THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

March 7, 2010

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. Let F_1 and F_2 be the foci of an ellipse and P be a point in the plane of the ellipse. Suppose that G_1 and G_2 are points on the ellipse for which PG_1 and PG_2 are tangents to the ellipse. Prove that $\angle F_1PG_1 = \angle F_2PG_2$.

2. Let $u_0 = 1$, $u_1 = 2$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Prove that, for every nonnegative integer n,
$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} .$$

3. Let $\mathbf{a}$ and $\mathbf{b}$, the latter nonzero, be vectors in $\mathbb{R}^3$. Determine the value of λ for which the vector equation
$$\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \lambda \mathbf{b}$$
is solvable, and then solve it.

4. The plane is partitioned into n regions by three families of parallel lines. What is the least number of lines to ensure that $n \geq 2010$?

5. Let m be a natural number, and let $c, a_1, a_2, \cdots, a_m$ be complex numbers for which $|a_i| = 1$ for $i = 1, 2, \cdots, m$. Suppose also that
$$\lim_{n\to\infty} \sum_{i=1}^{m} a_i^n = c .$$
Prove that $c = m$ and that $a_i = 1$ for $i = 1, 2, \cdots, m$.

6. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which
$$f(x)f(x+1) = g(h(x)) ,$$

7. Suppose that f is a continuous real-valued function defined on the closed interval $[0, 1]$ and that
$$\left(\int_0^1 xf(x)dx \right)^2 = \left(\int_0^1 f(x)dx \right) \left(\int_0^1 x^2 f(x)dx \right) .$$
Prove that there is a point $c \in (0, 1)$ for which $f(c) = 0$.

8. Let A be an invertible symmetric $n \times n$ matrix with entries $\{a_{i,j}\}$ in $\mathbb{Z}_2$. Prove that there is an $n \times n$ matrix M with entries in $\mathbb{Z}_2$ such that $A = M^{\mathbf{t}}M$ only if $a_{i,i} \neq 0$ for some i.

[$\mathbb{Z}_2$ refers to the field of integers modulo 2 with two elements $0, 1$ for which $1 + 1 = 0$. $M^{\mathbf{t}}$ refers to the transpose of the matrix M.]

9. Let f be a real-valued functions defined on $\mathbb{R}$ with a continuous third derivative, let $S_0 = \{x : f(x) = 0\}$, and, for $k = 1, 2, 3$, $S_k = \{x : f^{(k)}(x) = 0\}$, where $f^{(k)}$ denotes the kth derivative of f. Suppose also that $\mathbb{R} = S_0 \cup S_1 \cup S_2 \cup S_3$. Must f be a polynomial of degree not exceeding 2?

10. Prove that the set $\mathbb{Q}$ of rationals can be written as the union of countably many subsets of $\mathbb{Q}$ each of which is dense in the set $\mathbb{R}$ of real numbers.

Solutions.

1. Let F_1 and F_2 be the foci of an ellipse and P be a point in the plane of the ellipse. Suppose that G_1 and G_2 are points on the ellipse for which PG_1 and PG_2 are tangents to the ellipse. Prove that $\angle F_1PG_1 = \angle F_2PG_2$.

Solution. Let H_1 be the reflection of F_1 in the tangent PG_1, and H_2 be the reflection of F_2 in the tangent PG_2. We have that $PH_1 = PF_1$ and $PF_2 = PH_2$. By the reflection property, $\angle PG_1F_2 = \angle F_1G_1Q = \angle H_1G_1Q$, where Q is a point on PG_1 produced. Therefore, H_1F_2 intersects the ellipse in G_1. Similarly, H_2F_1 intersects the ellipse in G_2. Therefore

$$\begin{aligned} H_1F_2 &= H_1G_1 + G_1F_2 = F_1G_1 + G_1F_2 \\ &= F_1G_2 + G_2F_2 = F_1G_2 + G_2H_2 = H_2F_1 \ . \end{aligned}$$

Therefore, triangle PH_1F_2 and PF_1H_2 are congruent (SSS), so that $\angle H_1PF_2 = \angle H_2PF_1$. It follows that

$$2\angle F_1PG_1 = \angle H_1PF_1 = \angle H_2PF_2 = 2\angle F_2PG_2$$

and the desired result follows.

2. Let $u_0 = 1$, $u_1 = 2$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Prove that, for every nonnegative integer n,

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i, j, k \geq 0, i + j + 2k = n \right\} \ .$$

Solution 1. Suppose that we have a supply of white and of blue coaches, each of length 1, and of red coaches, each of length 2; the coaches of each colour are indistinguishable. Let v_n be the number of trains of total length n that can be made up of red, white and blue coaches of total length n. Then $v_0 = 1$, $v_1 = 2$ and $v_2 = 5$ (R, WW, WB, BW, BB). In general, for $n \geq 1$, we can get a train of length $n + 1$ by appending either a white or a blue coach to a train of length n or a red coach to a train of length $n - 1$, so that $v_{n+1} = 2v_n + v_{n-1}$. Therefore $v_n = u_n$ for $n \geq 0$.

We can count v_n in another way. Suppose that the train consists of i white coaches, j blue coaches and k red coaches, so that $i + j + 2k = n$. There are $(i + j + k)!$ ways of arranging the coaches in order; any permutation of the i white coaches among themselves, the j blue coaches among themselves and k red coaches among themselves does not change the train. Therefore

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i, j, k \geq 0, i + j + 2k = n \right\} \ .$$

Solution 2. Let $f(t) = \sum_{n=0}^{\infty} u_n t^n$. Then

$$\begin{aligned} f(t) &= u_0 + u_1 t + (2u_1 + u_0)t^2 + (2u_2 + u_1)t^3 + \cdots \\ &= u_0 + u_1 t + 2t(f(t) - u_0) + t^2 f(t) = u_0 + (u_1 - 2u_0)t + (2t + t^2)f(t) \\ &= 1 + (2t + t^2)f(t) \ , \end{aligned}$$

whence

$$\begin{aligned} f(t) &= \frac{1}{1 - 2t - t^2} = \frac{1}{1 - t - t - t^2} \\ &= \sum_{n=0}^{\infty} (t + t + t^2)^n = \sum_{n=0}^{\infty} t^n \left[\sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i, j, k \geq 0, i + j + 2k = n \right\} \right] \ . \end{aligned}$$

Solution 3. Let w_n be the sum in the problem. It is straightforward to check that $u_0 = w_0$ and $u_1 = w_1$. We show that, for $n \geq 1$, $w_{n+1} = 2w_n + w_{n-1}$ from which it follows by induction that $u_n = w_n$ for each n. By convention, let $(-1)! = \infty$. Then, for $i, j, k \geq 0$ and $i + j + 2k = n + 1$, we have that

$$\begin{aligned} \frac{(i+j+k)!}{i!j!k!} &= \frac{(i+j+k)(i+j+k-1)!}{i!j!k!} \\ &= \frac{(i+j+k-1)!}{(i-1)!j!k!} + \frac{(i+j+k-1)!}{i!(j-1)!k!} + \frac{(i+j+k-1)!}{i!j!(k-1)!} \ , \end{aligned}$$

whence

$$\begin{aligned} w_{n+1} = \sum &\left\{ \frac{(i+j+k-1)!}{(i-1)!j!k!} : i, j, k \geq 0, (i-1) + j + 2k = n \right\} \\ &+ \sum \left\{ \frac{(i+j+k-1)!}{i!(j-1)!k!} : i, j, k \geq 0, i + (j-1) + 2k = n \right\} \\ &+ \sum \left\{ \frac{(i+j+k-1)!}{i!j!(k-1)!} : i, j, k \geq 0, i + j + 2(k-1) = n-1 \right\} \\ &= w_n + w_n + w_{n-1} = 2w_n + w_{n-1} \end{aligned}$$

as desired.

3. Let $\mathbf{a}$ and $\mathbf{b}$, the latter nonzero, be vectors in $\mathbf{R}^3$. Determine the value of λ for which the vector equation

$$\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \lambda \mathbf{b}$$

is solvable, and then solve it.

Solution 1. If there is a solution, we must have $\mathbf{a} \cdot \mathbf{b} = \lambda |\mathbf{b}|^2$, so that $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$. On the other hand, suppose that λ has this value. Then

$$\begin{aligned} \mathbf{0} &= \mathbf{b} \times \mathbf{a} - \mathbf{b} \times (\mathbf{x} \times \mathbf{b}) \\ &= \mathbf{b} \times \mathbf{a} - [(\mathbf{b} \cdot \mathbf{b})\mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b}] \end{aligned}$$

so that

$$\mathbf{b} \times \mathbf{a} = |\mathbf{b}|^2 \mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b} \ .$$

A particular solution of this equation is

$$\mathbf{x} = \mathbf{u} \equiv \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b}|^2} \ .$$

Let $\mathbf{x} = \mathbf{z}$ be any other solution. Then

$$\begin{aligned} |\mathbf{b}|^2 (\mathbf{z} - \mathbf{u}) &= |\mathbf{b}|^2 \mathbf{z} - |\mathbf{b}|^2 \mathbf{u} \\ &= (\mathbf{b} \times \mathbf{a} + (\mathbf{b} \cdot \mathbf{z})\mathbf{b}) - (\mathbf{b} \times \mathbf{a} + (\mathbf{b} \cdot \mathbf{u})\mathbf{b}) \\ &= (\mathbf{b} \cdot \mathbf{z})\mathbf{b} \end{aligned}$$

so that $\mathbf{z} - \mathbf{u} = \mu \mathbf{b}$ for some scalar μ.

We check when this works. Let $\mathbf{x} = \mathbf{u} + \mu \mathbf{b}$ for some scalar μ. Then

$$\begin{aligned} \mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \mathbf{a} - (\mathbf{u} \times \mathbf{b}) &= \mathbf{a} - \frac{(\mathbf{b} \times \mathbf{a}) \times \mathbf{b}}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \frac{\mathbf{b} \times (\mathbf{b} \times \mathbf{a})}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \frac{(\mathbf{b} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{b})\mathbf{a}}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}|^2} \right) \mathbf{b} - \mathbf{a} = \lambda \mathbf{b} \ , \end{aligned}$$

as desired. Hence, the solution is

$$\mathbf{x} = \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b}|^2} + \mu \mathbf{b} ,$$

where μ is an arbitrary scalar.

Solution 2. [B. Yahagni] Suppose, to begin with, that $\{\mathbf{a}, \mathbf{b}\}$ is linearly dependent. Then $\mathbf{a} = [(\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2]\mathbf{b}$. Since $(\mathbf{x} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all $\mathbf{x}$, the equation has no solutions except when $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$. In this case, it becomes $\mathbf{x} \times \mathbf{b} = \mathbf{0}$ and is satisfied by $\mathbf{x} = \mu \mathbf{b}$, where μ is any scalar.

Otherwise, $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ is linearly independent and constitutes a basis for $\mathbb{R}^3$. Let a solution be

$$\mathbf{x} = \alpha \mathbf{a} + \mu \mathbf{b} + \beta(\mathbf{a} \times \mathbf{b}) .$$

Then

$$\mathbf{x} \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b}) + \beta[(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}] = \alpha(\mathbf{a} \times \mathbf{b}) + \beta(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - \beta(\mathbf{b} \cdot \mathbf{b})\mathbf{a}$$

and the equation becomes

$$(1 + \beta|\mathbf{b}|^2)\mathbf{a} - \beta(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - \alpha(\mathbf{a} \times \mathbf{b}) = \lambda \mathbf{b} .$$

Therefore $\alpha = 0$, μ is arbitrary, $\beta = -1/|\mathbf{b}|^2$ and $\lambda = -\beta(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$.

Therefore, the existence of a solution requires that $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$ and the solution then is

$$\mathbf{x} = \mu \mathbf{b} - \frac{1}{|\mathbf{b}|^2}(\mathbf{a} \times \mathbf{b}) .$$

Solution 3. Writing the equation in vector components yields the system

$$b_3 x_2 - b_2 x_3 = a_1 - \lambda b_1 ;$$

$$-b_3 x_1 + b_1 x_3 = a_2 - \lambda b_2 ;$$

$$b_2 x_1 - b_1 x_2 = a_3 - \lambda b_3 .$$

The matrix of coefficients of the left side is of rank 2, so that the corresponding homogeneous system of equations has a single infinity of solutions. Multiplying the three equations by b_1, b_2 and b_3 respectively and adding yields

$$0 = a_1 b_1 + a_2 b_2 + a_3 b_3 - \lambda(b_1^2 + b_2^2 + b_3^2) .$$

Thus, for a solution to exist, we require that

$$\lambda = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{b_1^2 + b_2^2 + b_3^2)}.$$

In addition, we learn that the corresponding homogeneous system is satisfied by

$$(x_1, x_2, x_3) = \mu(b_1, b_2, b_3)$$

where μ is an arbitrary scalar.

It remains to find a particular solution for the nonhomogeneous system. Multiplying the third equation by b_2 and subtracting the second multiplied by b_3, we obtain that

$$(b_2^2 + b_3^2)x_1 = b_1(b_2 x_2 + b_3 x_3) + (a_3 b_2 - a_2 b_3) .$$

Therefore, setting $b_1^2 + b_2^2 + b_3^2 = b^2$, we have that

$$b^2 x_1 = b_1(b_1 x_1 + b_2 x_2 + b_3 x_3) + (a_3 b_2 - a_2 b_3) .$$

Similarly

$$b^2x_2 = b_2(b_1x_1 + b_2x_2 + b_3x_3) + (a_1b_3 - a_3b_1) ,$$

$$b^2x_3 = b_3(b_1x_1 + b_2x_2 + b_3x_3) + (a_2b_1 - a_1b_2) .$$

Observing that $b_1x_1 + b_2x_2 + b_3x_3$ vanishes when

$$(x_1, x_2, x_3) = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2) ,$$

we obtain a particular solution to the system:

$$(x_1, x_2, x_3) = b^{-2}(a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2) .$$

Adding to this the general solution of the homogeneous system yields the solution of the nonhomogeneous system.

4. The plane is partitioned into n regions by three families of parallel lines. What is the least number of lines to ensure that $n \geq 2010$?

Solution. Suppose that there are x, y and z lines in the three families. Assume that no point is common to three distinct lines. The $x+y$ lines of the first two families partition the plane into $(x+1)(y+1)$ regions. Let λ be one of the lines of the third family. It is cut into $x+y+1$ parts by the lines in the first two families, so the number of regions is increased by $x + y + 1$. Since this happens z times, the number of regions that the plane is partitioned into by the three families of lines is

$$n = (x + 1)(y + 1) + z(x + y + 1) = (x + y + z) + (xy + yz + zx) + 1 .$$

Let $u = x + y + z$ and $v = xy + yz + zx$. Then (by the Cauchy-Schwarz Inequality for example), $v \leq x^2 + y^2 + z^2$, so that $u^2 = x^2 + y^2 + z^2 + 2v \geq 3v$. Therefore, $n \leq u + \frac{1}{3}u^2 + 1$. This takes the value 2002 when $u = 76$. However, when $(x, y, z) = (26, 26, 25)$, then $u = 77$, $v = 1976$ and $n = 2044$. Therefore, we need at least 77 lines, but a suitably chosen set of 77 lines will suffice.

5. Let m be a natural number, and let $c, a_1, a_2, \cdots, a_m$ be complex numbers for which $|a_i| = 1$ for $i = 1, 2, \cdots, m$. Suppose also that

$$\lim_{n\to\infty} \sum_{i=1}^{m} a_i^n = c .$$

Prove that $c = m$ and that $a_i = 1$ for $i = 1, 2, \cdots, m$.

Solution. If $a_i = e^{i\alpha_i}$, then either the sequence $\{a_i^n\}$ is periodic and assumes the value 1 infinitely often (when α_i is a rational multiple of π) or has a subsequence whose limit is 1 (when α_i is not a rational multiple of π). [In the latter case, we can find an increasing subsequence $\{n_k\}$ of natural numbers for which $a_i^{n_k}$ converges, so that $a_i^{n_{k+1}-n_k}$ converges to 1.]

We prove that there is a subsequence $\{n_k\}$ of natural numbers for which $\lim_k a_i^{n_k} = 1$ for each $1 \leq i \leq m$. Proceed by induction on m. When $m = 1$, the limit of any subsequence of $\{a_1^n\}$ is equal to the limit of the whole sequence, so that $c = 1$ in this case. In fact, we can go further: $a_1 = \lim a_1^{n+1} = \lim a_1^n = 1$.

Suppose the induction hypothesis holds for $m - 1$. Then there is a subsequence S_1 of natural numbers such that $\{a_i^n\}$ has limit 1 along this subsequence for $1 \leq i \leq m - 1$. We can find a subsequence S_2 along which the sequence $\{a_m^n\}$ converges to some limit b on the unit circle. Let S_3 be the sequence of quotients of consecutive terms in the sequence S_2. Then, for $1 \leq i \leq m$, the sequence $\{a_i^n\}$ converges to 1 along S_3.

It follows from this that $c = m$. Also, along S_3, we have that

$$\sum_{i=1}^{m} a_i = \lim \sum_{i=1}^{m} a_i^{n+1} = m .$$

Therefore $\sum_{i=1}^m \text{Re } a_i = m$. But the real part of each a_i does not exceed 1, with equality if and only if $a_i = 1$, it follows that $a_i = 1$ for each i.

This problem was contributed by Bamdad R. Yahaghi.

6. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which
$$f(x)f(x+1) = g(h(x)) \ ,$$

Solution 1. [A. Remorov] Let $f(x) = a(x-r)(x-s)$. Then

$$\begin{aligned} f(x)f(x+1) &= a^2(x-r)(x-s+1)(x-r+1)(x-s) \\ &= a^2(x^2+x-rx-sx+rs-r)(x^2+x-rx-sx+rs-s) \\ &= a^2[(x^2-(r+s-1)x+rs)-r][(x^2-(r+s-1)x+rs)-s] \\ &= g(h(x)) \ , \end{aligned}$$

where $g(x) = a^2(x-r)(x-s) = af(x)$ and $h(x) = x^2-(r+s-1)x+rs$.

Solution 2. Let $f(x) = ax^2+bx+c$, $g(x) = px^2+qx+r$ and $h(x) = ux^2+vx+w$. Then

$$f(x)f(x+1) = a^2x^4 + 2a(a+b)x^3 + (a^2+b^2+3ab+2ac)x^2 + (b+2c)(a+b)x + c(a+b-c)$$

$$\begin{aligned} g(h(x)) &= p(ux^2+vx+w)^2 + q(ux+vx+w) + r \\ &= pu^2x^4 + 2puvx^3 + (2puw+pv^2+qu)x^2 + (2pvw+qv)x + (pw^2+qw+r) \ . \end{aligned}$$

Equating coefficients, we find that $pu^2 = a^2$, $puv = a(a+b)$, $2puw+pv^2+qu = a^2+b^2+3ab+2ac$, $(b+2c)(a+b) = (2pw+q)v$ and $c(a+b+c) = pw^2+qw+r$. We need to find just one solution of this system. Let $p = 1$ and $u = a$. Then $v = a+b$ and $b+2c = 2pw+q$ from the second and fourth equations. This yields the third equation automatically. Let $q = b$ and $w = c$. Then from the fifth equation, we find that $r = ac$.

Thus, when $f(x) = ax^2+bx+c$, we can take $g(x) = x^2+bx+ac$ and $h(x) = ax^2+(a+b)x+c$.

Solution 3. [S. Wang] Suppose that

$$f(x) = a(x+h)^2 + k = a(t-(1/2))^2 + k \ ,$$

where $t = x+h+\frac{1}{2}$. Then $f(x+1) = a(x+1+h)^2+k = a(t+(1/2))^2+k$, so that

$$\begin{aligned} f(x)f(x+1) &= a^2(t^2-(1/4))^2 + 2ak(t^2+(1/4)) + k^2 \\ &= a^2t^4 + \left(-\frac{a^2}{2}+2ak\right)t^2 + \left(\frac{a^2}{16}+\frac{ak}{2}+k^2\right) \ . \end{aligned}$$

Thus, we can achieve the desired representation with $h(x) = t^2 = x^2+(2h+1)x+\frac{1}{4}$ and $g(x) = a^2x^2 + (\frac{-a^2}{2}+2ak)x + (\frac{a^2}{16}+\frac{ak}{2}+k^2)$.

Solution 4. [V. Krakovna] Let $f(x) = ax^2+bx+c = au(x)$ where $u(x) = x^2+dx+e$, where $b = ad$ and $c = ae$. If we can find functions $v(x)$ and $w(x)$ for which $u(x)u(x+1) = v(w(x))$, then $f(x)f(x+1) = a^2v(w(x))$, and we can take $h(x) = w(x)$ and $g(x) = a^2v(x)$.

Define $p(t) = u(x+t)$, so that $p(t)$ is a monic quadratic in t. Then, noting that $p''(t) = u''(x+t) = 2$, we have that

$$p(t) = u(x+t) = u(x) + u'(x)t + \frac{u''(x)}{2}t^2 = t^2 + u'(x)t + u(x) \ ,$$

from which we find that

$$\begin{aligned} u(x)u(x+1) = p(0)p(1) &= u(x)[u(x) + u'(x) + 1] \\ &= u(x)^2 + u'(x)u(x) + u(x) = p(u(x)) = u(x + u(x)) \ . \end{aligned}$$

Thus, $u(x)u(x+1) = v(w(x))$ where $w(x) = x + u(x)$ and $v(x) = u(x)$. Therefore, we get the desired representation with

$$h(x) = x + u(x) = x^2 + \left(1 + \frac{b}{a}\right)x + \frac{c}{a}$$

and

$$g(x) = a^2 v(x) = a^2 u(x) = af(x) = a^2x^2 + abx + ac \ .$$

Comment. The second solution can also be obtained by looking at special cases, such as when $a = 1$ or $b = 0$, getting the answer and then making a conjecture.

7. Suppose that f is a continuous real-valued function defined on the closed interval $[0, 1]$ and that

$$\left(\int_0^1 xf(x)dx\right)^2 = \left(\int_0^1 f(x)dx\right)\left(\int_0^1 x^2 f(x)dx\right) .$$

Prove that there is a point $c \in (0, 1)$ for which $f(c) = 0$.

Solution 1. Suppose, if possible, that f never vanishes on the interval, then it must be everywhere positive or negative. By replacing f by $-f$ is necessary, wolog we can assume that $f(x) > 0$ on $[0, 1]$. Let $f(x) = [g(x)]^2$ for some positive function g. Then the equation becomes

$$\left(\int_0^1 xg^2(x)dx\right)^2 = \left(\int_0^1 g^2(x)dx\right)\left(\int_0^1 (xg(x))^2 dx\right) .$$

This is the equality situation in the Cauchy-Schwarz Inequality, whence $xg(x)$ must be a constant multiple of $g(x)$. But this is not the case. Therefore, by a contradiction argument, the result follows.

Solution 2. The condition and conclusion is satisfied by the zero function. Suppose, henceforth, that f is not identically zero and let $\int_0^1 f(x)dx = u$, $\int_0^1 xf(x)dx = uv$. If $u = 0$, then f must assume both positive and negative values in $(0, 1)$, and so, by the Intermediate Value Theorem, must vanish. Assume $u \neq 0$. Then the condition implies that $\int_0^1 x^2 f(x)dx = uv^2$. Then

$$\begin{aligned} \int_0^1 (x - v)^2 f(x)dx &= \int_0^1 x^2 f(x)dx - 2v \int_0^1 xf(x)dx + v^2 \int_0^1 f(x)dx \\ &= uv^2 - 2uv^2 + uv^2 = 0 \ , \end{aligned}$$

whence $(x - v)^2 f(x)$, and also $f(x)$ must assume both positive and negative values on $(0, 1)$. The result follows.

8. Let A be an invertible symmetric $n \times n$ matrix with entries $\{a_{i,j}\}$ in $\mathbb{Z}_2$. Prove that there is an $n \times n$ matrix M with entries in $\mathbb{Z}_2$ such that $A = M^{\mathbf{t}} M$ only if $a_{i,i} \neq 0$ for some i.

[$\mathbb{Z}_2$ refers to the field of integers modulo 2 with two elements $0, 1$ for which $1 + 1 = 0$. $M^{\mathbf{t}}$ refers to the transpose of the matrix M.]

Solution 1. [A. Kim] Let the entries of M be $m_{i,j}$. Then

$$a_{i,i} = \sum_j m_{i,j} m_{j,i} = \sum_j m_{i,j}^2 = \sum_j m_{i,j} \ .$$

Suppose that $a_{i,i} = 0$ for each i. Then each column must have evenly many entries equal to 1. But then the sum of the row vectors of M must be the zero vector, and so the rows are linearly dependent. Hence the rank of M is less than n, and so M is not invertible.

Solution 2. Let $x = (x_1, x_2, \cdots, z_n)^{\mathbf{t}}$ be a column vector over $\mathbb{Z}_2$ and observe that

$$xAx^{\mathbf{t}} = \sum_{i=1}^{n} a_{i,i}x_i^2 = (\sum_{i=1}^{n} x_i)^2 \ .$$

Define $N = \{x : xAx^{\mathbf{t}} = 0\}$. Then $N = \{x : \sum_{i=1}^{n} a_{i,i}x_i = 0\}$.

Suppose that $A = M^{\mathbf{t}}M$. Then M is invertible and so $x \rightarrow Mx$ is a surjection (onto). Therefore the equation

$$0 = xAx^{\mathbf{t}} = xM^{\mathbf{t}}Mx^{\mathbf{t}} = (Mx^{\mathbf{t}})^{\mathbf{t}}(Mx^{\mathbf{t}})$$

is not satisfied for each x, so that N is a proper subspace of $(\mathbb{Z}_2)^n$. Therefore, there must exist i for which $a_{i,i} \neq 0$.

This problem was contributed by Franklin Vera Pachebo.

9. Let f be a real-valued functions defined on $\mathbb{R}$ with a continuous third derivative, let $S_0 = \{x : f(x) = 0\}$, and, for $k = 1, 2, 3$, $S_k = \{x : f^{(k)}(x) = 0\}$, where $f^{(k)}$ denotes the kth derivative of f. Suppose also that $\mathbb{R} = S_0 \cup S_1 \cup S_2 \cup S_3$. Must f be a polynomial of degree not exceeding 2?

Solution. Observe that, because f and its derivatives are continuous, each set S_k is closed. Note also that, if U is an open subset upon which f or any of its derivatives vanish, then all derivatives of higher order (in particular, $f^{(3)}$ must also vanish on U).

First, we show that S_3 is dense in $\mathbb{R}$. Otherwise, there is an open interval J upon which $f^{(3)}$ never vanishes, so that $J \subseteq S_0 \cup S_1 \cup S_2$. The set $J \setminus S_2$ must be a nonvoid open set, and so contain an open interval J_1 upon which f'' never vanishes. Similarly, there is a nonvoid open interval $J_2 \subseteq J_1$ open which f' never vanishes and a nonvoid open interval $J_3 \subseteq J_2$ upon which f never vanishes. But then, none of f, f'. f'', $f^{(3)}$ would vanish on J_3 contrary to hypothesis.

Let I be an open real interval and let $T_0 = I \setminus S_0$. If $T_0 = \emptyset$, then f, and so $f^{(3)}$, must vanish on I. Otherwise, T_0 is a nonvoid open set and there is an open interval $I_1 \subseteq T_0 \cap I$. Let $T_1 = I_1 \setminus S_1$. Then, as before, either f', and so $f^{(3)}$, vanishes on I_1 or else there is a nonvoid open interval $I_2 \subseteq T_1 \cap I_1 \subseteq I$. Let $T_2 = I_2 \setminus S_2$. Either f'', and so $f^{(3)}$, vanishes on I_2 or there is an nonvoid open subset $I_3 \subseteq T_2 \cap I_2 \subseteq I$. Then either $f^{(3)}$ vanishes on I_3 or there is a nonvoid open interval $I_4 \subseteq T_3 \cap I_3$, where T_3 is the open set $I_3 \setminus S_3$. But this would contradict the density of S_3 in $\mathbb{R}$.

Therefore, $f^{(3)}$ is identically 0 on $\mathbb{R}$, and so therefore, by the mean value theorem, f" must be constant, f' linear and f quadratic.

10. Prove that the set $\mathbb{Q}$ of rationals can be written as the union of countably many subsets of $\mathbb{Q}$ each of which is dense in the set $\mathbb{R}$ of real numbers.

Solution. Let $\{r_1, r_2, \cdots, r_i, \cdots\}$ be the increasing sequence of all positive integers that are not the mth power of any integer for any integer exponent exceeding 1. Let X_k be the set of all rationals of the form a/r_k^c where a is an integer coprime with r_k and c is a positive integer. We include also in X_1 all integers. Then it can be seen that every rational lies in one of the X_k and that the X_k are disjoint. Since each closed interval of length $1/r_k^c$ contains a number in X_k of the form

$$\frac{a_{c+1}r_k^{c+1} + a_c r_k^c + \cdots + a_1 r_k + 1}{r_k^{c+1}}$$

, it can be seen that X_k is dense in $\mathbb{R}$.