THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

Sunday, March 14, 2004

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

1. Prove that, for any complex numbers z and w,
$$(|z| + |w|)\left|\frac{z}{|z|} + \frac{w}{|w|}\right| \leq 2|z+w| \ .$$

2. Prove that
$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} + \cdots \ .$$

3. Suppose that u and v are positive integer divisors of the positive integer n and that $uv < n$. Is it necessarily so that the greatest common divisor of n/u and n/v exceeds 1?

4. Let n be a positive integer exceeding 1. How many permutations $\{a_1, a_2, \cdots, a_n\}$ of $\{1, 2, \cdots, n\}$ are there which maximize the value of the sum
$$|a_2 - a_1| + |a_3 - a_2| + \cdots + |a_{i+1} - a_i| + \cdots + |a_n - a_{n-1}|$$
over all permutations? What is the value of this maximum sum?

5. Let A be a $n \times n$ matrix with determinant equal to 1. Let B be the matrix obtained by adding 1 to every entry of A. Prove that the determinant of B is equal to $1 + s$, where s is the sum of the n^2 entries of A^{-1}.

6. Determine
$$\left(\int_0^1 \frac{dt}{\sqrt{1-t^4}}\right) \div \left(\int_0^1 \frac{dt}{\sqrt{1+t^4}}\right) \ .$$

7. Let a be a parameter. Define the sequence $\{f_n(x) : n = 0, 1, 2, \cdots\}$ of polynomials by
$$f_0(x) \equiv 1$$
$$f_{n+1}(x) = xf_n(x) + f_n(ax)$$
for $n \geq 0$.

   (a) Prove that, for all n, x,
$$f_n(x) = x^n f_n(1/x) \ .$$

   (b) Determine a formula for the coefficient of x^k $(0 \leq k \leq n)$ in $f_n(x)$.

8. Let V be a complex $n-$dimensional inner product space. Prove that
$$|u|^2|v|^2 - \frac{1}{4}|u-v|^2|u+v|^2 \leq |(u,v)|^2 \leq |u|^2|v|^2 \ .$$

9. Let $ABCD$ be a convex quadrilateral for which all sides and diagonals have rational length and AC and BD intersect at P. Prove that AP, BP, CP, DP all have rational length.

END

Solutions

1. Prove that, for any complex numbers z and w,

$$(|z|+|w|)\left|\frac{z}{|z|}+\frac{w}{|w|}\right| \le 2|z+w|\ .$$

Solution 1.

$$\begin{aligned}(|z|+|w|)&\left|\frac{z}{|z|}+\frac{w}{|w|}\right| \\ &= \left|z+w+\frac{|z|w}{|w|}+\frac{|w|z}{|z|}\right| \\ &\le |z+w| + \frac{1}{|z||w|}|\bar{z}zw+\bar{w}zw| \\ &= |z+w| + \frac{|zw|}{|z||w|}|\bar{z}+\bar{w}| = 2|z+w|\ .\end{aligned}$$

Solution 2. Let $z = ae^{i\alpha}$ and $w = be^{i\beta}$, with a and b real and positive. Then the left side is equal to

$$\begin{aligned}|(a+b)(e^{i\alpha}+e^{i\beta})| &= |ae^{i\alpha}+ae^{i\beta}+be^{i\alpha}+be^{i\beta}| \\ &\le |ae^{i\alpha}+be^{i\beta}| + |ae^{i\beta}+be^{i\alpha}|\ .\end{aligned}$$

Observe that

$$\begin{aligned}|z+w|^2 &= |(ae^{i\alpha}+be^{i\beta})(ae^{-i\alpha}+be^{-i\beta})| \\ &= a^2+b^2+ab[e^{i(\alpha-\beta)}+e^{i(\beta-\alpha)}] \\ &= |(ae^{i\beta}+be^{i\alpha})(ae^{-i\beta}+be^{-i\alpha})|\end{aligned}$$

from which we find that the left side does not exceed

$$|ae^{i\alpha}+be^{i\beta}| + |ae^{i\beta}+be^{i\alpha}| = 2|ae^{i\alpha}+be^{i\beta}| = 2|z+w|\ .$$

Solution 3. Let $z = ae^{i\alpha}$ and $w = be^{i\beta}$, where a and b are positive reals. Then the inequality is equivalent to

$$\left|\frac{1}{2}(e^{i\alpha}+e^{i\beta}\right| \le |\lambda e^{i\alpha}+(1-\lambda)e^{i\beta}|$$

where $\lambda = a/(a+b)$. But this simply says that the midpoint of the segment joining $e^{i\alpha}$ and $e^{i\beta}$ on the unit circle in the Argand diagram is at least as close to the origin as another point on the segment.

Solution 4. [G. Goldstein] Observe that, for each $\mu \in \mathbf{C}$,

$$\left|\frac{\mu z}{|\mu z|}+\frac{\mu w}{|\mu w|}\right| = \frac{z}{|z|}+\frac{w}{|w|}\ ,$$

$$|\mu|[|z|+|w|] = |\mu z+\mu w|\ ,$$

and

$$|\mu||z+w| = |\mu z+\mu w|\ .$$

So the inequality is equivalent to

$$(|t|+1)\left|\frac{t}{|t|}+1\right| \le 2|t+1|$$

for $t \in \mathbf{C}$. (Take $\mu = 1/w$ and $t = z/w$.)

Let $t = r(\cos\theta + i\sin\theta)$. Then the inequality becomes

$$(r+1)\sqrt{(\cos\theta+1)^2+\sin^2\theta} \leq 2\sqrt{(r\cos\theta+1)^2+r^2\sin^2\theta} = 2\sqrt{r^2+2r\cos\theta+1}\ .$$

Now,

$$\begin{aligned} 4(r^2+2r\cos\theta+1) - (r+1)^2(2+2\cos\theta) \\ &= 2r^2(1-\cos\theta)+4r(\cos\theta-1)+2(1-\cos\theta) \\ &= 2(r-1)^2(1-\cos\theta) \geq 0\ , \end{aligned}$$

from which the inequality follows.

Solution 5. [R. Mong] Consider complex numbers as vectors in the plane. $q = (|z|/|w|)w$ is a vector of magnitude z in the direction w and $p = (|w|/|z|)z$ is a vector of magnitude w in the direction z. A reflection about the angle bisector of vectors z and w interchanges p and w, q and z. Hence $|p+q| = |w+z|$. Therefore

$$\begin{aligned} (|z|+|w|)\left|\frac{z}{|z|}+\frac{w}{|w|}\right| \\ &= |z+q+p+w| \leq |z+w|+|p+q| \\ &= 2|z+w|\ . \end{aligned}$$

2. Prove that
$$\int_0^1 x^x\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} + \cdots\ .$$

Solution. First, let
$$I(m,n) = \int_0^1 x^m(\log x)^n\,dx$$
for nonnegative integers m and n. Then $I(0,0) = 1$ and $I(m,0) = 1/(m+1)$ for every nonnegative integer m. Taking the parts $u = (\log x)^n$, $dv = x^m dx$ and noting that $\lim_{x\to 0} x^{m+1}(\log x)^n = 0$, we find that $I(m,n) = -(n/(m+1))I(m,n-1)$ whence
$$I(m,n) = \frac{(-1)^n n!}{(m+1)^{n+1}}$$
for each nonnegative integer n. In particular,
$$I(k,k) = \frac{(-1)^k k!}{(k+1)^{(k+1)}}$$
for each nonnegative integer k.

Using the fact that the series is uniformly convergent and term-by-term integration is possible, we find that
$$\begin{aligned} \int_0^1 x^x\,dx &= \int_0^1 e^{x\log x}\,dx = \sum_{k=0}^{\infty}\int_0^1 \frac{(x\log x)^k}{k!}\,dx \\ &= 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^{(k+1)}} = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \cdots. \end{aligned}$$

3. Suppose that u and v are positive integer divisors of the positive integer n and that $uv < n$. Is it necessarily so that the greatest common divisor of n/u and n/v exceeds 1?

Solution 1. Let $n = ur = vs$. Then $uv < n \Rightarrow v < r, u < s$, so that $n^2 = uvrs \Rightarrow rs > n$. Let the greatest common divisor of r and s be g and the least common multiple of r and s be m. Then $m \leq n < rs = gm$, so that $g > 1$.

Solution 2. Let $g = \gcd\,(u, v)$, $u = gs$ and $v = gt$. Then $gst \leq g^2 st < n$ so that $st < n/g$. Now s and t are a coprime pair of integers, each of which divides n/g. Therefore, $n/g = dst$ for some $d > 1$. Therefore $n/u = n/(gs) = dt$ and $n/v = n/(gt) = ds$, so that n/u and n/v are divisible by d, and so their greatest common divisor exceeds 1.

Solution 3. $uv < n \Longrightarrow nuv < n^2 \Longrightarrow n < (n/u)(n/v)$. Suppose, if possible, that n/u and n/v have greatest common divisor 1. Then the least common multiple of n/u and n/v must equal $(n/u)(n/v)$. But n is a common multiple of n/u and n/v, so that $(n/u)(n/v) \leq n$, a contradiction. Hence the greatest common divisor of n/u and n/v exceeds 1.

Solution 4. Let P be the set of prime divisors of n, and for each $p \in P$. let $\alpha(p)$ be the largest integer k for which p^k divides n. Since u and v are divisors of n, the only prime divisors of either u or v must belong to P. Suppose that $\beta(p)$ is the largest value of the integer k for which p^k divides uv.

If $\beta(p) \geq \alpha(p)$ for each $p \in P$, then n would divide uv, contradicting $uv < n$. (Note that $\beta(p) > \alpha(p)$ may occur for *some* p.) Hence there is a prime $q \in P$ for which $\beta(q) < \alpha(q)$. Then $q^{\alpha(q)}$ is not a divisor of either u or v, so that q divides both n/u and n/v. Thus, the greatest common divisor of n/u and n/v exceeds 1.

Solution 5. [D. Shirokoff] If n/u and n/v be coprime, then there are integers x and y for which $(n/u)x + (n/v) = 1$, whence $n(xv + yu) = uv$. Since n and uv are positive, then so is the integer $xv + yu$. But $uv < n \Longrightarrow 0 < xv + yu < 1$, an impossibility. Hence the greatest common divisor of n/u and n/v exceeds 1.

4. Let n be a positive integer exceeding 1. How many permutations $\{a_1, a_2, \cdots, a_n\}$ of $\{1, 2, \cdots, n\}$ are there which maximize the value of the sum

$$|a_2 - a_1| + |a_3 - a_2| + \cdots + |a_{i+1} - a_i| + \cdots + |a_n - a_{n-1}|$$

over all permutations? What is the value of this maximum sum?

Solution. First, suppose that n is odd. Then

$$|a_{i+1} - a_i| \leq \left| a_{i+1} - \frac{n+1}{2} \right| + \left| a_i - \frac{n+1}{2} \right|$$

with equality if and only if $\frac{1}{2}(n+1)$ lies between a_{i+1} and a_i.

Hence

$$\begin{aligned}
\sum_{i=0}^{n-1} |a_{i+1} - a_i| &\leq 2\bigg(\sum_{i=1}^{n} \left| a_i - \frac{n+1}{2} \right| \bigg) - \bigg(\left| a_1 - \frac{n+1}{2} \right| + \left| a_n - \frac{n+1}{2} \right| \bigg) \\
&= \bigg(\sum_{i=1}^{n} |2a_i - (n+1)| \bigg) - \bigg(\left| a_1 - \frac{n+1}{2} \right| + \left| a_n - \frac{n+1}{2} \right| \bigg) \\
&= [(n-1) + (n-3) + \cdots + 2 + 0 + 2 + \cdots + (n-1)] - \bigg(\left| a_1 - \frac{n+1}{2} \right| + \left| a_n - \frac{n+1}{2} \right| \bigg) \\
&= 4\bigg(1 + 2 + \cdots + \frac{n-1}{2} \bigg) - \bigg(\left| a_1 - \frac{n+1}{2} \right| + \left| a_n - \frac{n+1}{2} \right| \bigg) \\
&= 4\bigg[\frac{((n-1)/2) \cdot ((n+1)/2)}{2} \bigg] - \bigg(\left| a_1 - \frac{n+1}{2} \right| + \left| a_n - \frac{n+1}{2} \right| \bigg) \\
&\leq \frac{n^2 - 1}{2} - 1 = \frac{n^2 - 3}{2}
\end{aligned}$$

since $|a_1 - ((n+1)/2)| + |a_n - ((n+1)/2)| \geq 1$. Equality occurs when one of a_1 and a_n is equal to $\frac{1}{2}(n+1)$ and the other is equal to $\frac{1}{2}(n+1) \pm 1$.

We get a permutation giving the maximum value of $\frac{1}{2}(n^2 - 3)$ if and only if the foregoing conditions on a_1 and a_n are satisfied (in four possible ways) and $\frac{1}{2}(n+1)$ lies between a_i and a_{i+1} for each i. For example, if $a_1 = \frac{1}{2}(n+1) + 1$, then we require that the $\frac{1}{2}(n-3)$ numbers $a_3, \cdots, a_{n-2}$ exceed $\frac{1}{2}(n+1) + 1$ and the $\frac{1}{2}(n-1)$ numbers $a_2, a_4, \cdots, a_{n-1}$ are less than $\frac{1}{2}(n+1)$. Thus, there are

$$4\left(\frac{n-3}{2}\right)!\left(\frac{n-1}{2}\right)!$$

ways of achieving the maximum.

Now suppose that n is even. As before,

$$|a_{i+1} - a_i| \leq \left|a_{i+1} - \frac{n+1}{2}\right| + \left|a_i - \frac{n+1}{2}\right|$$

with equality if and only if $\frac{1}{2}(n+1)$ lies between a_{i+1} and a_i.

We have that

$$\begin{aligned}\sum_{i=1}^{n} |a_{i+1} - a_i| &= 2(1 + 3 + \cdots + (n-1)) - \left[\left|a_{i+1} - \frac{n+1}{2}\right| + \left|a_n - \frac{n+2}{2}\right|\right] \\ &\leq \frac{n^2}{2} - 1 = \frac{n^2 - 2}{2} ,\end{aligned}$$

with the latter inequality becoming equality if and only if $\{a_1, a_n\} = \{n/2, (n+2)/2\}$. Suppose, say, that $a_1 = n/2$ and $a_n = (n+2)/2$. Then, to achieve the maximum, we require that $\{a_1, a_3, \cdots, a_{n-1}\} = \{1, 2. \cdots, n/2\}$ and $\{a_2, a_4, \cdots, a_n\} = \{(n/2) + 1, \cdots, n\}$. The maximum value of $(n^2 - 2)/2$ can be achieved with $2[(n-2)/2]!^2$ permutations.

5. Let A be a $n \times n$ matrix with determinant equal to 1. Let B be the matrix obtained by adding 1 to every entry of A. Prove that the determinant of B is equal to $1 + s$, where s is the sum of the n^2 entries of A^{-1}.

Solution 1. First, we make a general observation. Let $U = (u_{ij})$ and $V = (v_{ij})$ be two $n \times n$ matrices. Then det $(U + V)$ is the sum of the determinants of 2^n $n \times n$ matrices W_S, where S is a subset of $\{1, 2, \cdots n\}$ and the (i, j)th element of W_S is equal to u_{ij} when $i \in S$ and v_{ij} when $i \notin S$. In the special case that $U = A$ and $V = E$, the matrix whose every entry is equal to 1, W_S is equal to 0 except when $S = \{1, 2, \cdots, n\}$ or $S = \{1, 2, \cdots, n\} \backslash \{k\}$ for some integer k. In the former case, det S = det $A = 1$, and in the latter case, all rows of S except the kth agree with the corresponding rows of A and the kth row of S consists solely of 1s, so that the determinant of S is equal to $A_{k,1} + A_{k,2} + \cdots + A_{k,n}$. Thus,

$$\det(A + E) = \det A + \sum_{k=1}^{n}\left(\sum_{l=1}^{n} A_{kl}\right) = 1 + \sum_{i,j} A_{ij} ,$$

as desired.

Solution 2. [R. Barrington Leigh] Let $A^{-1} = (c_{ij})$ and $d_j = \sum_{i=1}^{n} c_{ij}$, the jth column sum of A^{-1} for $1 \leq j \leq n$. Let E be the $n \times n$ matrix all of whose entries are 1, so that $B = A + E$. Observe that $E = (1, 1, \cdots, 1)^{\mathbf{t}}(1, 1, \cdots, 1)$, where $\mathbf{t}$ denotes the transpose. Then

$$\begin{aligned}\det B = (\det B)(\det A)^{-1} &= (\det B)(\det A^{-1}) = \det\ (BA^{-1}) \\ &= \det\ [(A + E)A^{-1}] = \det\ [I + (1, 1, \cdots, 1)^{\mathrm{t}}(1, 1, \cdots, 1)(c_{ij}) \\ &= \det\ [I + (1, 1, \cdots, 1)^{\mathbf{t}}(d_1, d_2, \cdots, d_n)\ .\end{aligned}$$

Thus, we need to calculate the determinant of the matrix

$$\begin{pmatrix} 1+d_1 & d_2 & \cdots & d_n \\ d_1 & 1+d_2 & \cdots & d_n \\ & & \cdots & \\ d_1 & d_2 & \cdots & 1+d_n \end{pmatrix} .$$

This is equal to the determinant of the matrix

$$\begin{pmatrix} 1+d_1 & d_2 & \cdots & d_n \\ -1 & 1 & \cdots & 0 \\ & & \cdots & \\ -1 & 0 & \cdots & 1 \end{pmatrix} ,$$

which in turn is equal to the determinant of a matrix whose top row is $1 + d_1 + d_2 + \cdots + d_n \ \ 0 \ \ 0 \ \ \cdots \ \ 0)$ and for which the cofactor fo the top lect element is the $(n-1) \times (n-1)$ identity matrix. The result follows.

Solution 3. [M.-D. Choi] *Lemma:* If C is a rank 1 $n \times n$ matrix, then the determinant of I + C is equal to 1 + trace C.

Proof. The sum of the eigenvalues of C is equal to the trace of C (sum of the diagonal elements). Since 0 is an eigenvalue of algebraic multiplicity $n-1$, the remaining eigenvalue is equal to the trace of C. The eigenvalues of $I + C$ are 1 with algebraic multiplicity $n-1$ and 1 + trace C, so that the determinant of $I + C$, which is the product of its eigenvalues, is equal to 1 + trace C. ♠

In the problem, with E defined as in Solution 2,

$$\det(A+E) = \det(A) \det(I + A^{-1}E) = \det(I + A^{-1}E) .$$

Since E is of rank 1, so is $A^{-1}E$. The diagonal elements of $A^{-1}E$ are the row sums of A^{-1} and so the trace of A^{-1} is equal to the sum of all the elements of A^{-1}. The desired result follows.

6. Determine
$$\left(\int_0^1 \frac{dt}{\sqrt{1-t^4}} \right) \div \left(\int_0^1 \frac{dt}{\sqrt{1+t^4}} \right) .$$

Solution. The substitution $t^2 = \sin\theta$ leads to

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \int_0^{\pi/2} \frac{\cos\theta d\theta}{2\sqrt{\sin\theta}\cos\theta} = \frac{1}{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} .$$

The substitution $t^2 = \tan\alpha$ followed by the substitution $\beta = 2\alpha$ leads to

$$\begin{aligned} \int_0^1 \frac{dt}{\sqrt{1+t^4}} &= \int_0^{\pi/4} \frac{\sec^2\alpha d\alpha}{2\sqrt{\tan\alpha}\sec\alpha} \\ &= \frac{1}{2}\int_0^{\pi/4} \frac{d\alpha}{\sqrt{\sin\alpha\cos\alpha}} \\ &= \frac{1}{2}\int_0^{\pi/4} \frac{\sqrt{2}d\alpha}{\sqrt{\sin 2\alpha}} = \frac{1}{4}\int_0^{\pi/2} \frac{\sqrt{2}d\beta}{\sqrt{\sin\beta}} \\ &= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{d\beta}{\sqrt{\sin\beta}} . \end{aligned}$$

Thus the answer is $\sqrt{2}$.

7. Let a be a parameter. Define the sequence $\{f_n(x) : n = 0, 1, 2, \cdots\}$ of polynomials by

$$f_0(x) \equiv 1$$

$$f_{n+1}(x) = xf_n(x) + f_n(ax)$$

for $n \geq 0$.

(a) Prove that, for all n, x,

$$f_n(x) = x^n f_n(1/x) \ .$$

(b) Determine a formula for the coefficient of x^k $(0 \leq k \leq n)$ in $f_n(x)$.

Solution. The polynomial $f_n(x)$ has degree n for each n, and we will write

$$f_n(x) = \sum_{k=0}^{n} b(n,k)x^k \ .$$

Then

$$x^n f_n(1/x) = \sum_{k=0}^{n} b(n,k)x^{n-k} = \sum_{k=0}^{n} b(n,n-k)x^k \ .$$

Thus, (a) is equivalent to $b(n,k) = b(n,n-k)$ for $0 \leq k \leq n$.

When $a = 1$, it can be established by induction that $f_n(x) = (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^n$. Also, when $a = 0$, $f_n(x) = x^n + x^{n-1} + \cdots + x + 1 = (x^{n+1} - 1)(x-1)^{-1}$. Thus, (a) holds in these cases and $b(n,k)$ is respectively equal to $\binom{n}{k}$ and 1.

Suppose, henceforth, that $a \neq 1$. For $n \geq 0$,

$$\begin{aligned} f_{n+1}(k) &= \sum_{k=0}^{n} b(n,k)x^{k+1} + \sum_{k=0}^{n} a^k b(n,k)x^k \\ &= \sum_{k=1}^{n} b(n,k-1)x^k + b(n,n)x^{n+1} + b(n,0) + \sum_{k=1}^{n} a^k b(n,k)x^k \\ &= b(n,0) + \sum_{k=1}^{n} [b(n,k-1) + a^k b(n,k)]x^k + b(n,n)x^{n+1} \ , \end{aligned}$$

whence $b(n+1,0) = b(n,0) = b(1,0)$ and $b(n+1,n+1) = b(n,n) = b(1,1)$ for all $n \geq 1$. Since $f_1(x) = x+1$, $b(n,0) = b(n,n) = 1$ for each n. Also

$$b(n+1,k) = b(n,k-1) + a^k b(n,k) \tag{1}$$

for $1 \leq k \leq n$.

We conjecture what the coefficients $b(n,k)$ are from an examination of the first few terms of the sequence:

$$f_0(x) = 1; \qquad f_1(x) = 1 + x; \qquad f_2(x) = 1 + (a+1)x + x^2;$$

$$f_3(x) = 1 + (a^2 + a + 1)x + (a^2 + a + 1)x^2 + x^3;$$

$$f_4(x) = 1 + (a^3 + a^2 + a + 1)x + (a^4 + a^3 + 2a^2 + a + 1)x^2 + (a^3 + a^2 + a + 1)x^3 + x^4;$$

$$f_5(x) = (1 + x^5) + (a^4 + a^3 + a^2 + a + 1)(x + x^4) + (a^6 + a^5 + 2a^4 + 2a^3 + 2a^2 + a + 1)(x^2 + x^3) \ .$$

We make the empirical observation that

$$b(n+1,k) = a^{n+1-k} b(n,k-1) + b(n,k) \tag{2}$$

which, with (1), yields

$$(a^{n+1-k}-1)b(n,k-1)=(a^k-1)b(n,k)$$

so that

$$b(n+1,k)=\left[\frac{a^k-1}{a^{n+1-k}-1}+a^k\right]b(n,k)=\left[\frac{a^{n+1}-1}{a^{n+1-k}-1}\right]b(n,k)$$

for $n\geq k$. This leads to the conjecture that

$$b(n,k)=\left(\frac{(a^n-1)(a^{n-1}-1)\cdots(a^{k+1}-1)}{(a^{n-k}-1)(a^{n-k-1}-1)\cdots(a-1)}\right)b(k,k) \tag{3}$$

where $b(k,k)=1$.

We establish this conjecture. Let $c(n,k)$ be the right side of (3) for $1\leq k\leq n-1$ and $c(n,n)=1$. Then $c(n,0)=b(n,0)=c(n,n)=b(n,n)=1$ for each n. In particular, $c(n,k)=b(n,k)$ when $n=1$.

We show that

$$c(n+1,k)=c(n,k-1)+a^kc(n,k)$$

for $1\leq k\leq n$, which will, through an induction argument, imply that $b(n,k)=c(n,k)$ for $0\leq k\leq n$. The right side is equal to

$$\left(\frac{a^n-1}{a^{n-k}-1}\right)\cdots\left(\frac{a^{k+1}-1}{a-1}\right)\left[\frac{a^k-1}{a^{n-k+1}-1}+a^k\right]=\frac{(a^{n+1}-1)(a^n-1)\cdots(a^{k+1}-1)}{(a^{n+1-k}-1)(a^{n-k}-1)\cdots(a-1)}=c(n+1,k)$$

as desired. Thus, we now have a formula for $b(n,k)$ as required in (b).

Finally, (a) can be established in a straightforward way, either from the formula (3) or using the pair of recursions (1) and (2).

8. Let V be a complex $n-$dimensional inner product space. Prove that

$$|u|^2|v|^2-\frac{1}{4}|u-v|^2|u+v|^2\leq|(u,v)|^2\leq|u|^2|v|^2\ .$$

Solution. The right inequality is the Cauchy-Schwarz Inequality. We have that

$$\begin{aligned}
&4|(u,v)|^2-4|u|^2|v|^2+|u-v|^2|u+v|^2\\
&\quad=4(u,v)(v,u)-4(u,u)(v,v)+(u-v,u-v)(u+v,u+v)\\
&\quad=4(u,v)(v,u)-4(u,u)(v,v)+[\overline{(u,u)+(v,v)}-\overline{(u,v)+(v,u)}][\overline{(u,u)+(v,v)}+\overline{(u,v)+(v,u)}]\\
&\quad=4(u,v)(v,u)-4(u,u)(v,v)+(u,u)^2+(v,v)^2+2(u,u)(v,v)-[(u,v)^2+2(u,v)(v,u)+(v,u)^2]\\
&\quad=[(u,u)-(v,v)]^2-[(u,v)-(v,u)]^2\\
&\quad=[|u|^2-|v|^2]^2-[2i\ \mathrm{Im}\ (u,v)]^2\\
&\quad=[|u|^2-|v|^2]^2+4[\mathrm{Im}\ (u,v)]^2\geq0\ .
\end{aligned}$$

9. Let $ABCD$ be a convex quadrilateral for which all sides and diagonals have rational length and AC and BD intersect at P. Prove that AP, BP, CP, DP all have rational length.

Solution 1. Because of the symmetry, it is enough to show that the length of AP is rational. The rationality of the lengths of the remaining segments can be shown similarly. Coordinatize the situation by taking $A\sim(0,0)$, $B\sim(p,q)$, $C\sim(c,0)$, $D\sim(r,s)$ and $P\sim(u,0)$. Then, equating slopes, we find that

$$\frac{s}{r-u}=\frac{s-q}{r-p}$$

so that

$$\frac{sr - ps}{s - q} = r - u$$

whence $u = r - \frac{sr-ps}{s-q} = \frac{ps-qr}{s-q}$.

Note that $|AB|^2 = p^2 + q^2$, $|AC|^2 = c^2$, $|BC|^2 = (p^2 - 2pc + c^2) + q^2$, $|CD|^2 = (c^2 - 2cr + r^2) + s^2$ and $|AD|^2 = r^2 + s^2$, we have that

$$2rc = AC^2 + AD^2 - CD^2$$

so that, since c is rational, r is rational. Hence s^2 is rational.

Similarly

$$2pc = AC^2 + AB^2 - BC^2 \ .$$

Thus. p is rational, so that q^2 is rational.

$$2qs = q^2 + s^2 - (q - s)^2 = q^2 + s^2 - [(p - r)^2 + (q - s)^2] + p^2 - 2pr + r^2$$

is rational, so that both qs and $q/s = (qs)/s^2$ are rational. Hence

$$u = \frac{p - r(q/s)}{1 - (q/s)}$$

is rational.

Solution 2. By the cosine law, the cosines of all of the angles of the triangle ACD, BCD, ABC and ABD are rational. Now

$$\frac{AP}{AB} = \frac{\sin \angle ABP}{\sin \angle BPC}$$

and

$$\frac{CP}{BC} = \frac{\sin \angle PBC}{\sin \angle BPC} \ .$$

Since $\angle APC + \angle BPC = 180°$, therefore $\sin \angle APC = \sin \angle BPC$ and

$$\begin{aligned} \frac{AP}{CP} &= \frac{AB \sin \angle ABP}{BC \sin \angle PBC} = \frac{AB \sin \angle ABP \sin \angle PBC}{BC \sin^2 \angle PBC} \\ &= \frac{AB(\cos \angle ABP \cos \angle PBC - \cos(\angle ABP + \angle PBC))}{BC(1 - \cos^2 \angle PBC)} \\ &= \frac{AB(\cos \angle ABD \cos \angle DBC - \cos \angle ABC)}{BC(1 - \cos^2 \angle DBC)} \end{aligned}$$

is rational. Also $AP + CP$ is rational, so that $(AP/CP)(AP + CP) = ((AP/CP) + 1)AP$ is rational. Hence AP is rational.