THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh and Alfonso Gracia-Saz

March 12-13, 2022

Time: $3\frac{3}{4}$ hours

`No aids or calculators permitted.`

1. In how many ways can a 5×5 square be tiled with 3×1 and 4×1 rectangles in such a way that each point in the square is covered and there is no overlap?

2. Let $f(x)$ and $g(x)$ be two increasing real-valued functions defined on $[0, \infty)$ for which (1) $f(0) = g(0) = 0$ and (2) $f(x) + g(x) = x$ for $x \geq 0$.

   (a) Give an example of such a pair (f, g) of distinct functions.

   (b) Prove that $f(x)$ and $g(x)$ are continuous on $[0, \infty)$.

3. (a) S is a set of positive integers, the largest of which is n. The least common multiple of any pair of numbers in S is greater than n. Prove that the sum of the reciprocals of the numbers in S is less than 2.

   (b) Give two examples of sets S as described in (a) for which the sum of the reciprocals exceeds 1.

   (c) If, in (a), the words "least common multiple" is replaced by "product", does the conclusion still hold?

4. Let
$$h(x, y, z) = x^2 + y^2 + z^2 + 3(xy + yz + zx) + 5(x + y + z) + 1.$$
   Prove that the diophantine equation $h(x, y, z) = 0$ has infinitely many solutions for which x, y, z are all integers.

5. Suppose that $f(x)$ is a continuous real-valued function defined on $[0, 1]$ for which
$$1 = \int_0^1 f(x)\, dx = \int_0^1 x f(x)\, dx.$$

   (a) Prove that
$$\int_0^1 f(x)^2\, dx \geq 4.$$

   (b) Give an example of such a function for which equality occurs in (a).

6. Find all solutions of the differential equation
$$x^2 y'' + 4xy' + (x^2 + 2)y = 1$$
   that are continuous at $x = 0$.

7. The sequence $\{x_n\}$ is defined by the recursion
$$x_{n+1} = 2x_n - n^2$$
   for $n \geq 0$. For which values of the initial term x_0 are all the terms of the sequence positive?

8. What are the possible subsets U of the plane for which (1) U contains finitely many points, and (2) for each point x in the plane, there is exactly one point $y \in U$ of maximum distance, such that $d(x,y) > d(x,u)$ for $u \in U$, $u \neq y$. ($d(x,y)$ is the Euclidean distance between x and y.)

9. Determine the set of all real numbers r for which

$$\frac{a^r}{u} + \frac{b^r}{v} + \frac{c^r}{w} \geq \frac{(a+b+c)^r}{u+v+w},$$

whenever a, b, c, u, v, w are all positive. When does equality hold?

10. For a regular polygon $A_0A_1A_2\dots A_{n-1}$, let a_1 denote the length of a side and a_k the length of the diagonal A_0A_k for $2 \leq k \leq n-1$.

(a) For the regular heptagon $A_0A_1\dots A_6$, prove that

$$\frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_3}.$$

(b) For the regular 15−gon $A_0A_1\dots A_{14}$, prove that

$$\frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_4} + \frac{1}{a_7}.$$

(c) State and prove a generalization for parts (a) and (b).

SOLUTIONS

1. In how many ways can a 5×5 square be tiled with 3×1 and 4×1 rectangles in such a way that each point in the square is covered and there is no overlap?

Solution 1. Let the square be split into 25 cells. Denote the rows from top to bottom by A, B, C, D, E and the columns from left to right by $1, 2, 3, 4, 5$. Each rectangle must be laid parallel to one of the sides of the square, so it belongs entirely to one row or to one column.

The tile that covers cell $A1$ must extend along row A or down column 1. Suppose the former; then it must cover cell $A3$. The tile covering $A5$ must extend down column 5 and so cover $C5$. The tile cover $E5$ must also cover $E3$, and finally the tile covering $E1$ must cover $C1$. Thus cells $A3$, $C5$, $E3$ and $C1$ are all covered by tiles covering the corner cells.

Consider the tile covering $C3$. It can only be a 3×1 tile placed horizontally or vertically. Suppose that it is placed horizontally. In this case, $A4$ and $E2$ must be covered by the respective 4×1 tiles $A1234$ and $E2345$. Cell $B1$ is covered by either 4×1 tiles $B1234$ or $BCDE1$, and cell $D4$ by either $D1234$ or $ABCD4$. Thus, there are 4 possibilities. The tiles in the centre 3×3 array of cells are covered by three horizontal 3×1 tiles.

If the tile covering $C3$ is vertical, then again there are 4 possible configurations. Thus there are 8 configurations when $A1$ is covered by a horizontal tile.

If cell $A1$ is covered by a vertical tile, then a similar analysis verifies that there are 8 additional configurations. Therefore, altogether there are 16 configurations.

Solution 2, by Jing Wang. Adopt the notation of Solution 1. There are two possibilities for the numbers of rectangles: (1) there is one 4×1 rectangle and seven 3×1 rectangles; (2) there are four 4×1 rectangles and three 3×1 rectangles.

Note that each tile must cover at least one cell in either row C or column 3. A 3×1 tile covers one or three such cells and a 4×1 tiles one or four such cells. If there is only one 4×1 tiles (case (1)), then there are eight tiles altogether, so that one of the tiles must cover exactly two of the nine cells in row C and column 3. This is not possible. Therefore, case (2) obtains.

There are seven tiles that cover the nine cells in the central row and column. Since no tile covers exactly two of these central cells, one of the 3×1 cells must lie in either the central row or the central column. Suppose it occupies $C123$. Consider the tile covering cell $C4$. If it is a 3×1 tile, if forces a 3×1 tile in the first three columns of two other rows, forcing a total of four 3×1 tiles, one more than we have. If it is a 4×1 tile, then again it forces a total of four 3×1 tiles. Similarly it cannot occupy $C354$. Hence it must occupy $C234$.

Cells $C1$ and $C5$ must be covered by tiles placed vertically. If they are both 3×1 tiles, then we have used up all the available 3×1 tiles and so both tiles together cannot block any other than the middle row. If say the 3×1 tiles cover $ABC1$ and $CDE5$, then the 4×1 tiles must cover $A2345$, $B2345$, $D1234$ and $E1234$. The other possibility is that the 3×1 tiles cover $CDE1$ and $ABC5$. Thus there are two configurations here.

If both $C1$ and $C5$ are covered by 4×1 tiles, then either the two tiles must cover $ABCD1$ and $BCDE5$ or $BCDE1$ and $ABCD5$. In either case, there are two 4×1 tiles remaining, and they must be placed on the remaining cells in rows A and E. The two other 3×1 tiles fill up rows B and D. There are two configurations here.

Finally, cells $C1$ and $C5$ are covered by a 3×1 and a 4×1 cell. There are four ways this can happen, depending on the columns selected for the cells and whether $A5$ is covered or not by one of them. Suppose, for example, that a 3×1 tiles covers $CDE1$ and a 4×1 cell covers $ABCD5$. Then the remaining 3×1 tile covers $D234$ and the remaining three 4×1 tiles are placed in rows A, B and E.

Thus, there are 8 configuration when a 3×1 tile covers $C234$. Similarly, there are 8 when such a tile covers $BCD3$, for a total of 16 configurations altogether.

Solution 3. First, we note that a 4×1 tile cannot lie along row C. If, say, it covers $C1234$, then each other position $X1234$ must be covered by horizontal tiles, which forces column 5 to be covered by vertical tiles, an impossibility. Similarly a 4×1 tile cannot lie down column 3.

If there are four 4×1 horizontal tiles, then they must cover the first four cells of rows A and B and the last four of C and D, or the last four of A and B and the first four of C and C. The position of three 3×1 tiles is determined. This accounts for two configurations.

If there are three horizontal 4×1 tiles, they must lie in adjacent rows A and B and row D or in row A and adjacent rows C and D. The ones in the adjacent rows can touch one edge and the other one the other edge. There is one placement for three 3×1 tiles. This accounts for four configurations.

If there are two horizontal 4×1 tiles, they must be in rows A and E, and the remaining tiles in columns 1 and 5. The centre 3×3 square of cells can be covered by three 3×1 tiles in two ways. There are four configuration from the two alternative placings of the edge cells and the two alternative placing of the central cells.

Similarly, there are four configurations with one horizontal (and three vertical) 4×1 cell and two configurations with all vertical 4×1 cells. The total number of configurations is $2 + 4 + 4 + 4 + 2 = 16$.

Solution 4. If there is exactly one 4×1 tile, it must abut one edge and or be in a middle row or column. Suppose, say, it is in the top row occupying the first four positions, which we denote by $A1234$. Then the remaining rectangles occupy, in order, $ABC5$, $D345$, $E345$, $CDE1$ and $CDE2$, and it is impossible to place the remaining two. If the 1×4 rectangle is is position $B1234$ or $C1234$, then there is a rectangle in position $A123$ and it is impossible to cover cell $A4$. Thus, there are no possibilities for this situation. The remaining possibility is that there are four 4×1 and three 3×1 tiles.

There here are two possibilities: either a 1×3 tile covers a corner cell or it does not. In the latter case, a 4×1 tile must cover either $A1234$ or $ABCD1$, In either situation, it determines the position of the remaining three such rectangles. The three 3×1 tiles cover the central 3×3 array of cells, all either vertical or horizontal. Thus, there are 4 possibilities for this case.

On the other hand, suppose that a 3×1 rectangle covers $A123$. Since $ABC4$ and $ABC5$ are each covered in whole or part by a single tile, there must be a tile that covers either $E345$ or $E2345$.

If two 3×1 tiles cover $A123$ and $E345$, the remaining 1×3 tiles covers $BCD3$ and the four remaning tiles cover the remainder of the columns $1, 2, 4, 5$. There are 4 ways for this configuration to occur according as two of the three 3×1 tiles are vertical or horizontal and which diagonal pair of cells they cover.

Suppose two tiles exactly cover $A123$ and $E2345$. Then the remaining 3×1 tiles must cover $BCD2$ and $BCD3$ and the remaining 4×1 tiles occupy the remaining squares in columns $1, 4, 5$. There are 8 ways for this configuration to occur according as to which corner is covered by the 3×1 tile and whether it is vertical or horizontal.

Therefore there are 16 possible configurations.

2. Let $f(x)$ and $g(x)$ be two increasing real-valued functions defined on $[0, \infty)$ for which (1) $f(0) = g(0) = 0$ and (2) $f(x) + g(x) = x$ for $x \geq 0$.

   (a) Give an example of such a pair (f, g) of distinct functions.

   (b) Prove that $f(x)$ and $g(x)$ are continuous on $[0, \infty)$.

Note. In part (a), it was intended that the examples be nonlinear, but many students received full credit for $f(x) = cx$ and $g(x) = (1 - c)x$ for some value of c in $(0, 1)$. Some students also provided nontrivial examples.

(a) Here are some possible pairs:

$f(x) = \log(1 + x)$, $g(x) = x - \log(1 + x)$.

$f(x) = \tanh \frac{x}{2}$, $g(x) = x - \tanh \frac{x}{2}$.

$f(x) = \sqrt{x + 1} - 1$, $g(x) = x + 1 - \sqrt{x + 1} = \sqrt{x + 1}(\sqrt{x + 1} - 1)$.

(b) *Solution 1.* Let $x > y \geq 0$. Then

$$f(x) - f(y) \leq f(x) - f(y) + g(x) - g(y) = x - y.$$

Then, for any nonnegative values of x and y, we have that $|f(x) - f(y)| \leq |x - y|$. It follows from this that $f(x)$ is continuous. Therefore $g(x) = x - f(x)$ is also continuous.

Solution 2. Since both $f(x)$ and $g(x)$ are increasing functions, the left and right hand limits of the function exist at each point in their domains. Let $a > 0$ and let $L = \lim_{x \to a^-} f(x) = \sup\{f(x) : x < a\}$ and $R = \lim_{x \to a^+} f(x) = \inf\{f(x) : x > a\}$ be the right and left hand limits of f at a. Then $L \leq R$. Then, since g is increasing

$$\begin{aligned} a - L &= \lim_{x \to a^-} (x - f(x)) = \lim_{x \to a^-} g(x) \leq \lim_{x \to a^+} g(x) \\ &= \lim_{x \to a^+} (x - f(x)) = a - R, \end{aligned}$$

from which $R \leq L$. It follows that $L = R = f(a)$ and f is continuous, as is g. An adaptation of this argument replacing L by $f(0)$ establishes continuity of f and g at 0.

Solution 3. Let $x > 0$ and let $\epsilon > 0$. Suppose, if possible that $f(x + \epsilon) > f(x) + \epsilon$. Then

$$\begin{aligned} g(x + \epsilon) &= (x + \epsilon) - f(x + \epsilon) < (x + \epsilon) - f(x) - \epsilon \\ &= x - f(x) = g(x), \end{aligned}$$

which contradicts the hypothesis that $g(x)$ is increasing. Therefore, for each $\epsilon > 0$,

$$0 \leq f(x + \epsilon) - f(x) \leq \epsilon,$$

from which it follows that $f(x)$, and so $g(x)$, is continuous.

3. (a) S is a set of at least two positive integers, the largest of which is n. The least common multiple of any pair of numbers in S is greater than n. Prove that the sum of the reciprocals of the numbers in S is less than 2.

   (b) Give two examples of sets S as described in (a) for which the sum of the reciprocals exceeds 1.

   (c) If, in (a), the words "least common multiple" is replaced by "product", does the conclusion still hold?

Solution. (a) The elements of S are distinct members of $\{2, 3, \ldots, n\}$. Each element a in S has $\lfloor n/a \rfloor$ multiples that do not exceed n. The sets of multiples not exceeding n for the various elements in S are pairwise disjoint sets. Therefore

$$\sum_{a \in S} \left\lfloor \frac{n}{a} \right\rfloor \leq n,$$

whence

$$\sum_{a \in S} \frac{n}{a} < \sum_{a \in S} \left(\left\lfloor \frac{n}{a} \right\rfloor + 1 \right) \leq 2n.$$

Dividing this equation by n yields the result.

(b) If $S = \{2, 3, 5\}$, the sum of the reciprocals is $31/30$. Another example is $S = \{3, 4, 5, 7, 11\}$, whose reciprocals add to $4690/4620$.

Consider

$$S = \{7, 8, 9, 11, 13, 15, 17, 19, 21, \ldots, 43\},$$

consisting of all the odd integers between 7 and 43 inclusive along with 8. The smallest least common multiple of a pair is 45. We use that fact that, for any positive integer k,

$$\frac{1}{2k-1} + \frac{1}{2k+1} > \frac{1}{k}.$$

Then

$$\begin{aligned}
&\left(\frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{11}\right) + \left(\frac{1}{13} + \frac{1}{15}\right) + \cdots + \left(\frac{1}{41} + \frac{1}{43}\right) \\
&> \frac{1}{4} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} \\
&> \frac{1}{4} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{21} \\
&> \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.
\end{aligned}$$

(c) The conclusion does not hold and there are infinitely many examples. Let m be any positive integer, and let

$$S = \{m, m+1, m+2, \ldots, m^2 - 1, m^2\}.$$

For any integer $1 \leq k \leq m - 1$,

$$\begin{aligned}
&\frac{1}{km+1} + \frac{1}{km+2} + \frac{1}{km+3} + \ldots + \frac{1}{km+m} \\
&\qquad > m\left(\frac{1}{(k+1)m}\right) = \frac{1}{k+1}.
\end{aligned}$$

The sum of the reciprocals of the number in S is greater than $\sum_{k=1}^{m} \frac{1}{k+1}$, which exceeds 2 when m is sufficiently large.

Alternatively, note that the sum of the reciprocal of the numbers in S is greater that

$$\int_m^{m^2} \frac{1}{t}\, dt = \log m$$

which can be made arbitrarily large.

4. Let

$$h(x, y, z) = x^2 + y^2 + z^2 + 3(xy + yz + zx) + 5(x + y + z) + 1.$$

Prove that the diophantine equation $h(x, y, z) = 0$ has infinitely many solutions for which x, y, z are all integers.

Solution. We can write

$$h(x, y, z) = x^2 + (3y + 3z + 5)x + (y^2 + z^2 + 3yz + 5y + 5z + 1).$$

Suppose that $(x, y, z) = (u, v, w)$ is a triple of integers that satisfies $h(u, v, w) = 0$. Then $h(x, v, w) = 0$ is a quadratic equation in x, the sum of whose roots is the integer $-(3v+3w+5)$. Since u is one root, the other is $-(u + 3v + 3w + 5)$, also an integer. Thus

$$h(v, w, -(u + 3v + 3w + 5)) = 0,$$

and we can use this second solution to generate a third and so on. This allows us to generate a bilateral sequence $\{x_n\}$ of integers for which $x_{-1} = u$, $x_0 = v$, $x_1 = w$ and

$$x_{n+1} = -3x_n - 3x_{n-1} - x_{n-2} - 5,$$

for each integer n. We have $h(x_{n-1}, x_n, x_{n+1}) = 0$. It remains to find a solution to the equation and to show that it generates a nonperiodic bilateral sequence.

Note that

$$h(x, y, z) = (x + y + z)^2 + (xy + yz + zx) + 5(x + y + z) + 1.$$

We search for a solution of $h(x, y, z) = 0$ that satisfies in addition $x + y + z = 0$. In this case, we require $xy + yz + zx = -1$. Such a solution is $(x, y, z) = (-1, 0, 1)$, which generates the bilateral sequence:

$$\{\ldots, -16, 7, -3, -1, 0, 1, -7, 13, -24, \ldots$$

where $x_{-1} = -1$, $x_0 = 0$, $x_1 = 1$ and

$$x_{n+1} + 3x_n + 3x_{n-1} + x_{n-2} + 5 = 0,$$

for each integer n. Observe that $x_1 - x_{-1} = 2$, $x_2 - x_{-2} = -4$ and $x_3 - x_{-3} = 6$. It can be proved by induction that $x_n - x_{-n} = (-1)^{n-1}2n$ for each positive integer n. Therefore, the sequence $\{x_n\}$ cannot be periodic.

Comments. In search for a solution, we can try to find one for which $z = 0$. Then x and y have to satisfy

$$x^2 + y^2 + 3xy + 5x + 5y + 1 = 0.$$

This can be rewritten

$$(x - y)^2 + 5(x - 1)(y - 1) - 4 = 0,$$

and it can be seen the $(x, y) = (-1, 1)$ is a solution.

The recursion in the solution can be rewritten

$$(x_{n+1} + 2x_n + x_{n-1}) + (x_n + 2x_{n-1} + x_{n-2}) + 5 = 0.$$

It can be proved by induction that

$$x_{2m+1} + 2x_{2m} + x_{2m-1} = 0$$

and

$$x_{2m} + 2x_{2m-1} + x_{2m-2} = -5$$

for every integer m. There is an additional interesting feature: for $m \geq 2$, it appears that $x_{-(n+1)} + x_{n+1} = -2(x_{-n} + x_n)$, although this fails for $n = 1$.

If we change the order of the variables in the solution $(-1, 0, 1)$ to $(0, -1, 1)$ and $(0, 1, -1)$, we get different sequences

$$\{\ldots, -11, 5, -3, 0, -1, 1, -5, 8, -15, \ldots\}$$

and

$$\{\ldots, -29, 15, -7, 0, 1, -1, -5, 12, -25 \ldots .\}$$

In these sequences, $x_{n+1} + 2x_n + x_{n-1}$ alternates between different pairs of values $(-1, -4)$ and $(1, -6)$, respectively.

Lisa Yu, without any explanation, provided the set of five solutions: $(1, -1, 0)$, $(1, -7, 0)$, $(5, -17, 0)$, $(15, -43, 0)$, $(41, -111, 0)$. I do not know how this was arrived at, but if we denote the terms in the sequence as (x_n, y_n, z_n) we see that after the first entry $x_{n+1} + y_n + 2 = 0$. A little more headscratching suggests a formula. Let $\{f_n\}$ be the Fibonacci bilateral sequence determined by $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for each integer n. Then it appears that, except for $(1, -1, 0)$, we have

$$(x_n, y_n, z_n) = (2f_{2n} - 1, -(2f_{2n+2} - 1), 0),$$

and

$$x_{n+1} = 3x_n - x_{n-1} + 1, \qquad y_{n+1} = 3y_n - y_{n-1} + 1,$$

for each integer n.

Here is a table showing values of n, the corresponding solution (x_n, y_n, z_n) and related solutions with two of the three entries unchanged:

$$-3 : (-17, 5, 0); (-17, 5, 31), (-17, 41, 0), (-3, 5, 0)$$

$$-2 : (-7, 1, 0); (-7, 1, 13), (-7, -15, 0), (-1, 1, 0)$$

$$-1 : (-3, -1, 0); (-3, -1, 7), (-3, 5, 0), (1, -1, 0)$$

$$0 : (-1, -3, 0); (-1, -3, 0), (-1, 1, 0), (5, -3, 0)$$

$$1 : (1, -7, 0); (1, -7, 13), (1, -1, 0), (-15, -7, 0)$$

$$2 : (5, -17, 0); (5, -17, 31), (5, -3, 0), (41, -17, 0)$$

$$3 : (15, -43, 0); (15, -43, 79), (15, -7, 0), (109, -43, 0)$$

$$4 : (41, -111, 0); (41. - 111, 205)$$

$$5; (109, -289, 0).$$

The solutions associated with $(1, -1, 0$ are $(1, -1, -5)$, $(1, -7, 0)$, $(-3, -1, 0)$.

5. Suppose that $f(x)$ is a continuous real-valued function defined on $[0, 1]$ for which

$$1 = \int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx.$$

(a) Prove that

$$\int_0^1 f(x)^2\,dx \geq 4.$$

(b) Give an example of such a function for which equality occurs in (a).

Solution 1. We construct an example of a function that satisfies the conditions of the problem. Let $f(x) = ax + b$. Then we require that $(a/2) + b = (a/3) + (b/2) = 1$, whence

$$a + 2b = 2, \qquad \text{and} \qquad 2a + 3b = 6.$$

This is satisfied by $(a, b) = (6, -2)$, so that $f(x) = 2(3x - 1)$ satisfies the conditions. It also turns out that

$$\int_0^1 (6x-2)^2\,dx = 4,$$

giving us an answer to (b).

Let $f(x)$ satisfy the given conditions and let $g(x) = f(x) - 2(3x - 1)$. Then

$$\begin{aligned} 0 \leq \int_0^1 g(x)^2\,dx &= \int_0^1 f(x)^2\,dx - \int_0^1 4(3x-1)f(x)\,dx + \int_0^1 4(3x-1)^2\,dx \\ &= \int_0^1 f(x)^2\,dx - \int_0^1 12xf(x)\,dx + 4\int_0^1 f(x)\,dx + \int_0^1 4(3x-1)^2\,dx \\ &= \int_0^1 f(x)^2\,dx - 12 + 4 + 4, \end{aligned}$$

whence

$$\int_0^1 f(x)^2\,dx \geq 12 - 4 - 4 = 4.$$

Solution 2, by Nicholas Sullivan. By the Cauchy-Schwarz inequality, we have for any real numbers a and b satisfying $a + b = 1$:

$$\begin{aligned} \int_0^1 f(x)^2\,dx \cdot \int_0^1 (ax+b)^2\,dx &\geq \left[\int_0^1 f(x)(ax+b)\,dx\right]^2 \\ &= \left[a\int_0^1 xf(x)\,dx + b\int_0^1 f(x)\,dx\right]^2 = (a+b)^2 = 1. \end{aligned}$$

Since

$$\begin{aligned} \int_0^1 (ax+b)^2 dx &= \frac{(a+b)^2 - b^3}{3a} = \frac{1-b^3}{3(1-b)} \\ &= \frac{1+b+b^2}{3} = \frac{(b+\frac{1}{2})^2}{3} + \frac{1}{4} \geq \frac{1}{4}, \end{aligned}$$

it follows that

$$\int_0^1 f(x)^2\,dx \geq 4.$$

Equality occurs if and only if $b = -1/2$ and $a = 3/2$, *i.e.*, when $ax + b = \frac{1}{2}(3x - 1)$ and $f(x)$ is a multiple of $3x - 1$. Since $\int_0^1 (3x-1)\,dx = \int_0^1 x(3x-1)\,dx = \frac{1}{2}$ and

$$\int_0^1 (3x-1)^2\,dx = (3x^3 - 3x^2 + x)\Big|_0^1 = 1,$$

we find that $f(x)$ satisfies the conditions of the problem and the conclusion with equality.

6. Find all solutions to the differential equation

$$x^2y'' + 4xy' + (x^2 + 2)y = 1$$

that are continuous at 0?

Solution 1. For $x \neq 0$, let $z = x^2y$. Then $z' = x^2y' + 2xy$ and $z'' = x^2y'' + 4xy' + 2y$. Therefore the equation becomes $z'' + z = 1$. The general solution of this is

$$z = 1 + a\cos x + b\sin x,$$

so that

$$y = x^{-2}(1 + a\cos x + b\sin x).$$

When $x = 0$, the equation is satisfied by $y = 1/2$.

As $x \to 0$, $\cos x = 1 - \frac{1}{2}x^2 + o(x^3)$ and $\sin x = x + \frac{1}{6}x^3 + o(x^4)$, so that in order for the solution to be finite as $x \to 0$, we require that $(a, b) = (-1, 0)$. The desired solution is

$$y = \frac{1 - \cos x}{x^2} = \left(\frac{\sin(x/2)}{x/2}\right)^2.$$

it can be checked that this satisfies the equation.

Solution 2. Suppose that $y = \sum_{k=0}^{\infty} a_k x^k$. Then

$$1 = 2a_0 + 6a_1x + \sum_{k=2}^{\infty}(a_{k-2} + (k+2)(k+1)a_k)x^k,$$

so that $a_0 = \frac{1}{2}$, $a_{2m-1} = 0$ for each positive integer m, and

$$a_{2m} = \frac{-1}{(2m+2)(2m+1)}a_{2m-2} = \frac{(-1)^m a_0}{(2m+2)(2m+1)\cdots(4)(3)} = \frac{(-1)^m}{(2m+2)!}.$$

Therefore

$$\begin{aligned} y &= \frac{1}{2} + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{(2m+2)!} \\ &= \frac{1}{x^2}\left[\sum_{m=0}^{\infty}\frac{(-1)^m x_{2m+2}}{(2m+2)!}\right] \\ &= \frac{1}{x^2}\left[\sum_{m=1}^{\infty}\frac{(-1)^{m-1}x^{2m}}{(2m)!}\right] = \frac{1}{x^2}[1 - \cos x]. \end{aligned}$$

7. The sequence $\{x_n\}$ is defined by the recursion

$$x_{n+1} = 2x_n - n^2$$

for $n \geq 0$. For which values of the initial term x_0 are all the terms of the sequence positive?

Answer. All terms are positive if and only if $x_0 \geq 3$.

Solution 1. Observe that, if $\{u_n\}$ and $\{v_n\}$ are two sequences satisfying the recursion, then $v_{n+1} - u_{n+1} = 2(v_n - u_n)$ for $n \geq 0$, so that $v_n = (v_0 - u_0)2^n + u_n$, so that if we find a solution for one instance of the recursion, solutions for other instances can be found by adding a multiple of 2^n. The answer to the problem turns on finding a convenient solution for the recursion.

For $\{x_n\}$ satisfying $x_{n+1} = 2x_n - n^2$, set $y_n = 2x_n - x_{n+1}$ for $n \geq 0$. Then $y_n = n^2$, $y_{n+1} - y_n = 2n+1$,

$$y_{n+2} - 2y_{n+1} + y_n = (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n) = 2,$$

and

$$(y_{n+3} - 2y_{n+2} + y_{n+1}) - (y_{n+2} - 2y_{n+1} + y_n) = 0.$$

From the definition of y_n, we can see that this condition is satisfied when

$$\begin{aligned} 0 &= (x_{n+3} - 2x_{n+2} + x_{n+1}) - (x_{n+2} - 2x_{n+1} + x_n) \\ &= (x_{n+3} - 2x_{n+2}) - (x_{n+2} - 2x_{n+1}) + (x_{n+1} - 2x_n) + x_n \\ &= -(n+2)^2 + (n+1)^2 - n^2 + x_n, \end{aligned}$$

whence

$$x_n = (n+2)^2 - (n+1)^2 + n^2 = n^2 + 2n + 3 = (n+1)^2 + 2.$$

It can be checked that the recursion is satisfied for these values of x_n and that $x_0 = 3$.

In general, the recursion is satisfied by

$$x_n = (x_0 - 3)2^n + (n^2 + 2n + 3)$$

for $n \geq 0$. When $x_0 \geq 3$, each term of the recursion is positive. Suppose that $x_0 < 3$ and that $m = 3 - x_0$. Then $x_n = (n^2 + 2n + 3) - m2^n$, which is negative for n sufficiently large.

Solution 2, by Samuel Li. Let $y_n = 2^{-n}x_n$ for $n \geq 0$. Then

$$y_n = y_{n-1} - \frac{(n-1)^2}{2^n} = y_0 - \sum_{k=1}^{n} \frac{(k-1)^2}{2^k}.$$

Therefore $x_n \geq 0$ for all $n \geq 0$ if and only if $y_n \geq 0$ if and only if

$$x_0 = y_0 \geq \sum_{k=1}^{\infty} \frac{(k-1)^2}{2^k}.$$

It remains only to evaluate the series on the right.

Let $f(x) = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots$. Then

$$f'(x) = (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots + kx^{k-1}x^{k-1} + (k+1)x^k + \cdots$$

and

$$f''(x) = 2(1-x)^{-3} = 2 + 6x + \cdots + k(k-1)x^{k-2} + (k+1)kx^{k-1} + (k+2)(k+1)x^k + \cdots.$$

For each real x,

$$\begin{aligned} \sum_{k=1}^{\infty}(k-1)^2x^k = \sum_{k=1}^{\infty}[k(k-1) - k + 1]x^k &= x^2f''(x) - xf'(x) + f(x) - 1 \\ &= 2x^2(1-x)^{-3} - x(1-x)^{-2} + (1-x)^{-1} - 1. \end{aligned}$$

Setting $x = 1/2$ yields the sum $4 - 2 + 2 - 1 = 3$, and the answer follows.

Solution 3. By induction, it can be shown that

$$x_n = 2^n x_0 - \sum_{k=0}^{n-1} 2^{(n-1)-k} k^2$$

for $n \geq 1$. $x_n \geq 0$ for all $n \geq 0$ iff $2^n x_0 \geq \sum_{k=0}^{n-1} 2^{(n-1)-k} k^2$ for all $n \geq 0$ iff

$$x_0 \geq A \equiv \sum_{k=0}^{\infty} \frac{k^2}{2^{k+1}}.$$

$$\begin{aligned}
A &= \sum_{k=1}^{\infty} \frac{(k-1)^2}{2^k} \\
&= 2\sum_{k=1}^{\infty} \frac{k^2}{2^{k+1}} - 2\sum_{k=1}^{\infty} \frac{k}{2^k} + \sum_{k=1}^{\infty} \frac{1}{2^k} \\
&= 2A - 2\sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} \frac{1}{2^j} \right) + 1 \\
&= 2A - 2\sum_{i=1}^{\infty} \frac{1}{2^{i-1}} + 1 = 2A - 3.
\end{aligned}$$

Therefore $A = 3$, each $x_n \geq 0$ if and only if $x_0 \geq 3$.

8. What are the possible subsets U of the plane for which (1) U contains finitely many points, and (2) for each point x in the plane, there is exactly one point $y \in U$ of maximum distance, such that $d(x, y) > d(x, u)$ for $u \in U$, $u \neq y$. ($d(x, y)$ is the Euclidean distance between x and y.)

Solution 1. It is straightforward to see that any singleton U has the property. Suppose, if possible, that a set U with at least two elements has the property. Then for each point x in the plane, $f(x) \neq x$ (whether or not it is in U). Suppose that for each point x in the plane, $f(x)$ is the unique point in U of maximum distance from x. We show that $f(x)$ is continuous.

Let v be any point of the plane. Let $d(v, f(v)) = r > 0$ and let $\epsilon > 0$ be such that $d(v, u) < r - \epsilon$ for $u \in U$, $u \neq f(v)$. Suppose that $d(v, x) < \epsilon/2$. Then $d(x, f(v)) > r - \epsilon$, while

$$d(x, u) < (r - \epsilon) + (1/2)\epsilon = r - (1/2)\epsilon$$

for $u \in U$, $u \neq f(v)$. Thus, if $d(x, v) < \epsilon/2$, then $f(x) = f(v)$. Hence f is a continuous function on the plane.

Suppose that $f(y) \neq f(z)$. Let $g(t) = f((1-t)y + tz)$ for $0 \leq t \leq 1$. Since $g(t)$ is continuous and has a finite discrete image, $[0, 1]$ is the union of at least two relatively open subsets $g^{-1}(u)$ where $u \in U$, which is impossible.

Solution 2, based on an approach by Brendan Kelly. Suppose that U is a set with at least two points that satisfies the conditions of the problem. Let z be a point in the plane that is not in U and let w be the farthest point from z in U. For x in the plane, define

$$g(x) = \max \{d(x, u) : u \in U, u \neq w\}.$$

Since each $d(x, u)$ is a continuous function of x, $g(x)$ is continuous. For x in the plane, let

$$h(x) = d(x, w) - g(x).$$

For $0 \leq t \leq 1$, define $z_t = (1-t)z+tw$. Observe that $h(z_t)$ is a continuous function of t for which $h(z_0) = h(z) > 0$ and $h(z_1) = h(w) < 0$. Therefore, there exists $s \in (0,1)$ for which $h(z_s) = d(z_s, w) - g(z_s) = 0$. Let $v \in U$, $v \neq w$ be such that $g(z_s) = d(z_s, v)$. Then w and v are two points in U which exceed $d(x, u)$ for all u distinct from v and w. But this contradicts the conditions of the problem. Thus, U must be a singleton.

Solution 3, based on an approach by Fateme Sajadi. If U lies within a line segment $[p, q]$ (with $p, q \in U$), then, since any point on the right bisector of the segment joining the points is equidistant from the points, U does not satisfy the condition. Suppose that U is a set with at least three points satisfying the conditions of the problem. The convex hull of U is the intersection of all the closed half-planes that contain U. It is a closed convex set whose boundary is a (possibly degenerate) polygon P whose vertices are extreme points (*i.e.*, points not contained in an open interval lying entirely within the convex hull).

Let $[v, w]$ be an edge of P with endpoints v and w lying in U, and let L be the right bisector of $[v, w]$. For each point $u \in U$ distinct from v and w, when x is a point of L on the same side of $[v, w]$ as U and sufficiently far from $[v, w]$ we have $d(x, u) < d(x, v) = d(x, w)$. Just arrange that the angle $\angle vux$ is obtuse (*i.e*, further away from v and w than the intersection of this bisector with the line through u perpendicular to uv). If x is far enough out, then $d(x, u) < d(x, v) = d(x, w)$ for all $u \in U$ distinct from v and w. But this gives a contradiction, and so U must be a singleton.

Solution 4, based on the approach of Samuel Li. Suppose that U contains at least two points, and that a and b are two points in U maximum distance apart. The circle C of centre b and radius $d(a, b)$ contains a on its circumference and all of U within its closed disc. Now let a central similarity of centre a be applied to C that shrinks it to a circle C' with centre p whose circumference contains a second point $c \in U$ and all of U within its closed disc. Then $d(p, a) = d(p, c) \geq d(p, u)$ $\forall u \in U$. It follows that U does not satisfy the condition of the problem. [This problem was contributed by Bamdad R. Yahaghi.]

Solution 5, based on the approach of Chayim Lowen. Suppose that U contains at least two points. For $v \in U$, let

$$P_u = \{x : d(x, v) > d(x, u) \quad \text{for} \quad u \in U, u \neq v\}.$$

Then P_u is the intersection of finitely many open half planes bounded by the right bisectors of the segments joining v to each u in U, and so is an open set. Each point x in the plane belongs to exactly one of the sets P_u, so that the plane would be the union of finitely many open sets. But this is impossible, since it is connected.

9. Determine the set of all real numbers r for which

$$\frac{a^r}{u} + \frac{b^r}{v} + \frac{c^r}{w} \geq \frac{(a+b+c)^r}{u+v+w},$$

whenever a, b, c, u, v, w are all positive. When does equality hold?

Solution 1. When $a = b = c = u = v = w = 1$, the inequality becomes $3 \geq 3^{r-1}$. Therefore, it is necessary that $r \leq 2$.

Suppose that $r = -s < 0$. Then

$$\frac{a^r}{u} + \frac{b^r}{v} + \frac{c^r}{w} > \frac{1}{ua^s} > \frac{1}{(u+v+w)(a+b+c)^s} = \frac{(a+b+c)^r}{u+v+w}.$$

Consider the case $r = 0$. Since $(u+v+w)(uv+vw+wu) > 3uvw > uvw$,

$$\frac{a^r}{u} + \frac{b^r}{v} + \frac{c^r}{w} = \frac{1}{u} + \frac{1}{v} + \frac{1}{w} > \frac{1}{u+v+w} = \frac{(a+b+c)^r}{u+v+w}.$$

Finally, we turn to $0 < r \leq 2$. By the Cauchy-Schwarz Inequality

$$(x_1y_1 + x_2y_2 + x_3y_3)^2 \leq (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)$$

applied to $(x_1, x_2, x_3) = (u^{1/2}, v^{1/2}, w^{1/2})$ and $(y_1, y_2, y_3) = ((a^r/u)^{1/2}, (b^r/v)^{1/2}, (c^r/w)^{1/2})$, we find that

$$(a^{r/2} + b^{r/2} + c^{r/2})^2 \le (u + v + w)\left(\frac{a^r}{u} + \frac{b^r}{v} + \frac{c^r}{w}\right).$$

Observe that, when $0 \le s, d, e, f \le 1$ and $d + e + f = 1$, then $d^s + e^s + f^s \ge d + e + f = 1$. Applying this to $s = r/2$, $d = a(a + b + c)^{-1}$, $e = b(a + b + c)^{-1}$ and $f = a(a + b + c)^{-1}$, we find that $a^{r/2} + b^{r/2} + c^{r/2} \ge (a + b + c)^{r/2}$. Hence

$$(a + b + c)^r \le (a^{r/2} + b^{r/2} + c^{r/2})^2 \le (u + v + w)\left(\frac{a^r}{u} + \frac{b^r}{v} + \frac{c^r}{w}\right).$$

Dividing by $u+v+w$ yields the desired inequality. Equality occurs in the Cauchy-Schwarz Inequality when $a : b : c = u : v : w$ and in the power inequality when $s = 1$, *i.e.* when $r = 2$.

Solution 2. As in the previous solution, we see that $r \le 2$. If the inequality

$$\frac{a^r}{u} + \frac{b^r}{v} \ge \frac{(a + b)^r}{u + v}, \tag{1}$$

holds whenever $a, b, u, v > 0$, then

$$\frac{a^r}{u} + \frac{b^r}{v} + \frac{c^r}{w} \ge \frac{(a + b)^r}{u + v} + \frac{c^r}{w} \ge \frac{(a + b + c)^r}{u + v + w}.$$

Thus, it remains to establish that (1) holds for $r \le 2$.

If $r = -s \le 0$, since $u < u + v$, $a < a + b$,

$$\begin{aligned} \frac{a^r}{u} + \frac{b^r}{v} &= \frac{1}{ua^s} + \frac{1}{vb^s} > \frac{1}{ua^s} \\ &> \frac{1}{(u + v)(a + b)^s} = \frac{(a + b)^r}{u + v}. \end{aligned}$$

Let $0 < r \le 2$, $t = b/a$ and $k = v/u$. Wolog, we may suppose that $b \le a$, so that $0 < t \le 1$. Then (1) holds if and only if

$$1 + \frac{t^r}{k} \ge \frac{(1 + t)^r}{1 + k} \tag{2}$$

holds for $k > 0$ and $0 < t \le 1$.

Let $0 < r \le 1$ and

$$f(t) = 1 + \frac{t^r}{k} - \frac{(1 + t)^r}{1 + k}$$

for $0 \le t \le 1$. Then

$$f'(t) = \frac{rt^{r-1}}{k(k + 1)}[(k + 1) - k(t^{-1} + 1)^{r-1}].$$

Since $f(0) > 0$ and $f'(t) > 0$, $f(t)$ is positive on $[0, 1]$ and the desired result holds.

Let $1 < r \le 2$. Since $0 < t \le 1$, then $t^r \ge t^2$ and $(1 + t)^r \le (1 + t)^2$. Therefore

$$\begin{aligned} 1 + \frac{t^r}{k} - \frac{(1 + t)^r}{1 + k} &\ge 1 + \frac{t^2}{k} - \frac{(1 + t)^2}{1 + k} \\ &= \frac{(t - k)^2}{k(k + 1)} \ge 0. \end{aligned}$$

Equality occurs if and only if $r = 2$ and $t = k$, *i.e.*, $b/a = v/u$. In the original inequality of the problem, equality holds if and only if $a : b : c :: u : v : w$.

10. For a regular polygon $A_0A_1A_2 \ldots A_{n-1}$, let a_1 denote the length of a side and a_k the length of the diagonal A_0A_k for $2 \leq k \leq n-1$.

 (a) For the regular heptagon $A_0A_1 \ldots A_6$, prove that

$$\frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_3}.$$

 (b) For the regular 15−gon $A_0A_1 \ldots A_{14}$, prove that

$$\frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_4} + \frac{1}{a_7}.$$

 (c) State and prove a generalization for parts (a) and (b).

Solution 1, by Zhekai Pang. (a) The quadrilateral $A_0A_1A_2A_4$ is cyclic, and so by Ptolemy's Theorem,

$$a_1a_2 + a_1a_3 = a_1a_2 + a_1a_4 = a_2a_3,$$

whence

$$\frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_3}.$$

(b) Apply Ptolemy's Theorem to the cyclic quadrilaterals $A_0A_1A_2A_4$ and $A_0A_1A_4A_8$ to obtain $a_2a_3 = a_1a_2 + a_1a_4$ and $a_4a_7 = a_1a_4 + a_3a_8 = a_1a_4 + a_3a_7$. Hence

$$\begin{aligned}\frac{1}{a_7} &= \frac{1}{a_1} - \frac{a_3}{a_1a_4} = \frac{1}{a_1} - \frac{a_2a_3}{a_1a_2a_4} \\ &= \frac{1}{a_1} - \frac{a_1a_2 + a_1a_4}{a_1a_2a_4} = \frac{1}{a_1} - \frac{1}{a_4} - \frac{1}{a_2},\end{aligned}$$

from which the result follows.

(c) Let $n = 2^m - 1$. For $1 \leq k \leq m-2$, apply Ptolemy's theorem to obtain

$$a_{2^k}a_{2^{k+1}-1} = a_1a_{2^k} + a_{2^k-1}a_{2^{k+1}}.$$

In particular, when $k = m-2$,

$$a_{2^{m-2}}a_{2^{m-1}-1} = a_1a_{2^{m-2}} + a_{2^{m-2}-1}a_{2^{m-1}} = a_1a_{2^{m-2}} + a_{2^{m-2}-1}a_{2^{m-1}-1}.$$

It follows that

$$a_{2^{m-1}-1}(a_{2^{m-2}} - a_{2^{m-2}-1}) = a_1a_{2^{m-2}}$$

or

$$\frac{1}{a_{2^{m-1}-1}} = \frac{1}{a_1} - \frac{a_{2^{m-2}-1}}{a_1a_{2^{m-2}}} \qquad (1).$$

For $1 \leq k \leq m-3$. we have that

$$\frac{a_{2^{k+1}-1}}{a_1a_{2^{k+1}}} = \frac{a_{2^k}a_{2^{k+1}-1}}{a_1a_{2^k}a_{2^{k+1}}} = \frac{a_1a_{2^k} + a_{2^k-1}a_{2^{k+1}}}{a_1a_{2^k}a_{2^{k+1}}} = \frac{1}{a_{2^{k+1}}} + \frac{a_{2^k-1}}{a_1a_{2^k}}.$$

Applying this successively to equation (1) for $k = m-3, m-2, \ldots, 1$, we find that

$$\begin{aligned} \frac{1}{a_{2^{m-1}-1}} &= \frac{1}{a_1} - \frac{a_{2^{m-2}-1}}{a_1 a_{2^{m-2}}} = \frac{1}{a_1} - \frac{1}{2^{m-2}} - \frac{a_{2^{m-3}-1}}{a_1 a_{2^{m-3}}} = \cdots \\ &= \frac{1}{a_1} - \frac{1}{a^{2^{m-2}}} - \frac{1}{a^{2^{m-3}}} - \cdots - \frac{1}{a_4} - \frac{1}{a_2}. \end{aligned}$$

Therefore

$$\frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_4} + \cdots + \frac{1}{a_{2^{m-2}}} + \frac{1}{2_{2^{m-1}-1}}.$$

Solution 2. For any angle ϕ,

$$\csc 2\phi + \cot 2\phi = \frac{1+\cos 2\phi}{\sin 2\phi} = \frac{2\cos^2\phi}{2\sin\phi\cos\phi} = \cot\phi,$$

whence $\csc 2\phi = \cot\phi - \cot 2\phi$.

(a) Let r be the circumradius of the heptagon. Then $a_k = 2r\sin k\theta$, where $\theta = \pi/7$. Therefore

$$\begin{aligned} \frac{1}{a_2} + \frac{1}{a_3} &= \frac{1}{a_2} + \frac{1}{a_4} = \frac{1}{2r}(\csc 2\theta - \csc 4\theta) \\ &= \frac{1}{2r}(\cot\theta - \cot 2\theta + \cot 2\theta - \cot 4\theta) \\ &= \frac{1}{2r}(\cot\theta - \cot 4\theta) = \frac{1}{2r}(-\cot 6\theta + \cot 3\theta) = \frac{1}{2r}\csc 6\theta = \frac{1}{2r}\csc\theta = \frac{1}{a_1}, \end{aligned}$$

since $\cot k\theta = -\cot(7-k)\theta$ and $\csc k\theta = \csc(7-k)\theta$ is for $1 \le k \le 7$.

(b)(c) *Solution.* The result can be generalized to a regular polygon with $2^m - 1$ sides:

$$\frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_4} + \frac{1}{a_8} + \cdots + \frac{1}{a_{2^{m-1}-1}}.$$

Follow the argument in (a) where $\theta = \pi/(2^m - 1)$, note that $a_{2^{m-1}-1} = a_{2^{m-1}}$, and obtain

$$\begin{aligned} &\frac{1}{a_2} + \frac{1}{a_4} + \frac{1}{a_8} + \ldots + \frac{1}{a_{2^{m-1}-1}} \\ &\quad = \frac{1}{a_2} + \frac{1}{a_4} + \frac{1}{a_8} \ldots + \frac{1}{a_{2^{m-1}}} \\ &\quad = \frac{1}{2r}(\cot\theta - \cot 2^{m-1}\theta) = \frac{1}{2r}(-\cot(2^m-2)\theta + \cot(2^{m-1}-1)\theta/2) \\ &\quad = \frac{1}{2r}\csc(2^m-2)\theta = \frac{1}{2r}\csc\theta = \frac{1}{a_1}. \end{aligned}$$

Solution 3, by Samuel Li. Place the vertices of a general regular $n-$gon at the roots ω^{2k} of unity on the unit circle in the complex plane, where $\omega = \exp(\pi i/n)$. For $0 \le k \le n-1$, the length of the segment A_0A_k is equal to

$$|\omega_{2k} - 1| = |\omega^k - \omega^{-k}| = -i(\omega^k - \omega^{-k}) = -i\frac{\omega_{2k}-1}{\omega_k}.$$

Note that $\omega^n = -1$ and that $1 - \omega + \omega^2 - \cdots + (-1)^{n-1}\omega^{n-1} = 0$. If $n = 2r+1$ is odd, then

$$a_r = a_{r+1} = |\omega^{2r+2} - 1| = |-\omega - 1| = |1 + \omega|.$$

(a) When $n = 7$, it is required to show that

$$\frac{1}{\omega - \omega^{-1}} = \frac{1}{\omega^2 - \omega^{-2}} + \frac{1}{\omega^4 - \omega^{-4}}.$$

Here $\omega^7 = -1$ and

$$0 = \omega^6 - \omega^5 + \omega^4 - \omega^3 + \omega^2 - \omega + 1.$$

Therefore

$$\begin{aligned}\omega^2 + 1 + \omega^{-2} &= \omega^2 + 1 - \omega^5 = \omega^3 + \omega - \omega^6 - \omega^4 \\ &= \omega^3 + \omega + \omega^{-1} + \omega^{-3}.\end{aligned}$$

The right side of the equation is equal to

$$\begin{aligned}&\frac{1}{\omega - \omega^{-1}}\left[\frac{1}{\omega + \omega^{-1}} + \frac{1}{(\omega + \omega^{-1})(\omega^2 + \omega^{-2})}\right] \\ &\qquad = \frac{1}{\omega - \omega^{-1}}\left[\frac{\omega^2 + 1 + \omega^{-2}}{\omega^3 + \omega + \omega^{-1} + \omega^{-3}}\right] = \frac{1}{\omega - \omega^{-1}},\end{aligned}$$

as desired.

(b) In the case that $n = 15$, the corresponding right side of the equation is

$$\begin{aligned}&\frac{1}{\omega^2 - \omega^{-2}} + \frac{1}{\omega^4 - \omega^{-4}} + \frac{1}{\omega^8 - \omega^{-8}} \\ &= \frac{1}{\omega - \omega^{-1}}\left[\frac{1}{\omega + \omega^{-1}} + \frac{1}{(\omega + \omega^{-1})(\omega^2 + \omega^{-2})} + \frac{1}{(\omega + \omega^{-1})(\omega^2 + \omega^{-2})(\omega^4 + \omega^{-4})}\right] \\ &= \frac{1}{\omega - \omega^{-1}}\left[\frac{\omega^6 + \omega^2 + \omega^{-2} + \omega^{-6} + \omega^4 + \omega^{-4} + 1}{\omega^7 + \omega^3 + \omega^{-1} + \omega^{-5} + \omega^3 + \omega + \omega^{-3} + \omega^{-5}}\right] \\ &= \frac{1}{\omega - \omega^{-1}}.\end{aligned}$$

The desired result follows.

(c) In the general case, when $n = 2^{m+1} - 1$ $(m \geq 2)$, the generalization is

$$\frac{1}{a_1} = \frac{1}{a_2} + \frac{1}{a_4} + \cdots + \frac{1}{a_{2^m}}.$$

Let $\omega = \exp[\pi i/(2^{m+1} - 1)]$. Then $\omega^{2^{m+1}-1} = -1$ and

$$1 + \sum_{k=1}^{2^{m-2}-1} (\omega^{2k} + \omega^{-2k}) = \sum_{k=0}^{2^{2m-2}} (\omega^{2k+1} + \omega^{-(2k+1)}).$$

First, note that for any number λ and nonnegative integer r, we have that

$$g(\lambda, r) \equiv (\lambda + \lambda^{-1})(\lambda^2 + \lambda^{-2})(\lambda^4 + \lambda^{-4}) \ldots (\lambda^{2^r} + \lambda^{-2^r}) = \sum \lambda^s.$$

where the product on the left has $r + 1$ factors and the sum of the right has 2^{r+1} terms and the values of s run through (once each) all the 2^{r+1} odd numbers between $1 + 2 + \cdots + 2^r = 2^{r+1} - 1$ and $-(2^{r+1} - 1)$.

The right side of the equation to be verified is equal to $(\omega - \omega^{-1})^{-1}$ times a fraction whose denominator is equal to

$$\prod\{(\omega^{2^k} + \omega^{2^{-k}}) : 0 \leq k \leq 2^{m-2}\} = g(\omega, m - 2)$$

which is the sum of odd powers of ω and whose numerator is equal to the sum of 1 and products of terms

$$g(\omega^{2^t}, m-2-t) = (\omega^{2^t} + \omega^{2^{-t}})(\omega^{2^{t+1}} + \omega^{2^{-(t+1)}}) \cdots (\omega^{2^{m-2}} + \omega^{2^{-(m-2)}})$$

where $1 \leq t \leq m-2$; this product picks up powers of ω whose exponents are odd multiples of 2^t. Thus the numerator is the sum of even powers of ω, and the ratio of the numerator and denominator is 1.

Solution 4. (a) Suppose that the polygon has circumradius 1 and that $\theta = \pi/7$. $a_k = 2\sin k\theta$ and $\cos k\theta + \cos(7-k)\theta = 0$ for $1 \leq k \leq 6$. Therefore

$$0 = \cos 5\theta - \cos 3\theta + \cos 2\theta - \cos 4\theta,$$

whence

$$\cos\theta - \cos 5\theta = \cos\theta - \cos 3\theta + \cos 2\theta - \cos 4\theta.$$

Using the identity $2\sin A \sin B = \cos(A-B) - \cos(A+B)$, we find that

$$\sin 3\theta \sin 2\theta = \sin 2\theta \sin\theta + \sin 3\theta \sin\theta.$$

Dividing by $2(\sin\theta)(\sin 2\theta)(\sin 3\theta)$ yields the desired result.