A generalization of a certain Pythagorean relation

Every child is introduced to the Pythagorean relationship $3^2+4^2=5^2$, but not many will know also that $10^2+11^2+12^2=13^2+14^2$. Each of these can be established by simple verification; computes the squares and perform the necesssary additions on both sides of the equation. If we accept this argument as our proof, then we might regard both these equations as a fluke, much in the same way that we have the interesting result that $3^3+4^3+5^3=6^3$, which also seems to be a stand-alone novelty.

However, there is another way of proving that $3^2+4^2=5^2$ which indicates the way to broader fields, we might note that $5^2-4^2=(5-4)(5+4)=1\times 9=9=3^2$. This immediately suggests that the numerical equation is but one in a larger family of similar relationahips. Simply take two consecutive numbers that add up to a square, and find the difference of their squares. Thus, because $5^2=25=13+12$, we see easily, from a difference of squares factorization, that

$$13^2-12^2=(13-12)(13+12)=1\times 25=5^2\ .$$

it is atraightforward to construct further example. In fact, from the idea of this argument, we can give a parametic family of Pythagorean triples. Since any square which is the sum of two consecutives is odd, let it be $(2k+1)^2=4k^2+4k+1=(2k^2+2k+1)+(2k^2+2k)$. Then, it is easy to check that

$$(2k^2+2k+1)^2-(2k^2+2k)^2=(2k+1)^2\ ,$$

so that

$$(2k+1)^2+(2k^2+2k)^2=(2k^2+2k+1)^2\ .$$

But none of these reproduces the second numerical equation given above: $10^2+11^2+12^2=13^2+14^2$. However, we do notice the consecutive integers 12 and 13 straddling the equals sign whose sum is the square 25. Furthermore, we also see that the second numbers in from the equals sign, 11 and 14, also add up to 25. So we can exploit the difference of squares strategy again to get

$$\begin{aligned}(13^2-12^2)+(14^2-11^2)&=(13-12)(13+12)+(14-11)(14+11)\\&=1\times 25+3\times 25=(1+3)\times 25=2^2\times 5^2=10^2\ .\end{aligned}$$

A new ingredient has appeared in the form of the sum $1+3$ whose sum happens to be $4=2^2$.

Is there a productive way to look at the sum $1 + 3 = 2^2$? The square of 2 can be represented by a 2×2 array of four dots, consisting of a single dot in the upper left corner and a "gnomon" of three additional dots. We can add a further gnomon of five dots to augment it to a 3×3 square, and then a gnomon of seven dots to augment it to a 4×4 square. Looking at the situation from this vantage point, we can convince ourselves that

$$1 + 3 + 5 + \cdots + 2k - 1 = k^2 \ ,$$

or, in words, that the sum of the first k odd numbers is k^2.

Now we are in a position to suss out further examples. Start with the two equations

$$3^2 + 4^2 = 5^2 \ ;$$

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2 \ .$$

In the first equation, the difference of the largest two squares gave us 3^2. In the second, we computed two differences of squares, getting 1×25 and 3×25 for a sum of $4 \times 25 = (2 \times 5)^2$. This invites us to look at the second equation with new eye. The roots of the squares go from $2 \times 5 = 10$ up to $3 \times 5 - 1 = 14$; they constitude $(3 - 2) \times 5 = 5$ consecutive numbers, three of which are on the left side of the equation and the other two on the right.

Let us set about the construction of the third equation in our series. This time, it will involve seven terms, all squares; there would be four terms on the left and the right; the roots of the middle two squares will add up to $49 = 7^2$, and the square roots will run consecutively from $3 \times 7 = 21$ up to $4 \times 7 - 1 = 27$. Of these, we can form three pairs $(22, 27)$, $(23, 26)$ and $(24, 25)$ straddling the equals sign, one of which are consecutive integers and all having sum 7^2.

Thus, we should have

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2 \ ,$$

which can be established by notin that

$$(25^2 - 24^2) + (26^2 - 23^2) + (27^2 - 22^2) = (1 + 3 + 5) \times (49) = 3^2 \times 7^2 = 21^2 \ .$$

Having established the three equations in this way, we are now on the path of constructing an infinite sequence of further equations following the pattern, and, *more signficantly*, of seeing how they can all be established.

This is much more satisfying than having to verify each one with a pocket calculator where the underlying structure would remain hidden.