such that the five numbers a, b^2, c^3, d^4, e^5 constitute an arithmetic progression. (The difference between adjacent pairs is the same.)

3. Prove that the set $\{1, 2, \cdots, n\}$ can be partitioned into k subsets with the same sum if and only if k divides $\frac{1}{2}n(n+1)$ and $n \geq 2k - 1$.

4. Suppose that $f(x)$ is a continuous real-valued function defined on the interval $[0, 1]$ that is twice differentiable on $(0, 1)$ and satisfies (i) $f(0) = 0$ and (ii) $f''(x) > 0$ for $0 < x < 1$.

   (a) Prove that there exists a number a for which $0 < a < 1$ and $f'(a) < f(1)$;

   (b) Prove that there exists a unique number b for which $a < b < 1$ and $f'(a) = f(b)/b$.

5. For $x \leq 1$ and $x \neq 0$, let
$$f(x) = \frac{-8[1 - (1 - x)^{1/2}]^3}{x^2} \ .$$

   (a) Prove that $\lim_{x \to 0} f(x)$ exists. Take this as the value of $f(0)$.

   (b) Determine the smallest closed interval that contains the set of all values assumed by $f(x)$ on its domain.

   (c) Prove that $f(f(f(x))) = f(x)$ for all $x \leq 1$.

6. Let $h(n)$ denote the number of finite sequences $\{a_1, a_2, \cdots, a_k\}$ of positive integers exceeding 1 for which $k \geq 1$, $a_1 \geq a_2 \geq \cdots \geq a_k$ and $n = a_1 a_2 \cdots a_k$. (For example, if $n = 20$, there are four such sequences $\{20\}$, $\{10, 2\}$, $\{5, 4\}$ and $\{5, 2, 2\}$ and $h(20) = 4$.

   Prove that
$$\sum_{n=1}^{\infty} \frac{h(n)}{n^2} = 1 \ .$$

7. Find the Jordan canonical form of the matrix $\mathbf{u}\mathbf{v}^{\mathbf{t}}$ where $\mathbf{u}$ and $\mathbf{v}$ are column vectors in $\mathbf{C}^n$. (The superscript $\mathbf{t}$ denotes the transpose.)

8. Suppose that n points are given in the plane, not all collinear. Prove that there are at least n distinct straight lines that can be drawn through pairs of the points.

9. Which integers can be written in the form

$$\frac{(x+y+z)^2}{xyz}$$

where x, y, z are positive integers?

10. Solve the following differential equation

$$2y' = 3|y|^{1/3}$$

subject to the intial conditions

$$y(-2) = -1 \qquad \text{and} \qquad y(3) = 1 \ .$$

Your solution should be everywhere differentiable.

1. A $m \times n$ rectangular array of distinct real numbers has the property that the numbers in each row increase from left to right. The entries in each column, individually, are rearranged so that the numbers in each column increase from top to bottom. Prove that it the final array, the numbers in each row will increase from left to right.

Solution 1. We prove this by induction. Note that the permutation yielding the final arrangement of each column is uniquely determined, so that if we can perform a sequence of transposition (switches) resulting in the entries increasing from top to bottom, the composite of these transpositions is the required permutation.

We can arrange the rows so that the first column is increasing from top to bottom; all the rows will still be increasing from left to right. Suppose that we have performed a sequence of rearrangements of sets of columns so that (a) each row is increasing, and (b) the first k columns are increasing for $1 \leq k \leq n-1$. Let the entries in the kth column be $a_1, a_2, \cdots, a_m$ and in the $(k+1)$th column be $b_1, b_2, \cdots, b_m$. We have that $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_m$ and $a_i \leq b_i$ for $1 \leq i \leq m$.

Suppose that b_r is the minimum of all the b_i $(1 \leq i \leq m)$. We interchange the elements in the first and rth rows of the jth column for $k+1 \leq j \leq n$. Since $a_1 \leq a_r \leq b_r$ and $a_r \leq b_r \leq b_1$, the first and rth rows are still increasing.

Let b_s be the minimum of all the b_i except for b_r; interchange the elements in the second and sth rows of the jth column for $k+1 \leq j \leq n$. Since $a_2 \leq a_s \leq b_s$ and $a_s \leq b_s \leq b_2$, the new second and sth rows are increasing. Observe that $b_r \leq b_s \leq b_i$ for $i \neq r, s$, so that the $(k+1)$th column is increasing down to the third entry.

We can continue in this way, moving the third smallest b_i to the third row, and so on, ending up with changing the order of the columns from the $(k+1)$th to the nth and keeping the rows increasing. The result follows by induction on k.

Solution 2. [M. Cai] Let a_{ij} be the (i,j)th entry in the array before the rearrangement and b_{ij} and (i,j)th entry after the rearrangement. Then

$$b_{1j} \leq b_{2j} \leq \cdots \leq b_{mj}$$

for $1 \leq j \leq n$. We need to show that, for each i with $1 \leq i \leq m$,

$$b_{i1} \leq b_{i2} \leq \cdots \leq b_{in} \ .$$

Let $1 \leq j \leq n-1$. For $1 \leq r \leq m$, we have that

$$b_{m,j+1} \geq a_{r,j+1} \quad \text{and} \quad a_{rj} \leq a_{r,j+1} \ .$$

There exist s with $1 \leq s \leq m$ for which $b_{mj} = a_{sj}$. Hence, for $1 \leq j \leq n-1$,

$$b_{mj} = a_{sj} \leq a_{s,j+1} \leq b_{m,j+1} \ .$$

Suppose, as an induction hypothesis, it has been established that $b_{ij} \leq b_{i,j+1}$ for $2 \leq k+1 \leq i \leq m$ and $1 \leq j \leq n-1$. Then $b_{kj} = a_{tj} \leq a_{t,j+1}$ for some t with $1 \leq t \leq m$. Since the set of numbers $\{b_{i,j+1} : k \leq i \leq m\}$ is the set of the largest $m-k+1$ numbers of the set $\{a_{i,j+1} : 1 \leq i \leq m\}$, we must have that $a_{t,j+1} \leq b_{t,j+1}$. The result follows.

Solution 3. There is nothing to prove when $n = 1$. Suppose that $n \geq 2$, that a given column, not the last, is $(a_1, a_2, \cdots, a_m)$ to begin with and $(b_1, b_2, \cdots, b_m)$ after the rearrangement, so that $b_1 \leq b_2 \leq \cdots \leq b_m$. Let the elements in a column to the right of this be $(c_1, c_2, \cdots, c_m)$ to begin with and $(d_1, d_2, \cdots, d_m)$ after the rearrangement. We have that $d_1 \leq d_2 \leq \cdots \leq d_m$ and $a_i \leq c_i$ for $1 \leq i \leq m$. We need to show that $c_k \leq d_k$ for $1 \leq k \leq m$.

Since b_k is the kth largest number in the given column to the left, there are $n-k+1$ elements in that column not less than it. Hence there are at least $n-k+1$ elements in the column to the right that are not less than b_k, and these elements include $d_k, d_{k+1}, \cdots, d_m$. Hence $b_k \leq d_k$.

2. Determine distinct positive integers a, b, c, d, e such that the five numbers a, b^2, c^3, d^4, e^5 constitute an arithmetic progression. (The difference between adjacent pairs is the same.)

Solution 1. One example is obtained by taking the arithmetic progression $(1, 9, 17, 25, 33)$ and multiplying by $3^{24}5^{30}11^{24}17^{20}$ to obtain

$$(a, b, c, d, e) = (3^{24}5^{30}11^{24}17^{20}, 3^{13}5^{15}11^{12}17^{10}, 3^{8}5^{10}11^{8}17^{7}, 3^{6}5^{8}11^{6}17^{5}, 3^{5}5^{6}11^{5}17^{4}) .$$

Solution 2. [G. Siu] Let

$$(a, b, c, d, e) = (33 \times 97^{24} \times 65^{20}, 7 \times 97^{12} \times 65^{10}, 65^{7} \times 97^{8}, 3 \times 97^{6} \times 65^{5}, 97^{5} \times 65^{4}) .$$

Then

$$(a, b^2, c^3, d^4, e^5) = (33k, 49k, 65k, 81k, 97k)$$

where $k = 65^{20} \times 97^{24}$.

Comment. Two solvers found the loophole that the arithmetic progression itself could be constant, and gave the example $(a, b, c, d, e) = (n^{60}, n^{30}, n^{20}, n^{15}, n^{12})$ for an integer $n \geq 2$.

3. Prove that the set $\{1, 2, \cdots, n\}$ can be partitioned into k subsets with the same sum if and only if k divides $\frac{1}{2}n(n+1)$ and $n \geq 2k-1$.

Solution. The necessity of the conditions follows from the fact that the sum of all the numbers in the set is $\frac{1}{2}n(n+1)$ and there is at most one subset with a single element; hence the number k of subsets is at most $\frac{1}{2}(n-1)+1$. [*Alternatively:* Since n must lie in one of the subsets, the sum of each subset is at least n, and so $kn \leq \frac{1}{2}n(n+1)$. Therefore $n \geq 2k-1$.]

On the other hand, suppose that the conditions obtain. If $k = 1$, then the result holds for every positive integer n. Assume, as an induction hypothesis it holds for all numbers of subsets up to $k-1$ and relevant n. Consider the case where there are k subsets. Let $n = 2k-1$ and let $s = n(n+1)/(2k) = 2k-1$. Then a partition of the required type consists of the singleton $\{2k-1\}$ and the $k-1$ pairs $\{2k-1-i, i\}$ with $1 \leq i \leq k-1$. If $n = 2k$, then $s = n(n+1)/(2k) = 2k+1$ and we can use the k pairs $\{2k+1-i, i\}$ with $1 \leq i \leq k$. We use induction to establish the result for $n > 2k$.

Suppose that $2k < n < 4k-1$ and that $s = n(n+1)/(2k)$ is an integer. We have that $2 < n/k < (n+1)/k < 4$ so that

$$n+1 = \frac{2ks}{n} < s = 2n\left[\frac{n+1}{4k}\right] < 2n .$$

Suppose that $n' = s-n-1$, so that $0 < n' < n+1$.

First, let s be odd. The sum of the numbers from 1 to $s-n-1$ inclusive is equal to

$$\frac{(s-n)(s-n-1)}{2} = \frac{s^2-(2n+1)s}{2} + \frac{n(n+1)}{2} = s\left[\frac{s-1}{2} - n + k\right] .$$

Let $k' = \frac{1}{2}(s-1)$; note that $k' < k$ and that

$$n' - (2k'-1) = (s-n-1) - (s-1-2n+2k) = n-2k > 0 ,$$

so that $n' > 2k' - 1$. By the induction hypothesis, we can partition $\{1, 2, \cdots, s-n-1\}$ into k' subsets, each of whose sum is s. Let $r = \frac{1}{2}(s-1)$, and adjoin the $n - r = k - k'$ doubletons $\{n, s-n\}, \{n-1, s-n+1\}, \cdots, \{r+1, r\}$ to get the desired partition of $\{1, 2, \cdots, n\}$ into k subsets with the same sum s.

Now let s be even. Then

$$\frac{(s-n)(s-n-1)}{2} = \frac{s}{2}[s-1-2n+2k] \ .$$

Let $k' = 2k - 2n + s - 1$ and $s' = s/2$. Since $s < 2(n-1)$, $2s' = s > 2(s-n-1) = 2n'$. By the induction hypothesis, we can partition the set $\{1, 2, \cdots, s-n-1\}$ into k' subsets whose sums are all s'. Since k' is odd and since $s' > s-n-1$, we can augment this family of subsets by the singleton $\{s'\}$ to get evenly many subsets with sum s'. Pair them off to get $\frac{1}{2}(k'+1) = k - n + s'$ subsets with sum s, and further augment the family with the $n - s'$ doubletons $\{n, s-n\}$, $\{n-1, s-n+1\}$ $\cdots$, $\{s'+1, s'-1\}$ to get a partition of $\{1, 2, \cdots, n\}$ into k sets with sum s.

Finally, suppose that $n \geq 4k - 1$. Let $n' = n - 2k$, $k' = k$. Then, if $s = n(n+1)/(2k)$,

$$\frac{(n-2k)(n-2k+1)}{2k} = \frac{n(n+1)}{2k} - (2n+1) - 2k = s - (2n - 2k + 1) > 0 \ .$$

Note that $n' = n - 2k \geq 2k - 1 = 2k' - 1$. Determine a partition of $\{1, 2, \cdots, n-2k\}$ into k subsets with sum s', and adjoin to these subsets the k doubletons

$$\{n, n-2k+1\}, \{n-1, n-2k+2\}, \cdots, \{n-k+1, n-k\} \ .$$

The whole result is now established.

Comment. The case $2k < n < 4k - 1$ with k a divisor of $\frac{1}{2}n(n+1)$ is not realizable for $2 \leq k \leq 4$. For $k = 5$, there are two possibilities, where k', n', s' are determined as in the solution:

$$(k, n, s) = (5, 14, 21); (k', n', s') = (1, 6, 21)$$

with the partition

$$\{1, 2, 3, 4, 5, 6\}, \{14, 7\}, \{13, 8\}, \{12, 9\}, \{11, 10\} \ ,$$

and

$$(k, n, s) = (5, 15, 24); (k', n', s') = (3, 8, 12)$$

with the partition

$$\{1, 5, 6; 2, 3, 7\}, \{4, 8; 12\}, \{15, 9\}, \{14, 10\}, \{13, 11\} \ .$$

For $k = 6$, there is one possibility:

$$(k, n, s) = (6, 15, 20); (k', n', s') = (1, 4, 10)$$

with the partition

$$\{1, 2, 3, 4; 10\}, \{15, 5\}, \{14, 6\}, \{13, 7\}, \{12, 8\}, \{11, 9\} \ .$$

4. Suppose that $f(x)$ is a continuous real-valued function defined on the interval $[0, 1]$ that is twice differentiable on $(0, 1)$ and satisfies (i) $f(0) = 0$ and (ii) $f''(x) > 0$ for $0 < x < 1$.

   (a) Prove that there exists a number a for which $0 < a < 1$ and $f'(a) < f(1)$;

   (b) Prove that there exists a unique number b for which $a < b < 1$ and $f'(a) = f(b)/b$.

Solution. (a) By the Mean Value Theorem, there exists $c \in (0, 1)$ for which $f'(c) = f(1)$. Since $f'(x)$ is increasing, when $0 < a < c$, $f'(a) < 1$.

(b) Let $g(x) = f(x) - xf'(a)$. Then $g(0) = 0$ and $g'(x) = f'(x) - f'(a)$. For $0 < u < a < v < 1$, $g'(u) < 0 < g'(v)$ (since $f'(x)$ increases). Therefore $g(a) < 0$ and $g(1) > 0$. Hence there is a unique number b for which $g(b) = 0$, and the result follows.

5. For $x \leq 1$ and $x \neq 0$, let

$$f(x) = \frac{-8[1-(1-x)^{1/2}]^3}{x^2} .$$

(a) Prove that $\lim_{x\to 0} f(x)$ exists. Take this as the value of $f(0)$.

(b) Determine the smallest closed interval that contains the set of all values assumed by $f(x)$ on its domain.

(c) Prove that $f(f(f(x))) = f(x)$ for all $x \leq 1$.

Solution. Suppose that $x = 1 - t^2$ for $t \geq 0$ and let $g(t) = f(1-t^2)$. Then

$$g(t) = \frac{-8(1-t)^3}{(1-t)^2(1+t)^2} = \frac{8(t-1(}{(t+1)^2} = \frac{8}{(t+1)} - \frac{16}{(t+1)^2} .$$

(a) Since $g(1) = 0$, its follows that $\lim_{x\to 0} g(x) = 0$.

(b) Since

$$0 \leq \left(1 - \frac{4}{t+1}\right)^2 = \left[1 - \left(\frac{8}{t+1} - \frac{16}{(t+1)^2}\right)\right] ,$$

$g(t)$ assumes its maximum value of 1 at $t = 3$. Indeed,

$$g'(t) = -8(t+1)^{-2} + 32(t+1)^{-3} = 8(t+1)^{-2}(3-t) ,$$

so that $g(t)$ increases from -8 at $t = 0$ to its maximum at $t = 3$ and then decreases with limit 0 at t tends to infinity. Therefore the smallest closed interval containing the image of f is $[-8, 1]$. Observe that this interval gets mapped onto itself one-to-one.

(c) Let $x = 8(t-1)(t+1)^{-2}$. Then

$$1 - x = (t-3)^2(t+1)^{-2} ,$$

so that

$$(1-x)^{1/2} = \begin{cases} \frac{3-t}{1+t}, & \text{if } 0 \leq t \leq 3; \\ \frac{t-3}{1+t}, & \text{if } t > 3. \end{cases}$$

and

$$1 - (1-x)^{1/2} = \begin{cases} \frac{2(t-1)}{t+1,} & \text{if } 0 \leq t \leq 3; \\ \frac{4}{t+1}, & \text{if } t > 3. \end{cases}$$

Thus

$$f\left[\frac{8(t-1)}{(t+1)^2}\right] = \begin{cases} 1-t^2, & \text{if } t \leq 3; \\ \frac{-8(t+1)}{(t-1)^2}, & \text{if } t > 3. \end{cases}$$

Thus, $f(f(x)) = x$ when $-8 \leq x \leq 1$, *i.e.* when $x = 1 - t^2$ for $0 \leq t \leq 3$.

Since $-8 \leq f(x) \leq 1$ for all $x \leq 1$, it follows that $f(f(f(x))) = f(x)$ for all $x \leq 1$.

Comment. It is of slight interest to note that $f(-3) = f(-24) = 8/9$.

6. Let $h(n)$ denote the number of finite sequences $\{a_1, a_2, \cdots, a_k\}$ of positive integers exceeding 1 for which $k \geq 1$, $a_1 \geq a_2 \geq \cdots \geq a_k$ and $n = a_1 a_2 \cdots a_k$. (For example, if $n = 20$, there are four such sequences $\{20\}$, $\{10, 2\}$, $\{5, 4\}$ and $\{5, 2, 2\}$ and $h(20) = 4$.

Prove that

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^2} = 1 \ .$$

Solution 1. We have that

$$\begin{aligned}
1 + \sum_{n=2}^{\infty} \frac{h(n)}{n^2} &= \prod_{r=2}^{\infty} \left(\sum_{k=0}^{\infty} \frac{1}{r^{2k}} \right) \\
&= \prod_{r=2}^{\infty} \frac{1}{1 - 1/r^2} = \prod_{r=1}^{\infty} \frac{r^2}{r^2 - 1} \\
&= \lim_{n \to \infty} \prod_{r=1}^{n} \left(\frac{r}{r-1} \right) \left(\frac{r}{r+1} \right) = \lim_{n \to \infty} \frac{2n}{n+1} = 2 \ ,
\end{aligned}$$

from which the result follows.

Solution 2. [J. Kramar] Observe that

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^2} = \sum \{ (a_1 a_2 \cdots a_k)^{-2} : a_1 \geq a_2 \geq \cdots \geq a_k \geq 2 \} \ .$$

For $m \geq 2$, let

$$b_m = \sum \{ (a_1 a_2 \cdots a_k)^{-2} : a_1 = m \geq a_2 \geq \cdots \geq a_k \geq 2 \} \ .$$

Since

$$b_m = \frac{1}{m^2} + \frac{1}{m^2}(b_m + b_{m-1} + \cdots + b_2) \ ,$$

then, for $m \geq 3$,

$$(m^2 - 1) b_m = 1 + b_2 + \cdots + b_{m-1} \ .$$

Note that

$$b_2 = \frac{1}{2^2} + \frac{1}{4^2} + \cdots = \frac{1}{3} \ .$$

Assume as an induction hypothesis that, for $2 \leq k \leq m-1$, $b_k = 2/(k(k+1))$. Then

$$\begin{aligned}
(m+1)(m-1) b_m = 1 + b_2 + \cdots + b_{m-1} &= 1 + 2 \sum_{k=2}^{n-1} \frac{1}{k(k+1)} \\
&= 1 + 2 \left(\sum_{k=2}^{m-1} \frac{1}{k} - \frac{1}{k+1} \right) = 1 + 2 \left(\frac{1}{2} - \frac{1}{m} \right) \\
&= 2 - \frac{2}{m} = 2 \left(\frac{m-1}{m} \right) \ ,
\end{aligned}$$

so that $b_m = 2/(m(m+1))$.

Hence

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^2} = \sum_{m=2}^{\infty} b_m = 2 \sum_{m=2}^{\infty} \left(\frac{1}{m} - \frac{1}{m+1} \right) = 1 \ .$$

7. Find the Jordan canonical form of the matrix $\mathbf{u}\mathbf{v}^{\mathbf{t}}$ where $\mathbf{u}$ and $\mathbf{v}$ are column vectors in $\mathbf{C}^n$.

Solution. Suppose first that $\mathbf{v^t u} \neq 0$. Then

$$(\mathbf{u v^t})\mathbf{u} = \mathbf{u}(\mathbf{v^t u}) = (\mathbf{v^t u})\mathbf{u}$$

with $\mathbf{u} \neq \mathbf{O}$, so that $\mathbf{uv^t}$ has the nonzero characteristic value $\mathbf{v^t u}$. Since $\mathbf{uv^t}$ has rank 1, so also does its Jordan form, which is then

$$\begin{pmatrix} \mathbf{v^t u} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} .$$

If $\mathbf{v^t} = 0$, $\mathbf{u} \neq \mathbf{O}$, $\mathbf{v} \neq \mathbf{O}$, then $(\mathbf{uv^t})^2 = \mathbf{O}$ and $\mathbf{uv^t}$ has rank 1, so that its Jordan form is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} .$$

8. Suppose that n points are given in the plane, not all collinear. Prove that there are at least n distinct straight lines that can be drawn through pairs of the points.

Solution. This can be obtained as a corollary to Sylvester's theorem: *Suppose that n points are given in the plane, not all collinear. Then there is exists a line that contains exactly two of them.* To see this, suppose that n points are given. Pick a point P and line m through two points of the set that does not contain P for which the distance between P and m is minimum. If m contains exactly two points, then the result is established. Otherwise, let A, B, C be three points of m with B between A and C. Let u be the line AP and v the line CP and let Q be the foot of the perpendicular from P to AC. Suppose, wolog, that B is between A and Q. Then the distance from B to u is less than the length of PQ, and this contradicts the choice of the pair (P, m).

We can now solve the problem by an induction argument. It is clearly true for $n = 3$; suppose it holds for $n - 1 \geq 3$ and that n points are given. Pick a line that passes through exactly two points P and Q of the set. At least one of these points, say P, is not collinear with the rest. Remove this line and the point Q. We can find $n - 1$ distinct lines determined by pairs of the other $n - 1$ points, and restoring the line through PQ yields the nth line.

Comment. Perhaps some students will find an alternative approach to this problem.

9. Which integers can be written in the form

$$\frac{(x+y+z)^2}{xyz}$$

where x, y, z are positive integers?

Solution. Let $F(x, y, z)$ be the expression in question. Wolog, suppose that $x \leq y \leq z$. Suppose that $F(x, y, z) = n$. Then

$$nxyz = (x + y + z)^2 = (x + y)^2 + 2z(x + y) + z^2$$

from which $z | (x + y)^2$. Let $w = (x + y)^2/z$. Then

$$F(x, y, w) = \frac{(zx + zy + (x + y)^2)^2}{zxy(x + y)^2} = \frac{(z + x + y)^2}{zxy} = F(x, y, z) .$$

If $x+y \le z$, then $w \le x+y$. So, if there is a representation of n, we can find one for which $z \le x+y$. Then

$$\begin{aligned} n &= \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} + 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \\ &\le \frac{1}{z} + \frac{1}{x} + \left(\frac{1}{x} + \frac{1}{y}\right) + \frac{2}{x} + \frac{2}{y} + \frac{2}{z} \\ &\le \frac{7}{x} + \frac{3}{z} \ . \end{aligned}$$

If $(x,y,z) = (1,1,1)$, then $n = 9$. Otherwise $n < 9$. We have that $F(9,9,9) = 1$, $F(4,4,8) = 2$, $F(3,3,3) = 3$, $F(2,2,4) = 4$, $F(1,4,5) = 5$, $F(1,2,3) = 6$, $F(1,1,2) = 8$.

However, $F(x,y,z)$ cannot equal 7. Supposing that $2 \le x \le y \le z \le x+y \le 2y$, we have that

$$\begin{aligned} \frac{(x+y+z)^2}{xyz} &\le \frac{(x^2+y^2) + 2x(y+z) + z^2 + 2yz}{2yz} \\ &\le \frac{x^2}{2yz} + \frac{y}{2z} + \frac{x(y+z)}{yz} + \frac{z^2}{2yz} + 2 \\ &\le \frac{1}{2} + \frac{1}{2} + \frac{y(2z)}{yz} + \frac{z}{2y} + 2 \\ &= 1 + 2 + 1 + 2 = 6 \ . \end{aligned}$$

Now let $x = 1$ and $y \le z \le 1+y$. Since $F(1,1,1)$ and $F(1,1,2)$ differ from 7, $y \ge 2$. But then

$$F(1,y,y) = \frac{(2y+1)^2}{y^2} = 4 + \frac{4}{y} + \frac{1}{y^2} < 7 \ ,$$

and

$$F(1,y,y+1) = \frac{(2y+2)^2}{y(y+1)} = \frac{4(y+1)}{y} = 4 + \frac{4}{y} < 7 \ .$$

Hence $F(x,y,z) = 7$ is not possible. Therefore, only the integers 1, 2, 3, 4, 5, 6, 8 can be represented.

10. Solve the following differential equation

$$2y' = 3|y|^{1/3}$$

subject to the intial conditions

$$y(-2) = -1 \qquad \text{and} \qquad y(3) = 1 \ .$$

Your solution should be everywhere differentiable.

Solution. Depending on the sign of y in any region, separation of variables leads to the solution

$$y^{2/3} = x + c \qquad \text{or} \qquad y = (x+c)^{3/2}$$

when $y \ge 0$ and to

$$y^{2/3} = -(x+c) \qquad \text{or} \qquad y = -[-(x+c)]^{3/2}$$

when $y < 0$. The desired solution is

$$y(x) = \begin{cases} -[-(x+1)]^{3/2}, & \text{if } x < -1; \\ 0, & \text{if } -1 \le x \le 2; \\ (x-2)^{3/2}, & \text{if } x > 2. \end{cases}$$