THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

Saturday, March 10, 2012

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. An equilateral triangle of side length 1 can be covered by five equilateral triangles of side length u. Prove that it can be covered by four equilateral triangles of side length u. (A triangle is a closed convex set that contains its three sides along with its interior.)

2. Suppose that f is a function defined on the set Z of integers that takes integer values and satisfies the condition that $f(b) - f(a)$ is a multiple of $b - a$ for every pair a, b, of integers. Suppose also that p is a polynomial with integer coefficients such that $p(n) = f(n)$ for infinitely many integers n. Prove that $p(x) = f(x)$ for every positive integer x.

3. Given the real numbers a, b, c not all zero, determine the real solutions x, y, z, u, v, w for the system of equations:
$$\begin{aligned} x^2 + v^2 + w^2 &= a^2 \\ u^2 + y^2 + w^2 &= b^2 \\ u^2 + v^2 + z^2 &= c^2 \\ u(y+z) + vw &= bc \\ v(x+z) + wu &= ca \\ w(x+y) + uv &= ab. \end{aligned}$$

4. (a) Let n and k be positive integers. Prove that the least common multiple of $\{n, n+1, n+2, \cdots, n+k\}$ is equal to
$$rn\binom{n+k}{k}$$
for some positive integer r.

   (b) For each positive integer k, prove that there exist infinitely many positive integers n, for which the number r defined in part (a) is equal to 1.

5. Let $\mathfrak{C}$ be a circle and Q a point in the plane. Determine the locus of the centres of those circles that are tangent to $\mathfrak{C}$ and whose circumference passes through Q.

6. Find all continuous real-valued functions defined on R that satisfy $f(0) = 0$ and
$$f(x) - f(y) = (x-y)g(x+y)$$
for some real valued function $g(x)$.

There are more problems overleaf.

7. Consider the following problem:

Suppose that $f(x)$ is a continuous real-valued function defined on the interval $[0,2]$ for which

$$\int_0^2 f(x)dx = \int_0^2 (f(x))^2 dx \ .$$

Prove that there exists a number $c \in [0,2]$ for which either $f(c) = 0$ or $f(c) = 1$.

(a) Criticize the following solution:

Solution. Clearly $\int_0^2 f(x)dx \geq 0$. By the extreme value theorem, there exist numbers u and v in $[0,2]$ for which $f(u) \leq f(x) \leq f(v)$ for $0 \leq x \leq 2$. Hence

$$f(u) \int_0^2 f(x)dx \leq \int_0^2 f(x)^2 dx \leq f(v) \int_0^2 f(x)dx \ .$$

Since $\int_0^2 f(x)^2 dx = 1 \cdot \int_0^2 f(x)dx$, by the intermediate value theorem, there exists a number $c \in [0,2]$ for which $f(c) = 1$. □

(b) Show that there is a nontrivial function f that satisfies the conditions of the problem but that never assumes the value 1.

(c) Provide a complete solution of the problem.

8. Determine the area of the set of points (x, y) in the plane that satisfy the two inequalities:

$$x^2 + y^2 \leq 2$$
$$x^4 + x^3y^3 \leq xy + y^4.$$

9. In a round-robin tournament of $n \geq 2$ teams, each pair of teams plays exactly one game that results in a win for one team and a loss for the other (there are no ties).

(a) Prove that the teams can be labelled $t_1, t_2, \cdots, t_n$, so that, for each i with $1 \leq i \leq n-1$, team t_i beats t_{i+1}.

(b) Suppose that a team t has the property that, for each other team u, one can find a chain $u_1, u_2, \cdots, u_m$ of (possibly zero) distinct teams for which t beats u_1, u_i beats u_{i+1} for $1 \leq i \leq m-1$ and u_m beats u. Prove that *all* of the n teams can be ordered as in (a) so that $t = t_1$ and each t_i beats t_{i+1} for $1 \leq i \leq n-1$.

(c) Let T denote the set of teams who can be labelled as t_1 in an ordering of teams as in (a). Prove that, in any ordering of teams as in (a), all the teams in T occur before all the teams that are not in T.

10. Let A be a square matrix whose entries are complex numbers. Prove that $A^* = A$ if and only if $AA^* = A^2$.

Notes. For any $m \times n$ matrix M with entries m_{ij}, the *hermitian transpose* M^* is the $n \times m$ matrix M^* obtained by taking the complex conjugates of entries of M and transposing; thus, the (i,j)th element of M^* is $\overline{m_{ji}}$. In particular, for the complex column vector x with ith entry x_i, x^* is a row vector whose ith entry is $\bar{x}_i$. The inner product $\langle x, y \rangle$ of two column vectors is $\sum \bar{x}_i y_i = x^* y$, and we have that $\langle x, Ay \rangle = \langle A^* x, y \rangle$. A matrix for which $A^* = A$ is said to be *hermitian*.

Solutions

1. An equilateral triangle of side length 1 can be covered by five equilateral triangles of side length u. Prove that it can be covered by four equilateral triangles of side length u. (A triangle is a closed convex set that contains its three sides along with its interior.)

Solution. Consider the set of six points consisting of the vertices of the triangle and the midpoints of the three sides. By the Pigeonhole Principle, one of the five covering equilateral triangles must cover two of these six points. Since the distance between any two of the sixpoints is at least $\frac{1}{2}$ and any two points within an equilateral triangle cannot be further apart than two of its vertices, we must have that $u \geq \frac{1}{2}$.

However, it is clear that the given equilateral triangle can be covered by four such triangles of side-length $\frac{1}{2}$ determined by joining pairs of midpoints of the sides. Each of these four triangles can be included within a triangle of side-length u.

2. Suppose that f is a function defined on the set Z of integers that takes integer values and satisfies the condition that $f(b) - f(a)$ is a multiple of $b - a$ for every pair a, b, of integers. Suppose also that p is a polynomial with integer coefficients such that $p(n) = f(n)$ for infinitely many integers n. Prove that $p(x) = f(x)$ for every positive integer x.

Solution. Let $q(x) = f(x) - p(x)$ for each integer x. The function $q(x)$ vanishes on an infinite subset M of integers and $q(a) - q(b)$ is divisible by $a - b$ for every pair a, b of integers (since the same is true of p). [If $p(x) = \sum c_i x_i$, then $p(a) - p(b) = \sum c_i(a^i - b^i) = (a - b) \sum c_i(a^{i-1} + a^{i-2}b + \cdots + ab^{i-2} + b^{i-1})$.]

Let n be any integer. Then $q(n) = q(n) - q(m)$ is divisible by $n - m$ for each $m \in M$. But then either $q(n) = 0$ or $|q(n)| \geq |n - m|$ for infinitely many integers m. However, the latter case cannot occur.

Note. The function $f(x)$ itself need not be a polynomial. For example, we might have $f(x) = x + \sin \pi x$.

3. Given the real numbers a, b, c not all zero, determine the real solutions x, y, z, u, v, w for the system of equations:

$$\begin{aligned} x^2 + v^2 + w^2 &= a^2 \\ u^2 + y^2 + w^2 &= b^2 \\ u^2 + v^2 + z^2 &= c^2 \\ u(y + z) + vw &= bc \\ v(x + z) + wu &= ca \\ w(x + y) + uv &= ab. \end{aligned}$$

Solution 1. Evaluating b^2c^2 in two ways, we find that

$$\begin{aligned} 0 &= (u^2 + y^2 + w^2)(u^2 + v^2 + z^2) - (uy + uz + vw)^2 \\ &= (u^2 - yz)^2 + (uv - wz)^2 + (wu - vy)^2. \end{aligned}$$

Hence $u^2 = yz$. Similarly, $v^2 = zx$ and $w^2 = xy$. Inserting these three values into the first three equations yields that

$$\begin{aligned} x(x + y + z) &= a^2 \\ y(x + y + z) &= b^2 \\ z(x + y + z) &= c^2. \end{aligned}$$

It follows that not all of x, y, z can vanish and that $(x + y + z)^2 = a^2 + b^2 + c^2$. Also

$$x : a^2 = y : b^2 = z : c^2 = 1 : (x + y + z) = (x + y + z) : (a^2 + b^2 + c^2).$$

Therefore

$$(x,y,z,u,v,w) = (a^2d, b^2d, c^2d, bcd, cad, abd)$$

where $d^2(a^2+b^2+c^2) = 1$.

Solution 2. [Y. Wu; P.J. Zhao] From the given equations, we see that

$$(x,v,w)\cdot(w,u,y) = w(x+y)+uv = ab = \sqrt{x^2+v^2+w^2}\sqrt{w^2+u^2+y^2};$$

$$(w,u,y)\cdot(v,z,w) = u(y+z)+vw = bc = \sqrt{w^2+u^2+y^2}\sqrt{v^2+z^2+w^2};$$

$$(v,z,w)\cdot(x,v,w) = v(x+z)+uw = ca = \sqrt{v^2+z^2+u^2}\sqrt{x^2+v^2+w^2}.$$

It follows from the conditions for equality in the Cauchy-Schwarz Inequality that (x,v,w), (w,u,y) and (v,z,u) are all constant multiples of the same unit vector (p,q,r). Indeed

$$(x,v,w) = a(p,q,r): \qquad (w,u,y) = b(p,q,r); \qquad (v,z,u) = c(p,q,r).$$

Thus, $x = ap$, $y = br$, $z - cq$, $u = bq = cr$, $v = aq = cp$ and $w = ar = bp$, whence

$$u^2 = bcqr = yz; \qquad v^2 = bcpq = xz; \qquad w^2 = abpr = xy.$$

The solution now can be completed as in Solution 1.

4. (a) Let n and k be positive integers. Prove that the least common multiple of $\{n, n+1, n+2, \cdots, n+k\}$ is equal to

$$rn\binom{n+k}{k}$$

for some positive integer r.

(b) For each positive integer k, prove that there exist infinitely many positive integers n, for which the number r defined in part (a) is equal to 1.

Solution 1. (a) Observe that

$$n\binom{n+k}{k} = \frac{n(n+1)(n+2)\cdots(n+k)}{(1)(2)(3)\cdots(k)} \ .$$

Let p be an arbitrary prime and suppose that p^b is the highest power of p that divides any of n, $n+1$, $\cdots$, $n+k$; thus, p^b is the power of p that occurs in the prime factorization of the least common multiple of n, $n+1$, $\cdots$, $n+k$. We need to show that p divides $n\binom{n+k}{k}$ to no greater a power than p^b.

Suppose that $n+u$ is divisible by p^b where $0 \le u \le k$. Write $n\binom{n+k}{k} = (n+u)\times A \times B$ where

$$A = \frac{(n+u+1)(n+u+2)\cdots(n+k)}{(1)(2)(3)\cdots(k-u)}$$

and

$$B = \frac{(n+u-1)(n+u-2)\cdots(n)}{(k-u+1)(k-u+2)\cdots(k)}.$$

Since $n+u+i \equiv i \pmod{p^b}$ it follows that p divides $n+u+i$ and i to the same power, for $1 \le i \le k-u$. Therefore A, written in lowest terns, has a numerator that is not divisible by p. Since $n+u-i \equiv -i$ (mod p^b) for $1 \le i \le u$ and $k-u+1$, $k-u+2$, $\cdots$, k constitutes a set of u consecutive integers, for each power p^a with $a \le b$, p^a divides a number among $k-u+1$, $\cdots$, k at most once more (either $\lfloor u/p^a \rfloor$ or $\lfloor u/p^a \rfloor + 1$) than it divides one of the numbers -1, -2, $\cdots$, $-u$ or one of the numbers $n+u-1$, $n+u-2$, $\cdots$, n. It follows that the power of p that divides the denominator of B cannot exceed the power that divides the numerator of B by more than b. Since $n+u$ is exactly divisible by p^b, it follows that the highest power of p that divides $n\binom{n+k}{k}$ is no greater than b. The desired result follows.

(b) Let $n = r \times k!$ for any positive integer r. Then $n\binom{n+k}{k} = rn(n+1)(n+2)\cdots(n+k)$ which is clearly a common multiple of n, $n+1$, $\cdots$, $n+k$. Since this number is a divisor of the least common multiple of n, $n+1$, $\cdots$, $n+k$, it must be equal to the least common multiple.

Solution 2. [J. Zung] (a) For a prime p, let $f_p(m)$ denote the exponent of the highest power of p that is a divisor of the integer m. It is required to prove that

$$f_p(n(n+1)\cdots(n+k)) - f_p(k!) = f_p\left(n\binom{n+k}{k}\right) \leq$$
$$f_p(\text{lcm }(n, n+1, \cdots, n+k)) = \max\{f_p(m) : n \leq m \leq n+k\}.$$

Suppose that M is the maximum of $f_p(m)$ for $n \leq m \leq n+k$. Then $f_p(n(n+1)\cdots(n+k))$ is the sum over i from 1 to M of the number of multiples of p^i that occur in the set $\{n, n+1, \cdots, n+k\}$. This is at most once more than the number of multiples of p^i that occur in the set $\{1, 2, \cdots, k\}$ with equality occurring when n is a multiple of p^i. Therefore

$$f_p(n(n+1)\cdots(n+k)) \leq \sum_{1\leq i\leq M}\left(\left\lfloor\frac{k}{p^i}\right\rfloor + 1\right) = f_p(k!) + M,$$

which yields the desired result.

(b) Equality occurs when n is a multiple of $p^{f_p(k!)}$ for every prime p.

5. Let $\mathfrak{C}$ be a circle and Q a point in the plane. Determine the locus of the centres of those circles that are tangent to $\mathfrak{C}$ and whose circumference passes through Q.

Solution 1. There are several cases to consider. Let O be the centre and r the radius of the given circle. Let P be the centre of the variable circle.

(i) Q lies on $\mathfrak{C}$. In this case, the variable circle is tangent to $\mathfrak{C}$ at Q and the locus is the set of points on the line OQ with the exception of Q itself.

(ii) Q lies in the interior of $\mathfrak{C}$. Suppose that the variable circle is tangent to $\mathfrak{C}$ at T and has radius a. Then

$$|QP| + |PO| = a + (r-a) = r$$

so that P lies on the ellipse with foci O and Q and major axis of length $2r$.

Conversely, if P is a point on the ellipse just described, let OP produced meet $\mathfrak{C}$ at T and construct a circle with centre P and radius $|PT|$. This circle will be tangent to $\mathfrak{C}$ and $|QP| = r - |OP| = |TP|$, so that it will pass through Q.

(iii) Q lies outside of $\mathfrak{C}$ and the variable circle touches $\mathfrak{C}$ at T, say, with the two circles external to each other. Let a be the radius of the variable circle. The segment OP contains T and

$$|OP| = r + a = r + |PQ|$$

so that $|OP| - |PQ| = r$, a constant. Hence P lies on one branch of a hyperbola with foci O, Q whose asymptotes are the lines that pass through O and the contact points of the tangents drawn to $\mathfrak{C}$ from Q.

(iv) Q lies outside of $\mathfrak{C}$ and the variable circle touches $\mathfrak{C}$ at T, say, with $\mathfrak{C}$ contained inside the variable circle. Then, with a the radius of the variable circle,

$$|QP| - |OP| = a - (a-r) = r$$

so that P lines on a branch of the hypebola with foci O, Q.

Solution 2. Let $\mathfrak{C}$ have equation $x^2 + y^2 = 1$ and $Q \sim (q, 0)$, with $q \geq 0$. Suppose that (X, Y) is the centre of a circle that passes through Q and touches $\mathfrak{C}$. Then

$$\sqrt{(X-q)^2 + Y^2} = |\sqrt{X^2+Y^2} - 1|$$

$$\Longrightarrow X^2 - 2qX + q^2 + Y^2 = X^2 + Y^2 + 1 - 2\sqrt{X^2+Y^2}$$

$$\Longrightarrow 2\sqrt{X^2+Y^2} = (1-q^2) + 2qX$$

$$\Longrightarrow 4X^2 + 4Y^2 = (1-q^2)^2 + 4q(1-q^2)X + 4q^2X^2.$$

$$\Longrightarrow (1-q^2)(2X-q)^2 + 4Y^2 = (1-q^2).$$

When $q = 0$, the locus is a circle concentric with the given circle with half the radius. When $0 < q < 1$, the locus is an ellipse with centre at $(q/2, 0)$, semi-major axis $1/2$ and semi-minor axis $\sqrt{1-q^2}/2$. When $q = 1$, the locus is the line $Y = 0$.

Suppose that $q > 1$. The equation of the locus can be written

$$(2X-q)^2 - \frac{4Y^2}{q^2-1} = 1,$$

from which it can be seen that the locus is a hyperbola with asymptotes given by the equations

$$\sqrt{q^2-1}(2X-q) = \pm 2Y.$$

These are lines passing through the point $(q/2, 0)$ with slopes $\pm\sqrt{q^2-1}$.

The tangents to the circle $\mathfrak{C}$ from $(q, 0)$ touch the circle at the points $(1/q, \pm\sqrt{q^2-1}/q)$. When the centre of the variable circle is on the right branch of the hyperbola, the two circles touch externally at a point that lies on the minor arc between these points. When the centre is on the left branch of the hyperbola, $\mathfrak{C}$ touches the variable circle internally at a point on the major arc joining these two points. It can be checked that there are no points on the hyperbola that lies on the lines $y = \pm(\sqrt{q^2-1})x$ parallel to the asymptotes and passing through the points $(1/q, \pm\sqrt{q^2-1}/q)$. Note that the vertex of the left branch of the hyperbola is $((q-1)/2, 0)$ which lies on the positive $x-$axis; this is the centre of a circle of radius $(q+1)/2 > 1$ that touches $\mathfrak{C}$ at the point $(-1, 0)$.

6. Find all continuous real-valued functions defined on $\mathbf{R}$ that satisfy $f(0) = 0$ and

$$f(x) - f(y) = (x-y)g(x+y)$$

for some real valued function $g(x)$.

Solution 1. Setting $y = 0$, we find that $f(x) = xg(x)$ for all real x, so that $g(x)$ is continuous for all nonzero values of x. Therefore

$$(x+y)[f(x) - f(y)] = (x-y)f(x+y)$$

for all real x, y. Hence, when $(x+y)(x-y) \neq 0$,

$$\frac{f(x)-f(y)}{x-y} = \frac{f(x+y)}{x+y}.$$

Suppose that $x \neq 0$. Then

$$f'(x) = \lim_{y \to x} \frac{f(x)-f(y)}{x-y} = \frac{f(2x)}{2x}.$$

Taking $y = 1 - x$, we find that

$$f(x) - f(1-x) = (2x-1)f(1).$$

Therefore, when $x \neq 0$,

$$f'(x) + f'(1-x) = 2f(1),$$

so that

$$\frac{f(2x)}{2x} + \frac{f(2-2x)}{2(1-x)} = 2f(1) \Longrightarrow$$

$$(1-x)f(2x) + xf(2-2x) = 4x(1-x)f(1).$$

Substituting $(2x, 2-2x)$ for (x, y) in the equation for f, we find that

$$f(2x) - f(2-2x) = (4x-2)\frac{f(2)}{2} = (2x-1)f(2).$$

Hence

$$f(2x) = 4x(1-x)f(1) + x(2x-1)f(2)$$

so that

$$f(x) = x(2-x)f(1) + \frac{1}{2}x(x-1)f(2) = \left[\frac{f(2)}{2} - f(1)\right]x^2 + \left[2f(1) - \frac{f(2)}{2}\right]x$$

for $x \neq 0$. Since $f(x)$ is continuous, this should hold for all real x, so that $f(x)$ is a linear combination of the functions x and x^2.

It is straighforward to check that $f(x) = x$ and $f(x) = x^2$ satisfy the equation (with $g(x)$ respectively equal to 1 and x), along with any linear combination of these. Hence the general solution is

$$f(x) = ax^2 + bx$$

where a and b are arbitrary constants.

Solution 2. [J. Zung] Since

$$g(x) = \frac{1}{2}\left[f\left(\frac{x}{2}+1\right) - f\left(\frac{x}{2}-1\right)\right],$$

we see that $g(x)$ is a continuous function. As in Solution 1, we show that $f'(x) = g(2x)$.

We verify that the linear space generated by the polynomials x and x^2 consists of solutions of the equation. Given a solution, we can add a polynomial of degree not exceeding 2 to it to get a solution for which $f(-1) = f(0) = f(1) = 0$, so there is no loss of generality in assuming this to be the case.

Since $f(-1) = f(0) = f(1)$, we have that $g(0) = 0$, $f(x) = xg(x)$ and $g(-x) = -g(x)$ for each x. We have that

$$f(x) - f(1-x) = (2x-1)g(1)$$

and

$$f(x) - f(-1-x) = (2x+1)g(-1) = -(2x+1)g(1),$$

whence

$$f(1-x) - f(-1-x) = -4xg(1).$$

However, from the given equation, we have also that

$$f(1-x) - f(-1-x) = 2g(-2x) = -2g(2x),$$

so that $g(2x) = 2xg(1)$ for all x. Therefore $f(x) = g(1)x^2$. Since $f(1) = 0$, we must have $g(1) = 0$, so that $f(x) \equiv 0$.

The significance of this is that the polynomial we add to a solution f of the equation to get a solution that vanishes at -1, 0, 1 is the negative of the solution itself, so that the solution must be itself a polynomial of degree 1 or 2.

7. Consider the following problem:

Suppose that $f(x)$ is a continuous real-valued function defined on the interval $[0, 2]$ for which

$$\int_0^2 f(x)dx = \int_0^2 (f(x))^2 dx \ .$$

Prove that there exists a number $c \in [0, 2]$ for which either $f(c) = 0$ or $f(c) = 1$.

(a) Criticize the following solution:

Clearly $\int_0^2 f(x)dx \geq 0$. By the extreme value theorem, there exist numbers u and v in $[0, 2]$ for which $f(u) \leq f(x) \leq f(v)$ for $0 \leq x \leq 2$. Hence

$$f(u)\int_0^2 f(x)dx \leq \int_0^2 f(x)^2 dx \leq f(v)\int_0^2 f(x)dx \ .$$

Since $\int_0^2 f(x)^2 dx = 1 \cdot \int_0^2 f(x)dx$, by the intermediate value theorem, there exists a number $c \in [0, 2]$ for which $f(c) = 1$. $\square$

(b) Show that there is a nontrivial function f that satisfies the conditions of the problem but that never assumes the value 1.

(c) Provide a complete solution of the problem.

Solution. (a) The solution does not apply when $f(x)$ is identically zero. However, there is a more fundamental difficulty with it. The solution is correct for a nonzero function which is nonnegative at every point of the interval. However, it fails whenever $f(x)$ assumes a negative value. In this case, there are values of x for which $f(u) \leq f(x) < 0$, so that $f(u)f(x) \geq f(x)^2$ and so

$$f(u)\int_N f(x)dx \geq \int_N (f(x))^2 dx$$

where N is the subset of $[0, 2]$ upon which $f(x)$ is negative. This puts into question the displayed inequality in the purported solution.

(b) *Solution 1.* We observe that when $f(x) = \frac{1}{2}x$, then $\int_0^2 f(x)dx > \int_0^2 (f(x))^2 dx$, while when $f(x) = \frac{1}{2}(2 - x)$, then $\int_0^2 f(x)dx < 0 < \int_0^2 (f(x))^2 dx$. This suggests that we can satisfy the condition of the problem with a function of the form $f(x) = \frac{1}{2}(x - \lambda)$ for some value of λ in the interval $(0, 2)$. Any such function will never assume the value 1 on the interval.

We have that

$$\int_0^2 f(x)dx = 1 - \lambda$$

and

$$\int_0^2 (f(x)^2 dx = \frac{1}{6}[3(1 - \lambda)^2 + 1].$$

These two expressions are equal when $\lambda = \sqrt{2/3}$.

Solution 2. [J. Love] The function

$$f(x) = \frac{1}{2} - \frac{\sqrt{3}}{4}x = \frac{1}{4}(2 - \sqrt{3}x)$$

satisfies the condition of the problem but does not assume the value 1.

Solution 3. [K. Ng] The function

$$f(x) = \frac{x}{2} - 6^{-1/3}$$

satisfies the condition of the problem but does not assume the value 1.

(c) Either we can complete the proof in (a) by considering the possibility that $f(x)$ takes negative values. Since the integral of $f(x)$ over $[0, 2]$ is nonnegative, $f(x)$ must assume positive values as well. Therefore, by the intermediate value theorem, it must vanish somewhere within the interval.

Alternatively, we can get a more direct argument by considering the function $g(x) = f(x)^2 - f(x)$. By hypothesis, $g(x)$ is a continuous function for which $\int_0^2 g(x) = 0$. Therefore, either $g(x)$ vanishes identically or it takes both positive and negative values. By the intermediate value theorem, there exists a number c for which $g(c) = 0$.

One can also argue by contradiction. Suppose that $f(x)$ takes neither of the values 0 and 1. Then one of the following must occur for each x in the interval $[0, 2]$:

Case 1: $f(x) < 0$, in which case $f(x) < f(x)^2$;

Case 2: $0 < f(x) < 1$, in which case $f(x)^2 < f(x)$;

Case 3: $1 < f(x)$, in which case $f(x) < f(x)^2$.

In Cases 1 and 3, the left side of the given equation is less than the right, while in Case 2, the left side is greater than the right. In all cases, the equality cannot hold, and we obtain the result.

8. Determine the area of the set of points (x, y) in the plane that satisfy the two inequalities:

$$x^2 + y^2 \le 2$$
$$x^4 + x^3y^3 \le xy + y^4.$$

Solution. The second inequality can be rewritten as

$$0 \le y^4 + xy - x^4 - x^3y^3 = (y^3 + x)(y - x^3).$$

The locus of the equation $y^3 + x = 0$ can be obtained from the locus of the equation $y - x^3 = 0$ by a $90°$ rotation about the origin that relates the points (u, v) and $(v, -u)$. Thus the two curves partition the disc $x^2 + y^2 \le 2$ into four regions of equal area, and the inequality is satisfied by the two regions that cover the $y-$axis. Thus the area of the set is half the area of the disc, namely π.

9. In a round-robin tournament of $n \ge 2$ teams, each pair of teams plays exactly one game that results in a win for one team and a loss for the other (there are no ties).

(a) Prove that the teams can be labelled $t_1, t_2, \cdots, t_n$, so that, for each i with $1 \le i \le n - 1$, team t_i beats t_{i+1}.

(b) Suppose that a team t has the property that, for each other team u, one can find a chain $u_1, u_2, \cdots, u_m$ of (possibly zero) distinct teams for which t beats u_1, u_i beats u_{i+1} for $1 \le i \le m - 1$ and u_m beats u. Prove that *all* of the n teams can be ordered as in (a) so that $t = t_1$ and each t_i beats t_{i+1} for $1 \le i \le n - 1$.

(c) Let T denote the set of teams who can be labelled as t_1 in an ordering of teams as in (a). Prove that, in any ordering of teams as in (a), all the teams in T occur before all the teams that are not in T.

Solution. (a) Let $a > b$ denote that team a beats team b. The result is clear when $n = 2$. Assume that the result holds for all numbers of teams up to $n - 1 \geq 2$.

(a) *Solution 1.* By the induction hypothesis, we can order $n-1$ of the team so that $t_1 > t_2 > \cdots > t_{n-1}$. Suppose that t is the nth team not included in this list. If t either beats t_1 or is beaten by t_{n-1} then it can be appended at one end of the list. Suppose therefore that $t_1 > t > t_{n-1}$. Let i be the largest index for which $t_i > t$. Then $i < n - 1$ and we can insert t between t_i and t_{i+1}.

Solution 2. Let t be any team, A be the set of teams that beat t and B be the set of teams that are beaten by t. We use induction. If A is nonvoid, then the teams in A can be ordered as in (a) so that each team beats the next; similarly, if B is nonvoid, its teams can be ordered in the same way. The orderings in A and B with t interpolated give the required ordering of all the teams.

(b) *Solution 1.* Suppose that $t > u_1 > u_2 > \cdots > u_r$ is a list of teams, each beating the next, of maximum length. Suppose, if possible, that a team v is not included in the list. Then $v > u_r$. If there is a team u_i with $u_i > v$, then by taking such a team with the maximum value of i, we can insert v between u_i and u_{i+1}, contradicting the maximality of the list. Therefore $v > u_i$ for each i.

By the hypothesis on t, we can find teams $v_1, v_2, \cdots, v_s$ for which $t > v_1 > v_2 > v_3 > \cdots > v_s = v$. Consider team v_{s-1}. This team is not one of the u_i since it beats $v_s = v$. Therefore, arguing as for v, we find that $v_{s-1} > u_r$. If, for some i, $u_i > v_{s-1}$, then we could insert $> v_{s-1} > v_s >$ between u_i and u_{i+1} in the list, contradicting maximality. Hence $v_{s-1} > u_i$ for each i. We can continue on in this way to argue in turn that v_{s-2}, $\cdots$, v_1 are all distinct from the u_i and beat each of the u_i.

But then, we could form a chain

$$t > v_1 > v_2 > \cdots > v_s > u_1 > \cdots > u_r = u$$

again contradicting the maximality. Hence any maximum chain contains tall the teams.

Solution 2. [J. Zung] We can consider the set T of teams as a directed graph with root t where each pair of nodes is connected by a directed edge $a \rightarrow b$, corresponding to team a beating team b. Let T be a minimum spanning tree of T with root T, (*i.e*, T and T_1 have the same nodes, T_1 is connected, but T_1 has no loops, and there is a path from t to any other node). If every node in T_1 has at most one exiting edge, then T_1 is a path of all the nodes. Otherwise, there are nodes u, v, w for which $u \rightarrow v$, $u \rightarrow w$. Suppose, wolog, in T , that $v \rightarrow w$. We convert T_1 to a new tree T_2 with one fewer node with more than one exiting edge by removing the edge $u \rightarrow w$ and inserting the edge $v \rightarrow w$.

We can continue the same process to get a succession of spanning trees from which the sum over all nodes of the number of edges required to go from t to those nodes increases. The process must terminate with a spanning path.

(c) *Solution.* First note that any team x that beats a team t in T must itself belong to T. For let an ordering of the all the teams be given and append x at the beginning. We must than have an ordering

$$x > t > \cdots > x > \cdots$$

in which x appears twice and every other team once. It follows from this that any team is at the end of a chain that begins with one of the x and consists of distinct teams. Thus, any team beating a member of T must lie in T. The desired result follows from this.

10. Let A be a square matrix whose entries are complex numbers. Prove that $A^* = A$ if and only if $AA^* = A^2$.

Notes. For any $m \times n$ matrix M with entries m_{ij}, the *hermitian transpose* is the $n \times m$ matrix M^* obtained by taking the complex conjugates of entries of M and transposing; thus, the (i, j)th element of M^* is $\overline{m_{ji}}$. In particular, for the complex column vector x with ith entry x_i, x^* is a row vector whose ith entry is $\overline{x_i}$. The inner product $\langle x, y \rangle$ of two column vectors is $\sum \overline{x_i} y_i = x^* y$, and we have that $\langle x, Ay \rangle = \langle A^* x, y \rangle$. A matrix for which $A^* = A$ is said to be *hermitian.*

Comment. If $A = A^*$, then clearly $AA^* = A^2$. Henceforth, we assume that $AA^* = A^2$ and that A is a $n \times n$ matrix.

Solution 1. Let $B = i(A - A^*)$, so that B is hermitian and $AB = O$. Then B is unitarily equivalent to a diagonal matrix, so that it suffices to show that all the eigenvalues of B are 0.

Suppose, if possible, that B has a nonzero eigenvalue λ with nontrivial eigenvector x. Then $O = ABx = \lambda Ax$, so that

$$O = Ax = A^* x - i\lambda x.$$

Therefore

$$0 = x^* Ax = x^* A^* x - i\lambda x^* x.$$

Since

$$x^* A^* x = \langle x, A^* x \rangle = \overline{\langle A^* x, x \rangle} = \overline{\langle x, Ax \rangle} = \overline{x^* Ax} = 0,$$

we have that $i\lambda \|x\|^2 = i\lambda x^* x = 0$, which yields a contradiction. Therefore $B = O$ and $A = A^*$.

Solution 2. There exists a unitary matrix U (with $UU^* = U^* U = I$) for which $U^* TU$ for some upper triangular matrix T. Let the diagonal of T be $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and its other entries be t_{ij} for $1 \leq i < j \leq n$.

We have that

$$TT^* = UAU^* UA^* U^* = UAA^* U^* = UA^2 U^* = UAU^* UAU^* = T^2,$$

so that the trace of TT^* equals the trace of T^2. Thus,

$$\sum_{k=1}^{n} |\lambda_k|^2 + \sum_{1 \leq i < j \leq n} |t_{ij}|^2 = \sum_{k=1}^{n} \lambda_k^2.$$

The right side of this equation must be real, and

$$\sum_{k=1}^{n} \lambda_k^2 \leq |\sum_{k=1}^{n} \lambda_k|^2 \leq \sum_{k=1}^{n} |\lambda_k|^2,$$

so that each t_{ij} is zero and $\lambda_k^2 = |\lambda_k|^2$ for each k. Thus T is a diagonal matrix with real eigenvalues, and so hermitian. Therefore A is hermitian.

Solution 3. Observe that

$$A^{*2} = (A^2)^* = (AA^*)^* = AA^* = A^2.$$

Suppose that it has been established that $A^{*k} = A^k$ for some integer k not less than 2. Then

$$A^{k+1} = A(A^{*k}) = (AA^*)(A^{*k-1}) = A^{*2}(A^{*k-1}) = A^{*k+1}$$

so that $A^{*m} = A^m$ for all integers $m \geq 2$.

Let p be the minimal polynomial for A. Since every eigenvalue of $A^2 = AA^*$ is nonnegative, it follows that every eigenvalue of A is real, so that p has real coefficients and $p(A^*) = (p(A))^* = O$. Thus p is the minimal polynomial for A^*.

Suppose, if possible, that $p(t) = t^2 q(t)$, for some polynomial $q(t)$. Let x be any vector, and let $y = q(A^*)x$. Then

$$AA^* y = A^2 y = A^{*2} y = 0$$

so that

$$\langle A^*y, A^*y \rangle = \langle y, AA^*y \rangle = 0$$

from which $A^*q(A^*)x = A^*y = 0$. But this contradicts the fact that p is the minimal polynomial for A^*. Thus $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ where at least one of a_0 and a_1 is nonzero.

Suppose that $a_0 = 0$. Then

$$O = p(A) - p(A^*) = a_1(A - A^*)$$

so that $A = A^*$. On the other hand, if a_0 is nonzero, then

$$O = Ap(A) - A^*p(A^*) = a_0(A - A^*) + a_1 A^2 - a_1 {A^*}^2 = a_0(A - A^*)$$

and again $A = A^*$. Thus, the desired result holds.

Solution 4. Suppose that $M = (m_{ij})$ is an arbitrary $n \times n$ matrix. Then $\text{tr}(M)$, the trace of M (the sum of its diagonal elements), is equal to the complex conjugate of $\text{tr}(M^*)$. Since the ith diagonal element of MM^* is equal to $\sum_j |m_{ij}|^2$, it follows that $\text{tr}(MM^*) = \sum_{i,j} |m_{ij}|^2$ and we have that $\text{tr}(MM^*) = \text{tr}(M^*M)$.

It follows from this that $M = O$ if and only if $\text{tr}(MM^*) = 0$. If $M = M^*$, then its trace is real and $\text{tr}(M) = \text{tr}(M^*)$.

Consider the situation of the problem where $AA^* = A^2$. We have that

$$(A - A^*)(A - A^*)^* = (A - A^*)(A^* - A) = (AA^* - A^2) + (A^*A - {A^*}^2) = A^*A - {A^2}^*,$$

from which we find that

$$\text{tr}((A - A^*)(A - A^*)^*) = \text{tr}(A^*A) - \text{tr}({A^2}^*) = \text{tr}(AA^*) - \overline{\text{tr}(A^2)} = \text{tr}(AA^*) - \overline{\text{tr}(AA^*)} = 0$$

since the trace of AA^* is real. It follows that $A - A^* = O$ so that $A = A^*$.

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