THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh and Alfonso Gracia-Saz

March 11, 2023

Time: 4 hours

`No aids or calculators permitted.`

1. When a patron in a restaurant is presented with the bill, they like to pay a whole number of dollars to cover the bill and a tip between 15% and 20% inclusive. (For example, if the bill is \$11.20, they will pay \$13.00, leaving a tip that is more than \$1.68 and less than \$2.24.) Find the largest amount of a bill for which this is not possible.

2. Let X be a set with n elements. Prove that the number of pairs (A, B) of subsets of X for which $A \subset B \subseteq X$ $(A \neq B)$ is equal to $3^n - 2^n$.

3. Solve for integers x, y the equation
$$y^2 = 1 + x + x^2 + x^3 + x^4.$$

4. Determine all polynomials $f(x)$ that satisfy $g'(x) = 4xf(x)$ where $g(x) = f(f(x))$. Are there any non-polynomial solutions?

5. For which positive integers x is the following inequality true?
$$\sqrt{x}^{\sqrt{x}} < \sqrt{x+1}.$$

6. Let $p(z) = z^3 + yz + x$ be a cubic polynomial with real coefficients $1, x, y$. Sketch in the cartesian plane the locus of points (x, y) for which $p(z)$ has at least one root whose absolute value is 1.

7. Let a and b be positive reals and $\{x_n\}$ be a sequence for which
$$\lim_{n\to\infty}\left(ax_n + \frac{b}{x_n}\right) = 2\sqrt{ab}.$$
Must the sequence $\{x_n\}$ converge? If so, what are the possible limits?

8. Let f be a real-valued function defined on the plane for which $f(x, y) = (x^2 - y^2)e^{-(x^2+y^2)}$.
   (a) Prove that f attains its maximum and minimum.
   (b) Determine all the points (x, y) where f has a global or local extremum (maximum or minimum).

9. Determine all the primes p for which $(2^{p-1} - 1)/p$ is a perfect square.

10. Suppose that $a_0 = 1$, $a_1 = 2$ and
$$a_{n+1} = a_n + \frac{a_{n-1}}{1 + a_{n-1}^2},$$
for $n \geq 1$. Determine integers u and v for which $v - u < 23$ and $u < a_{2023} < v$.

11. Prove that $x + y + z$ divides the polynomial

$$f(x, y, z) = 2(x^7 + y^7 + z^7) - 7xyz(x^4 + y^4 + z^4).$$

12. (a) Determine real numbers a, b, c for which

$$abc = a + b + c = 1.$$

Is it possible for a, b, c to be all positive?

(b) Let $\epsilon > 0$. Prove that there are non-zero rational numbers a, b, c, d for which

$$abc = a + b + c = 1 + d,$$

where $|d| < \epsilon$.

(c) Determine rational numbers a, b, c for which $abc = a + b + c$ and $|abc - 1| < 0.1$

13. $ABCD$ is a convex quadrilateral whose diagonals AC and BD intersect at P. Suppose that $PA = PD$, $PB = PC$ and O is the centre of the circumcircle of triangle APB. Prove that $OP \perp CD$.

14. Suppose that $a_1, a_2, \ldots, a_n$ are real numbers. Prove that

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{a_i a_j}{i + j} \right) \geq 0.$$

SOLUTIONS

1. When a patron in a restaurant is presented with the bill, they likes to pay a whole number of dollars to cover the bill and a tip between 15% and 20% inclusive. (For example, if the bill is \$11.20, they will pay \$13.00, leaving a tip that is more that \$1.68 and less that \$2.24.) Find the largest amount of a bill for which this is not possible.

Solution 1. Let b be the amount of the bill. Then the process is possible if and only if there is an integer c for which $1.15b \leq c \leq 1.2b$. Thus, given a integer c, the range of bills b including a suitable tip payable by c satisfies

$$\frac{c}{1.2} \leq b \leq \frac{c}{1.15}.$$

Two adjacent intervals overlap when $c/(1.2) \leq (c-1)/(1.15)$ or $0.05c \geq 1.2$ ($c \geq 24$). Thus, if these intervals fail to overlap, then $c \leq 23$, in which case there is a bill b for which the process is not possible.

When $c = 23$, the possible bills satisfy $19.17 \leq b \leq 20.00$ and when $c = 22$, they satisfy $18.34 \leq b \leq 19.13$. Therefore the process is not possible for $b = 19.14, 19.15, 19.16$ and the answer is \$19.16.

Solution 2. First, we show that this is possible whenever the bill is at least twenty dollars. Suppose that the amount of the bill is b. Then he can do the task if there is an integer between $(23/20)b$ and $(6/5)b$. This will occur when $(6/5)b \geq (23/20)b + 1$, that is, when $b \geq 20$ (because each closed unit real interval contains an integer).

When the bill is 20 dollars, then the amount payable is between 23 and 24 dollars. When the bill b is reduced, so proportionately is the payable range. The patron can pay 23 dollars as long is $(6/5)b \geq 23$, or $b > 19.166$. When $b = 19.16$, the total payable should be between $(1.15)(19.16) > 22.03$ and $(1.2)(19.16) < 23$. But there is no integer between these limits, and the patron cannot do as they wish.

Solution 3, by Yuhan Guo. As in Solution 1, we see that the process is possible when $b \geq 20$. Suppose that the bill amounts to $19+c$ where $0 \leq c < 1$. Then the desired tip lies between $(1.15)(19+c) = 21.85+1.15c$ and $(1.2)(19+c) = 22.80 + 1.2c$. The process is impossible if and only if both these quantities lie strictly between 22 and 23, *i.e.* $1.15c > 0.15$ and $1.2c < 0.2$. This is so iff $0.13 < c < 0.17$, so that the process is not possible for bills equal to 19.14, 19.15 and 19.16, so that the required answer is 19.16.

2. Let X be a set with n elements. Prove that the number of pairs (A, B) of subsets of X for which $A \subset B \subseteq X$ $(A \neq B)$ is equal to $3^n - 2^n$.

Solution 1. For each element of X, there are three options: leave it out of both A and B, include it in A but not in B, include it in both A and B. There are thus 3^n ways if choosing A and B for which $A \subseteq B \subseteq X$. However, this includes the 2^n ways of selecting a subset of X which is equal to both A and B. Therefore the required number is equal to $3^n - 2^n$.

Solution 2. For $0 \leq k \leq n$, the number of ways of selecting the set A with exactly k elements is $\binom{n}{k}$. The number of ways of choosing a set B that properly contains A is the number of ways of selecting a subset of $X \setminus A$ with at least one elements, namely $2^{n-k} - 1$. Therefore the total number of ways of selecting the pair (A, B) is

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k}(2^{n-k} - 1) &= \sum_{k=0}^{n} \binom{n}{k}(2^{n-k}) - \sum_{k=0}^{n} \binom{n}{k} \\ &= (2+1)^n - (1+1)^n = 3^n - 2^n.\end{aligned}$$

Note. A variant of this selects the set B in $\binom{n}{k}$ was and then choosing one of $2^k - 1$ proper subsets A.

Solution 3, by Justin Fus. Let $X = \{x_1, x_2, \ldots, x_n\}$. Let C be the set of $n-$tples $S = (s_1, s_2, \ldots, s_n)$ where, for each i, s_i is associated with x_i and s_i is equal to one of 0, 1 or 2. The set C contains 3^n elements, of which 2^n elements have no entries equal to 1. We define a function f defined on pairs (A, B) of subsets of$X$ that satisfy the conditions of the problem: $f(A, B) = S$ where $a_i = 0$ when $x_i \not\in B$, $a_i = 1$ when $x_i \in B \setminus A$ and $a_i = 2$ when $x_i \in A \subset B$. The function f is one-one onto the subset of C of vectors having no entry equal to 1. Thus, there are $3^n - 2^n$ pairs (A, B) that satisfy the conditions of the problem.

Solution 4. We prove the result by induction. When $n = 1$ and $X = \{x\}$, there is only $1 = 3 - 2$ possibilities, namely $A = \emptyset$ and $B = \{x\}$. Suppose the result holds for $n \geq 1$ and that X has $n+1$ elements. Let z be a particular one of these.

In selecting the sets A and B, there are four possibilities:

(1) Neither A nor B contains z. There are $3^n - 2^n$ possibilities.

(2) Both A and B contain z. Since $A \setminus \{z\} \subset B \setminus \{z\}$ if and only if $A \subset B$, there are $3^n - 2^n$ possibilities.

(3) $z \in B \setminus A$ and $B \neq A \cup \{z\}$. Since $A \subset B$ if and only if $A \subset B \setminus \{z\}$, there are $3^n - 2^n$ possibilities.

(4) $z \in B \setminus A$ and $B = A \cup \{z\}$. There are 2^n was of picking the set $A = B \setminus \{z\}$.

Thus, the total number of possibilities is $3(3^n - 2^n) + 2^n = 3(3^n) - 2(2^n) = 3^{n+1} - 2^{n+1}$.

3. Solve for integers x, y the equation

$$y^2 = 1 + x + x^2 + x^3 + x^4.$$

Solution 1. When $x = 0$, then $y = \pm 1$. More generally, we have that

$$\begin{aligned}(2x^2 + x)^2 = 4x^4 + 4x^3 + x^2 &< 4x^4 + 4x^3 + 4x^2 + 4x + 4 = (2y)^2 \\ &\leq 4x^4 + 4x^3 + 9x^2 + 4x + 4 = (2x^2 + x + 2)^2,\end{aligned}$$

with equality on the right if and only if $x = 0$.

It follows that, when $x \neq 0$, then $2x^2 + x + 1 = 2y$, so that

$$\begin{aligned} 0 &= (4x^4 + 4x^3 + 5x^2 + 2x + 1) - (4x^4 + 4x^3 + 4x^2 + 4x + 4) \\ &= x^2 - 2x - 3 = (x-3)(x+1). \end{aligned}$$

This leads to the solutions $(x, y) = (3, \pm 11), (-1, \pm 1)$ as well as the solutions $(x, y) = (0, \pm 1)$.

Soution 2, by Hshmat Sahak. Since (x, y) satisfies the equation if and only if $(x, -y)$ does, we will assume that $y \geq 0$. There is no solution with $x = 1$ and we have the solutions $(x, y) = (-1, 1), (0, 1)$. Suppose x and y are positive integers with $x \geq 2$ that satisfy the equation. Then, by the arithmetic-geometric means inequality,

$$(x^2 + 1)^2 = x^4 + 2x^2 + 1 \leq x^4 + x^3 + x + 1 < y^2.$$

Also,

$$(x^2 + x + 1)^2 = y^2 + x(x+1)^2 > y^2.$$

Therefore $y = x^2 + z$ where $2 \leq z \leq x$. Since $x^4 + 2zx^2 + z^2 = y^2 = x^4 + x^3 + x^2 + x + 1$, $z^2 \equiv x + 1 \pmod{x^2}$. Since $0 < z^2 \leq x^2$ and $x + 1 + kx^2$ is either negative or exceeds x^2 when $k \neq 0$, we must have $z^2 = x + 1$ and $2zx^2 = x^3 + x^2$. Since $x \neq 0$, $2z = x + 1$ and

$$(x+1)^2 = 4z^2 = 4(x+1)$$

$x = 3$. This yields the solutions $(x, y) = (3, 11)$.

Now suppose that $x = -w \leq -2$, so that $y^2 = w^4 - w^3 + w^2 - w + 1$. Then $y^2 = w^4 - (w^2+1)(w-1) < w^4$ and

$$(w^2 - w - 1)^2 = y^2 - (w^3 + 2w^2 - 3w) = y^2 - w(w+3)(w-1) < y^2.$$

Therefore $y = w^2 - v$ where $1 \leq v \leq w$. Therefore

$$w^4 - 2vw^2 + v^2 = w^4 - w^3 + w^2 - w + 1,$$

whence $v^2 \equiv -w + 1 \pmod{w^2}$. Since v^2 is positive and $w^2 - w + 1 = (w-1)^2 + w < w$, this congruence has no solution for which $1 \leq v \leq w$. Therefore there are no additional solutions.

Therefore the complete set of solutions is $(x, y) = (0, \pm 1), (-1, \pm 1), (3, \pm 11)$.

4. Determine all polynomials $f(x)$ that satisfy $g'(x) = 4xf(x)$ where $g(x) = f(f(x))$. Are there any non-polynomial solutions?

Solution. One obvious solution is $f(x) = 0$, and no other constant function satisfies the equation. Let $n > 0$ be the degree of f. Then the degree of g is n^2 and so of g' is $n^2 - 1$. Therefore $n^2 - 1 = n + 1$, whence $0 = n^2 - n - 2 = (n+1)(n-2)$. Therefore the degree of f is 2. Note that the equation of the problem is

$$f'(f(x))f'(x) = 4xf(x). \tag{1}$$

Suppose that $f(x) = ax^2 + bx + c$. Then $g(x) = af(x)^2 + bf(x) + c$ and

$$4xf(x) = g'(x) = 2f(x)f'(x) + bf'(x).$$

Hence $f(x)$ divides $bf'(x)$, so that $b = 0$. Therefore $4x = 2af'(x) = 4a^2x$. It follows that $f(x) = \pm x^2 + c$, and it can be checked that each of these satisfies the equation.

A different solution to the equations is $f(x) = 1/4x$. (Note that $f(f(x)) = x$.)

5. For which positive integers x is the following inequality true?

$$\sqrt{x}^{\sqrt{x}} < \sqrt{x+1}.$$

Solution 1, by Louis Ryan Tan. Suppose that the inequality holds for $x > 2$. Then

$$\sqrt{x}\log x < \log(x+1) < \log x^2 = 2\log x,$$

whence $x < 4$. The inequality clearly holds for $x = 1$. Now

$$\sqrt{2}^{\sqrt{2}} < (2^{1/2})^{3/2} = 8^{1/4} < 9^{1/4} = \sqrt{3}$$

and

$$\sqrt{3}^{\sqrt{3}} > (3^{1/2})^{3/2}) = 27^{1/4} > 16^{1/4} = \sqrt{4}.$$

Therefore the inequalily holds for $x = 1, 2$ and fails for $x \geq 3$.

Solution 2. The inequality is equivalent to $x^{\sqrt{x}} < x + 1$. When $x \geq 4$, $x(x-1) > 1$ so that $x^{\sqrt{x}} \geq x^2 > x+1$, and the inequality is false. Since $3^{\sqrt{3}} > 3^{1.5} = 3\sqrt{3} > 4$, the inequality is false for $x = 3$. Since $2^3 < 3^2$, $2^{\sqrt{2}} < 2^{3/2} < 3$ and the inequality holds. It clearly holds when $x = 1$. Therefore the inequality holds if and only if $x = 1, 2$.

Solution 3. We begin with some preliminary observations. Since $2^3 < 3^2$ and $3^5 < 2^8$, then $3/2 < \log_2 3 < 8/5$. Also $4/3 < \sqrt{2} < 3/2$ and $5/3 < \sqrt{3} < 7/4$. Therefore

$$\sqrt{2}\log_2(\sqrt{2}) = (1/2)\sqrt{2} < 3/4 < (1/2)\log_2 3 = \log_2\sqrt{3},$$

whence $\sqrt{2}^{\sqrt{2}} < \sqrt{3}$. Also

$$\sqrt{3}\log_2(\sqrt{3}) > (5/3)(1/2)(3/2) = 5/4 > 1 = \log_2(\sqrt{4}),$$

whence $\sqrt{3}^{\sqrt{3}} > 2 = \sqrt{4}$. We see that the inequality holds for $x = 1$ and $x = 2$ but not for $x = 3$.

Let $x \geq 4$. Then

$$\begin{aligned}
\sqrt{x}\log\sqrt{x} - \log\sqrt{x+1} &= \frac{1}{2}\log x\left[\sqrt{x} - \frac{\log(x+1)}{\log x}\right] \\
&= \frac{1}{2}\log x\left[\sqrt{x} - \frac{\log x + \log(1+x^{-1})}{\log x}\right] \\
&= \frac{1}{2}\log x\left[(\sqrt{x}-1) - \frac{\log(1+x^{-1})}{\log x}\right] \\
&> \frac{1}{2}\log x\left[\sqrt{x} - 1 - \frac{1}{x\log x}\right] \\
&> \frac{1}{2}\log x\left[2 - \frac{5}{4}\right] > 0.
\end{aligned}$$

The only positive integers x for which the inequality holds are 1 and 2.

Solution 4, by Lucas Jacobs. The inequality $\sqrt{x}^{\sqrt{x}} < \sqrt{x+1}$ is equivalent to

$$\sqrt{x}\log x < \log x + \int_x^{x+1} \frac{1}{t}\,dt$$

or

$$(\sqrt{x} - 1) \log x < \int_x^{x+1} \frac{1}{t}\, dt.$$

When $x \geq 4$, the left side exceeds 1 while the right side is less than $1/x \leq 1/4$. Therefore the inequality is false when $x \geq 4$. The cases $x = 1, 2, 3$ can be handled as before.

Solution 5, by Chaim Lowen. It can be checked that the inequality holds for $x = 1, 2$. Let $x \geq 3$. Then $\sqrt{x} > 1 + (1/x)$; this can be verified for $x = 3$ and when $x \geq 4$, the left side is not less than 2. Hence

$$\begin{aligned} x^{\sqrt{x}-1} &= \exp((\sqrt{x} - 1) \log x > \exp(\sqrt{x} - 1) > \exp(1/x) \\ &> \left(1 + \frac{1}{x}\right) = \frac{x+1}{x}, \end{aligned}$$

from which $x^{\sqrt{x}} > x + 1$. Therefore the given inequality is false when $x \geq 3$.

Solution 6. Let $t = x^2$ and $f(t) = 2t \log t - \log(t^2 + 1)$. Then

$$f'(t) = 2\left(1 - \frac{t}{t^2+1}\right) + 2 \log t$$

is positive for $t \geq 1$. Therefore $f(t)$ is increasing. Now

$$f(\sqrt{3}) = 2\sqrt{3} \log 3 - \log 4 = 2 \log 2(\sqrt{3} \log_2 3 - 1) > 2 \log 2(\sqrt{3} - 1) > 0.$$

This proves that the inequality is false for $x \geq 3$. However, it does hold for $x = 1, 2$.

Note. The given inequality is equivalent to

$$x < \left[\frac{\log(x+1)}{\log x}\right]^2$$

which clearly is false when x becomes suitably large. Alternatively, if we let $x = u^2$, then the inequality becomes $u^{2u} < u^2 + 1$, again false when u is large.

6. Let $p(z) = z^3 + yz + x$ be a cubic polynomial with real coefficients $1, x, y$. Sketch in the cartesian plane the locus of points (x, y) for which $p(z)$ has at least one root whose absolute value is 1.

Solution 1. 1 is a root of $p(z)$ if and only $y = -(1 + x)$, and -1 is a root if and only if $y = x - 1$. Suppose that $p(z)$ has a nonreal root $e^{i\theta}$. Then $e^{-i\theta}$ is also a root, and the remaining root is a real number $-r$. Therefore

$$p(z) = (z - e^{i\theta})(z - e^{-i\theta})(z + r) = z^3 - (-r + e^{i\theta} + e^{-i\theta})z^2 + (-r(e^{i\theta} + e^{-i\theta}) + 1)z + r.$$

Hence $x = r = e^{i\theta} + e^{-i\theta} = 2 \cos \theta$ and

$$y = 1 - r(e^{i\theta} + e^{-i\theta}) = 1 - 2r \cos \theta = 1 - 4 \cos^2 \theta = 1 - x^2.$$

Conversely, if $|x| \leq 2$, we can find θ for which $x = \cos 2\theta$ and determine y so that $p(e^{i\theta}) = 0$.

The set of points (x, y) consists of two straight lines of slopes 1 and -1 passing through $(0, -1)$ and the portion of the parabola with equation $y = 1 - x^2$ where $-2 \leq z \leq 2$.

Solution 2. Suppose that $p(z)$ has the root $u+iv$ where u and v are real numbers for which $u^2+v^2=1$. Then

$$0 = p(u+iv) = (u^3 - 3uv^2 + uy + x) + iv(3u^2 - v^2 + y).$$

Either $v=0$, $|u|=1$ and $y = \pm x - 1$, or

$$y = v^2 - 3u^2 = 1 - 4u^2$$

and

$$x = 3uv^2 - u^3 - uy = 3uv^2 - u^3 - u + 4u^3 = 3u(u^2+v^2) - u = 2u.$$

Hence $y = 1 - x^2$, where $|x| \leq 2$.

Solution 3. Suppose that $z = e^{i\theta}$ is a root of $p(z)$. Then $e^{3i\theta} + ye^{i\theta} + x = 0$, whence $(e^{-3i\theta})x + (e^{-2i\theta})y = -1$. Separating out the real and imaginary parts, we obtain

$$(\cos 3\theta)x + (\cos 2\theta)y = -1,$$
$$(\sin 3\theta)x + (\sin 2\theta)y = 0;$$

a linear system whose determinant of coefficients is $\cos 3\theta \sin 2\theta - \sin 3\theta \cos 2\theta = -\sin\theta$. Thus, it has a unique solution except when $\theta = 0, \pi$.

Using the facts that $\cos 3\theta = \cos\theta(4\cos^2\theta - 1)$ and $\sin 3\theta = \sin\theta(4\cos^2\theta - 3)$, when $\theta \neq 0, \pi$,

$$(x, y) = (2\cos\theta, 1 - 4\cos^2\theta).$$

Therefore, part of the desired locus is the portion of the parabola with equation $y = 1 - x^2$ where $-2 < x < 2$.

It remains to deal with the possible roots ± 1. These occur when $x + y + 1 = 0$ and $x - y - 1 = 0$. Thus the locus is the union of these two lines and the portion of the parabola.

Comment. The two lines intersect at $(0, -1)$ corresponding to the polynomial $z^3 - z = z(z-1)(z+1)$. One of the lines intersection the full parabola at $(-1,0), (1,0), (-2,-3), (2,-3)$ corresponding to the respective polynomials $z^3 - 1 = (z-1)(z^2+z+1)$, $x^3+1 = (z+1)(z^2-z+1)$, $z^3 - 3z - 2 = (z+1)(z^2-z-2) = (z+1)^2(z-2)$ and $z^3 - 3z + 2 = (z-1)(z^2+z-2) = (z-1)^2(z+2)$.

7. Let a and b be positive reals and $\{x_n\}$ be a sequence for which

$$\lim_{n\to\infty} \left(ax_n + \frac{b}{x_n} \right) = 2\sqrt{ab}.$$

Must the sequence $\{x_n\}$ converge? If so, what are the possible limits?

Solution 1. Observe that

$$\left(ax_n + \frac{b}{x_n} \right)^2 - \left(ax_n - \frac{b}{x_n} \right)^2 = 4ab,$$

whence, letting $n \to \infty$, we see that $ax_n - (b/x_n) \to 0$. Since

$$2ax_n = \left(ax_n + \frac{b}{x_n} \right) + \left(ax_n - \frac{b}{x_n} \right),$$

it follows that $\lim_{n\to\infty} x_n = \sqrt{b/a}$. Therefore $\{x_n\}$ must converge and the only limit is $\sqrt{b/a}$.

Solution 2. Wolog, we can assume that $x_n > 0$ for all n, Since $ax_n < ax_n + (b/x_n) < 2\sqrt{ab} + 1$ for large values of n, the sequence $\{x_n\}$ is bounded. Let $u = \liminf x_n$ and $v = \limsup x_n$. Since $\{x_n\}$ have subsequences with these limits, both u and v are solutions of $at + (b/t) = \sqrt{ab}$ or

$$0 - at^2 - 2\sqrt{ab}t + b = (\sqrt{a}t - \sqrt{b})^2.$$

Since this equation has a single solution, $u = v = \sqrt{b/a}$ and this is the limit of $\{x_n\}$.

Solution 3. When n is sufficiently large, $0 < ax_n < ax_n + (b/x_n) < 3\sqrt{ab}$. Therefore

$$\left| ax_n + \frac{b}{x_n} - 2\sqrt{ab} \right| = \frac{(\sqrt{a}x_n - \sqrt{b})^2}{x_n} \geq \frac{a(\sqrt{a}x_n - \sqrt{b})^2}{3\sqrt{ab}},$$

As $n \to \infty$, the left side goes to 0, and so then must the right side. Therefore $\lim_{n\to\infty} x_n = \sqrt{b/a}$.

Solution 4. Suppose that $y_n = ax_n + (b/x_n)$. Then $ax_n^2 - y_n x_n + b = 0$, from which

$$x_n = \frac{y_n \pm \sqrt{y_n^2 - 4ab}}{2a}.$$

Let $n \to \infty$. Then, the quantity under the square root tends to 0 no matter which sign precedes it, so that $\lim_{n\to\infty} x_n = 2\sqrt{ab}/2a = \sqrt{b/a}$.

Solution 5, by Louis Ryan Tan. Since x_n and $ax_n + (b/x_n)$ have the same sign, there exists an index u for which $x_n > 0$ for $n \geq u$. There exists an index $v > u$ for which

$$ax_n < ax_n + \frac{b}{x_n} < 3\sqrt{ab},$$

or $c > x_n$ where $c = 3\sqrt{b/a}$.

Let $\epsilon > 0$ and choose $w > v$ such that, for $n \geq w$,

$$\left| ax_n + \frac{b}{x_n} - 2\sqrt{ab} \right| < \frac{a\epsilon^2}{c}.$$

Then, for $n \geq w$,

$$\begin{aligned} \frac{|\sqrt{a}x_n - \sqrt{b}|^2}{c} &\leq \frac{|ax_n^2 - 2\sqrt{ab}x_n + b|}{x_n} \\ &= \left| ax_n + \frac{b}{x_n} - 2\sqrt{ab} \right| < \frac{a\epsilon^2}{c}, \end{aligned}$$

whence

$$|x_n - \sqrt{b/a}| = (1/\sqrt{a})|\sqrt{a}x_n - \sqrt{b}| < \epsilon.$$

The result follows.

Note. In solution 3, we can see that if $2\sqrt{ab}$ is replaced by a larger number, that the sequence $\{x_n\}$ could have two limit points. The graph of the function $ax + (b/x)$ is above and tangent to the line $y = 2\sqrt{ab}$ when $x = \sqrt{b/a}$. It can be seen from a graphical picture that the limit in the problem forces the convergence of $\{x_n\}$. However, if, say, $2\sqrt{ab}$ is replaced by $4\sqrt{ab}$, then one can find a bounded nonconvergent sequence $\{x_n\}$ that makes this limit possible. This is problem 4163 from *Crux Mathematicorum* 43:7.

8. Let f be a real-valued function defined on the plane for which $f(x, y) = (x^2 - y^2)e^{-(x^2+y^2)}$.

(a) Prove that f attains its maximum and minimum.

(b) Determine all the points (x,y) where f has a global or local extremum (maximum or minimum).

Solution 1. Fix $t > 0$ and let $x^2 + y^2 = t$. Then $-t \leq x^2 - y^2 \leq x^2 + y^2 \leq t$ so that

$$-te^t \leq f(x,y) \leq te^t$$

with equality on the left iff $(x,y) = (0, \pm\sqrt{t})$ and on the right iff $(x,y) = (\pm\sqrt{t}, 0)$. The function te^{-t} achieves its maximum value of e^{-1} when $t = 1$. It follows that, when $(x,y) \neq (0,0)$, $f(x,y)$ assumes its maximum value when $(x,y) = (\pm 1, 0)$ and its minimum value when $(x,y) = (0, \pm 1)$.

Since $f(0,0) = (0,0)$ and f assumes negative values along the $y-$axis and positive values along the $x-$axis, f assumes neither a maximum nor a minimum value at the origin.

Solution 2. (a) Observe that $f(1,0) = e^{-1}$ and $f(0,1) = -e^{-1}$. It is readily checked that the function te^{-t} is decreasing when $t > 1$. Hence, for $t \geq 2$, $te^{-t} \leq 2e^{-2} < e^{-1}$. It follows that $|f(x,y)| \leq (x^2 + y^2)e^{-(x^2+y^2)} < e^{-1}$ whenever $x^2 + y^2 \geq 2$. Hence, if f achieves an extreme value, it must do so on the closed disc $\{(x,y) : x^2 + y^2 \leq 2\}$. Since this disc is compact, it in fact achieves both its maximum and minimum values within the disc (and evidently not on the boundary).

It must achieve its extreme values in the interior of the disc where both partial derivatives vanish. We have that

$$\frac{\partial f}{\partial x}(x,y) = 2(x - x^3 + xy^2)e^{-(x^2+y^2)};$$

$$\frac{\partial f}{\partial y}(x,y) = 2(-y - x^2 y + y^3)e^{-(x^2+y^2)}.$$

To find the critical points, we need to solve

$$x(1 - x^2 + y^2) = y(1 + x^2 - y^2) = 0.$$

The solutions are $(x,y) = (0,0), (0,1), (0,-1), (1,0), (-1,0)$.

Since $f(x,0) > 0$ for $x \neq 0$ and $f(0,y) < 0$ for $y \neq 0$, $(0,0)$ is neither a local or global maximum or minimum for $f(x,y)$. Since $f(1,0) = f(-1,0) = e^{-1}$ and $f(0,1) = f(0,-1) = -e^{-1}$ and the extreme values can be assumed among only these four points, $f(x,y)$ mus assume its maximum value and $(\pm 1, 0)$ and its minimum at $(0, \pm 1)$.

9. Determine all the primes p for which $(2^{p-1} - 1)/p$ is a perfect square.

Solution. When $p = 2, 3, 5, 7$, the quantity in question is respectively equal to $1/2$, 1, 3 and 9, so the primes 3 and 7 yield squares. Suppose that $p > 7$ is such that $2^{p-1} - 1$ is p times a square.

Suppose that $p = 4k + 1$. Then $2^{p-1} - 1 = (2^{2k} - 1)(2^{2k} + 1)$. Since the factors are coprime, the factor not divisible by p must be a square. But two positive squares cannot differ by 1, so this case is impossible.

The remaining case is that $p = 4k + 3$. In this case, $2^{p-1} - 1 = (2^{2k+1} - 1)(2^{2k+1} + 1)$, and one of the factors must be square. If $2^{2k+1} + 1 = u^2$, then $2^{2k+1} = (u-1)(u+1)$. Since each factor must be a power of 2, the only possibility is $k = 1$ and $u = 3$. This gives the known case $p = 7$.

Otherwise, we must have $2^{2k+1} - 1 = v^2$ and $2^{2k+1} + 1 = pw^2$. But then $v^2 \equiv -1 \pmod 8$, which is impossible. So we have found all the possibilities.

10. Suppose that $a_0 = 1$, $a_1 = 2$ and

$$a_{n+1} = a_n + \frac{a_{n-1}}{1 + a_{n-1}^2},$$

for $n \geq 1$. Determine integers u and v for which $v - u < 23$ and $u < a_{2023} < v$.

Solution. Observe that

$$\frac{a_{n-1}^2 + 1}{a_{n-1}} = a_{n-1} + \frac{1}{a_{n-1}}.$$

We note that $a_1 = a_0 + \frac{1}{a_0}$. Suppose that for $k \geq 1$, it has been established that

$$a_k = a_{k-1} + \frac{1}{a_{k-1}} = \frac{1 + a_{k-1}^2}{a_{k-1}}.$$

Then

$$a_{k+1} = a_k + \frac{a_{k-1}}{1 + a_{k-1}^2} = a_k + \frac{1}{a_k}.$$

Thus, by induction, for $n \geq 1$,

$$a_{n+1} = a_n + \frac{1}{a_n}.$$

Squaring, we find that

$$\begin{aligned} a_{n+1}^2 &= 2 + a_n^2 + \frac{1}{a_n^2} = 4 + a_{n-1}^2 + \frac{1}{a_{n-1}^2} + \frac{1}{{a_n}^2} \\ &= \cdots = 2n + a_0^2 + \sum_{k=0}^{n} \frac{1}{{a_k}^2}. \end{aligned}$$

Hence $a_{n+1}^2 > 2n + a_0^2 = 2n + 1$, and for each index k, $a_k^2 \geq 2k - 1$, and so $\frac{1}{a_k^2} \leq \frac{1}{2k-1}$.

Therefore

$$\begin{aligned} a_{n+1}^2 &\leq 2n + 1 + 1 + 1 + \sum_{k=2}^{n} \frac{1}{2k-1} \\ &= 2n + 3 + \left[\frac{1}{3} + \frac{1}{5} + \cdots \frac{1}{2n-1}\right] \\ &\leq 2n + 3 + \int_1^{2n-1} \frac{1}{t}\, dt = 2n + 3 + \log_e(2n-1) \\ &< 2n + 3 + \log_2(2n-1). \end{aligned}$$

Setting $n = 2022$ we find that

$$4044 < a_{2023}^2 < 4047 + \log_2 4045 < 4047 + 12 = 4059.$$

Clearly, $60 < a_{2023} < 70$. [More precisely, $63.5925 < a_{2023} < 63.7103$.]

11. Prove that $x + y + z$ divides the polynomial

$$f(x, y, z) = 2(x^7 + y^7 + z^7) - 7xyz(x^4 + y^4 + z^4).$$

Solution 1. By the Factor Theorem, $f(x, y, z)$ is divisible by $x + y + z$ if and only if $f(x, y, z) = 0$ when $x + y + z = 0$. Suppose that x, y, z are the roots of the polynomial $t^3 - at^2 + bt - c$.

For $n \geq 0$, let $s_n = x^n + y^n + z^n$ and suppose $s_1 = a = 0$. Note that $t^n = at^{n-1} - bt^{n-2} + ct^{n-3}$ when $n \geq 3$ and $t = x, y, z$. Then

$$\begin{aligned}
s_0 &= 3 \\
s_1 &= 0 \\
s_2 &= s_1^2 - 2b = -2b \\
s_3 &= 3c \\
s_4 &= -bs_2 = 2b^2 \\
s_5 &= -bs_3 + cs_2 = -3bc - 2bc = -5bc \\
s_6 &= -bs_4 + cs_3 = -2b^3 + 3c^2 \\
s_7 &= -bs_5 + cs_4 = 5b^2c + 2b^2c = 7b^2c
\end{aligned}$$

Hence

$$f(x, y, z) = 14b^2c^2 - 7c(2b^2) = 0$$

as desired.

Solution 2, by George Mu-Zhao. We establish the result by showing that $f(x, y, z) \equiv 0$ (modulo $x+y+z$). In what follows, $\equiv$ will refer to congruences with this modulus. Begin by noting that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \equiv 0$$

and

$$x^2 \equiv [-(y + z)]^2 = y^2 + z^2 + 2yz.$$

Thus $x^3 + y^3 + z^3 \equiv 3xyz$, $y^2 + z^2 \equiv x^2 - 2yz$ and $y + z \equiv -x$, with similar congruences for other permutations of the variables.

We have that

$$\begin{aligned}
3xyz(x^4 + y^4 + z^4) &\equiv (x^3 + y^3 + z^3)(x^4 + y^4 + z^4) \\
&= x^7 + y^7 + z^7 + x^3y^3(x + y) + y^3z^3(y + z) + z^3x^3(z + x) \\
&\equiv x^7 + y^7 + z^7 - xyz(x^2y^2 + y^2z^2 + z^2x^2),
\end{aligned}$$

whence

$$x^7 + y^7 + z^z \equiv xyz[3(x^4 + y^4 + z^4) + (x^2y^2 + y^2z^2 + z^2x^2)].$$

Therefore

$$\begin{aligned}
f(x, y, z) &\equiv xyz(2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4) \\
&= xyz[x^2(y^2 + z^2) + y^2(z^2 + x^2) + z^2(x^2 + y^2) - (x^4 + y^4 + z^4)] \\
&\equiv xyz(x^4 - 2x^2yz + y^4 - 2xy^2z + z^4 - 2xyz^2 - x^4 - y^4 - z^4) = -2xyz[xyz(x + y + z)] \equiv 0,
\end{aligned}$$

as desired.

Note. By the Factor Theorem, it can be shown that $x + y + z$ is a factor by going through the laborious process of checking that $f(x, y, -(x + y)) = 0$. This problem appears in Crux Mathematicorum, #2012 in January, 1996.

12. (a) Determine real numbers a, b, c for which

$$abc = a + b + c = 1.$$

Is it possible for a, b, c to be all positive?

(b) Let $\epsilon > 0$. Prove that there are non-zero rational numbers a, b, c, d for which

$$abc = a + b + c = 1 + d,$$

where $|d| < \epsilon$.

(c) Determine rational numbers a, b, c for which $abc = a + b + c$ and $|abc - 1| < 0.1$

Solution. (a) Pick an arbitrary $a \neq 0$. Then $b + c = 1 - a$ and $bc = 1/a$, so that b and c are the roots of the quadratic equation $x^2 + (a-1)x + 1/a = 0$. The discriminant $(a-1)^2 - 4/a$ is nonnegative whenever $a < 0$, or $a > 0$ and $a(a-1)^2 \geq 4$. Any of these values of a generates a solution. We note that in all cases exactly one of a, b, c is positive.

When $a = -1$, we get the equation $x^2 - 2x - 1 = 0$ and the solution $(a, b, c) = (-1, 1 + \sqrt{2}, 1 - \sqrt{2})$.

(b) Suppose that $b = c$. Then we examine the equation $ab^2 = a + 2b$, or $ab^2 - 2b - a = 0$. Therefore $b = (1 \pm \sqrt{1 + a^2})/a$. Consider the rational pythagorean triple $(u^2 - 1, 2u, u^2 + 1)$ where u is rational. Set $a = (u^2 - 1)/2u$, so that

$$b = \frac{2u \pm (1 + u^2)}{u^2 - 1}.$$

Taking the minus sign, we can take

$$(a, b) = \left(\frac{u^2 - 1}{2u}, -\frac{u-1}{u+1} \right).$$

The common value of $a + 2b$ and ab^2 is equal to

$$f(u) = \frac{(u-1)^3}{2u(u+1)}.$$

Since $f(4) = \frac{27}{40} < 1$ and $f(5) = \frac{16}{15} > 1$, there is a number $r \in (4, 5)$ for which $f(r) = 1$.

Given $\epsilon > 0$, $\exists \delta > 0$ for which $|u - r| < \delta$ implies that $|f(u) - 1| = |f(u) - f(r)| < \epsilon$. There is a rational u that satisfies this condition, and this generates a triple $(a, b, c) = (a, b, b)$ that satisfies the problem.

(c) In particular, when

$$(a, b, c) = \left(\frac{12}{5}, \frac{-2}{3}, \frac{-2}{3} \right),$$

then

$$|abc - 1| = |a + b + c - 1| = \left| \frac{16}{15} - 1 \right| = \frac{1}{15} < 0.1.$$

Comment. There are other possibilities in (c). If we take

$$(a, b) = \left(\frac{u^2 - 1}{2u}, \frac{u+1}{u-1} \right),$$

then $f(u) = ab^2 = a + 2b = (u+1)^3/(2u(u-1))$. If we take

$$(a, b) = \left(\frac{2u}{u^2 - 1}, u \right),$$

then $f(u) = 2u^3/(u^2 - 1)$. When $u = -2/3$, then $f(u) = 16/15$ and $(a, b, c) = (12/5, -2/3, -2/3)$. If we take

$$(a, b) = \left(\frac{2u}{u^2 - 1}, \frac{-1}{u} \right),$$

then $f(u) = 2/(u(u^2-1))$.

Other solution for $abc = a+b+c = 1$ were given, to wit $(a,b,c) = (4, -(3/2)+\sqrt{2}, -(3/2)-\sqrt{2})$ from Berjer Ding, $(a,b,c) = (3, -1+\sqrt{3/2}, -1-\sqrt{3/2})$ from Benjamin Edian, and $(a,b,c) = (-2, \frac{1}{2}(3+\sqrt{11}), \frac{1}{2}(3-\sqrt{11})$ from Haylyn Fung. The one given in the solution was also obtained from Zhiyuan Li. When $a,b,c > 0$ and $abc = 1$, the arithmetic-geometric means inequality shows that $a+b+c \geq 3$. On the other hand, if $a+b+c = 1$, then

$$\begin{aligned} 1 = (a+b+c)^3 &= a^3+b^3+c^3+\sum_{(6)} a^2b + 6abc \\ &\geq 3abc + 6(3(a^6b^6c^6)^{1/6} + 6abc = 27abc. \end{aligned}$$

whence $abc \leq 27$ with equality if and only if $a = b = c = 1/3$.

If $a,b,c > 0$ and $abc = a+b+c = 1$, then, since $\log t \leq t-1$ for $t > 0$, we find that

$$1 = (a - \log a) + (b - \log b) + (c - \log c) \geq 1+1+1 = 3,$$

a contradiction.

13. $ABCD$ is a convex quadrilateral whose diagonals AC and BD intersect at P. Suppose that $PA = PD$, $PB = PC$ and O is the centre of the circumcircle of triangle APB. Prove that $OP \perp CD$.

Solution 1, by Louis Ryan Tan. Because $\angle APD = \angle BPC$, the isosceles triangles APD and BPC have equal base angles, and AD and BC have a common right bisector through P. The reflection through this right bisector fixes P and switches A and D as well as B and C. Therefore $\angle PCD = \angle ABP$.

Let PO be produced to meet the circumcircle of ABP at R. Then $\angle RAP = 90°$. We have that

$$\angle ARQ = \angle ARP = \angle ABP = \angle PCD = \angle ACQ.$$

Therefore $ARCQ$ is a concyclic quadrilateral with equal angles subtended at A and Q by the side RC. Hence

$$\angle OQC = \angle RQC = \angle RAC = \angle RAP = 90°,$$

as desired.

Solution 2. Let OP produced meet CD at Q. First, suppose that O lies outside triangle PAB. Since triangles PDC and PAB are congruent (SAS), $\angle PDC = \angle PAB$. Being opposite, $\angle DPQ = \angle OPB$. Using the fact that triangle OAP, OPB and OAB are isosceles, we find that

$$\begin{aligned} \angle PDQ + \angle DPQ &= \angle PAB + \angle OPB = \angle OAP - \angle OAB + \angle OBP \\ &= \angle OPA - \angle OBA + \angle OBP = \angle OPA + \angle PBA. \end{aligned}$$

Since

$$\begin{aligned} 180° &= \angle PAB + \angle APB + \angle PBA = \angle PAB + \angle OPB + \angle OPA + \angle PBA \\ &= 2(\angle PAB + \angle OPB), \end{aligned}$$

it follows that $\angle PDQ + \angle DPQ = 90°$, and so $\angle OQD = \angle PQD = 90°$, as desired.

If O lies within triangle PAB, as before $\angle PDQ = \angle PAB$ and $\angle DPQ = \angle OPB$.

$$\begin{aligned} \angle PDQ + \angle DPQ &= \angle PAB + \angle OPB = \angle OAP + \angle OAB + \angle OBP \\ &= \angle OPA + \angle PBA. \end{aligned}$$

The remainder of the argument is as before.

Solution 3, by George Mu-Zhao. Suppose O is internal to triangle APB; an adapted argument is possible when O is external. Wolog, let $BC < AD$. Triangles PBA and PCD are isosceles, and the common right bisector m of BC and AD passes through P. Let OP produced meet CD at N. The reflection in m interchanges triangles PBA and PCD and carries OPN to a line through P that meets AB at M. We have that $\angle BPM = \angle CPN = \angle APO$.

Let $\angle APO = \angle BPM = \alpha$, $\angle OAB = \angle OBA = \beta$, and $\angle OPM = \gamma$. Since $OA = OB = OP$,

$$\begin{aligned} 180^\circ &= \angle APB + \angle ABP + \angle BAP \\ &= (2\alpha + \gamma) + (\beta + \alpha + \gamma) + (\beta + \alpha) = 2(2\alpha + \beta + \gamma), \end{aligned}$$

and

$$180^\circ = \angle PMA + (\alpha + \beta) + (\alpha + \gamma) = \angle PMA + (2\alpha + \beta + \gamma),$$

whence $\angle PND = \angle PMA = 90^\circ$ and the result follows.

Solution 4, by Murhammadrizo Madjidov. Since triangles PBC and PCD are congruent (SAS), $\angle ABP = \angle DCP$. Drop a perpendicular from O to meet AP at R, and produce OP to meet CD at Q. Then $\angle ROP = \frac{1}{2}\angle AOP = \angle ABP = \angle PCQ$. Also the opposite angles OPR and CPQ are equal. Hence triangles ROP and QCP are similar. Therefore $\angle PQC = \angle PRO = 90^\circ$, as desired.

Solution 5, by Jing Wang. Assign coordinates, placing P at the origin, B at $(2a, 2b)$ and C at $(2a, -2b)$. Then A is at $(-2ma, 2mb)$ and D is at $(-2ma, -2mb)$ for some positive real m. Let O be placed at (u, v). Since O lies on the right bisectors of both PB and PA, we require that

$$\frac{v-b}{u-a} = \frac{-a}{b} \qquad \text{and} \qquad \frac{v-mb}{u+ma} = \frac{a}{b}.$$

Hence

$$au + bv = a^2 + b^2 \qquad \text{and} \qquad au - bv = -ma^2 - mb^2,$$

whence

$$2au = (1-m)(a^2+b^2) \qquad \text{and} \qquad 2bv = (1+m)(a^2+b^2).$$

The slope of OQ is equal to

$$\frac{v}{u} = \frac{1-m}{2a} \Big/ \frac{1+m}{2b} = \frac{b(1+m)}{a(1-m)}.$$

Since this is the negative reciprocal of the slope of CD, the desired result follows.

Solution 6, by Zhiyuan Li. Assign coordinates, putting A at $(0, 0)$ and D at $(4a, 0)$, where $a > 0$. Since P lies on the right bisector of AD, P is at $(2a, 2b)$ for some $b > 0$. Then C is at $(2(1+t)a, 2(1+t)b)$ for some $t > 0$, and, since $BP = CP$, B is at $(2(1-t)a, 2(1+t)b)$.

Suppose O is at (u, v). The point O is at the intersection of the right bisectors of BP and CP through their respective midpoints $((2-t)a, (2+t)b)$ and (a, b). Since the respective slopes of AP and BP are b/a and $-b/a$,

$$\frac{v-b}{u-a} = \frac{-a}{b} \qquad \text{and} \qquad \frac{v-(2+t)b}{u-(2-t)a} = \frac{a}{b}.$$

Hence $au + bv = a^2 + b^2$ and $au - bv = (2-t)a^2 - (2+t)b^2$. so that

$$2au = (3-t)a^2 - (1+t)b^2 \qquad \text{and} \qquad 2bv = -(1-t)a^2 + (3+t)b^2.$$

The slope of OP is equal to

$$\frac{2b-v}{2a-u} = \left(\frac{2a}{2b}\right)\left[\frac{4b^2+(1-t)a^2-(3+t)b^2}{4a^2-(3-t)a^2+(1+t)b^2}\right]$$
$$= \left(\frac{a}{b}\right)\left[\frac{(1-t)a^2+(1-t)b^2}{(1+t)a^2+(1+t)b^2}\right] = \frac{a(1-t)}{b(1+t)}.$$

Since the slope of CD is $\frac{(1+t)b}{(1-t)a}$, the result follows.

Solution 7. Suppose that the circumcircle of triangle APB is the unit circle in the complex plane centered at 0. Let the points P, A, B be represented respectively by the points 1, $e^{i\alpha}$, $e^{-i\beta}$ in the complex plane, where $1 \leq \alpha, \beta \leq \pi$. Then

$$|PA| = |1-e^{i\alpha}| = \sqrt{(1-e^{i\alpha})(1-e^{-i\alpha})} = 2\sin(\alpha/2)$$

and

$$|PB| = |1-e^{-i\beta}| = 2\sin(\beta/2).$$

Let Q be a point on OP produced. Note that

$$\angle QPC = \angle APO = \frac{\pi-\alpha}{2}$$

and

$$\angle QPD = \angle BPO = \frac{\pi-\beta}{2}.$$

Therefore C and D are represented respectively by the points

$$1+|1-e^{i\alpha}|\exp(-i(\pi-\alpha)/2) = [1+2\sin(\beta/2)\sin(\alpha/2)] - 2i\sin(\beta/2)\cos(\alpha/2)$$

and

$$1+|1-e^{-i\beta}|\exp(i(\pi-\beta)/2) = [1+2\sin(\alpha/2)\sin(\beta/2)] + 2i\sin(\alpha/2)\cos(\beta/2).$$

Since these complex numbers have the same real part, it follows that $CD \perp OP$.

14. Suppose that $a_1, a_2, \ldots, a_n$ are real numbers. Prove that

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{a_i a_j}{i+j}\right) \geq 0.$$

Solution. Let $p(x) = \sum_{i=1}^{n} a_i x^i$. Then

$$0 \leq p(x)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j x^{i+j}.$$

Suppose that $x \geq 0$. Then, dividing this inequality by x, we find that $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j x^{i+j-1} \geq 0$. Therefore

$$0 \leq \int_0^1 \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j x^{i+j-1}\,dx$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_i a_j x^{i+j}}{i+j}\bigg|_0^1 = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_i a_j}{i+j},$$

Note. The sum is equal to the matrix product $a^{\mathbf{t}}Ma$ when M is a positive-definite matrix with $m_{ij} = (i+j)^{-1}$. This is similar to the Hilbert matrix. However, not every matric with positive entries is positive definite; the 2×2 matrix $(1,2;2,3)$ is a counterexample.