Solutions - CMO 2008

1. $ABCD$ is a convex quadrilateral in which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

Solution. Since $[MADC] = \frac{1}{2}[ABCD] = [NADC]$, it follows that $[ANC] = [AMC]$, so that $MN \| AC$. Let m be a line through D parallel to AC and MN and let BA produced meet m at P and BC produced meet m at Q. Then

$$[MPC] = [MAC] + [CAP] = [MAC] + [CAD] = [MADC] = [BMC]$$

whence $BM = MP$. Similarly $BN = NQ$, so that MN is a midline of triangle BPQ and must bisect BD.

2. Determine all functions f defined on the set of rationals that take rational values for which

$$f(2f(x) + f(y)) = 2x + y$$

for each x and y.

Solution 1. The only solutions are $f(x) = x$ for all rational x and $f(x) = -x$ for all rational x. Both of these readily check out.

Setting $y = x$ yields $f(3f(x)) = 3x$ for all rational x. Now replacing x by $3f(x)$, we find that

$$f(9x) = f(3f(3f(x)) = 3[3f(x)] = 9f(x) ,$$

for all rational x. Setting $x = 0$ yields $f(0) = 9f(0)$, whence $f(0) = 0$.

Setting $x = 0$ in the given functional equation yields $f(f(y)) = y$ for all rational y. Thus f is one-one onto. Applying f to the functional equation yields that

$$2f(x) + f(y) = f(2x + y)$$

for every rational pair (x, y).

Setting $y = 0$ in the functional equation yields $f(2f(x)) = 2x$, whence $2f(x) = f(2x)$. Therefore $f(2x) + f(y) = f(2x + y)$ for each rational pair (x, y), so that

$$f(u + v) = f(u) + f(v)$$

for each rational pair (u, v).

Since $0 = f(0) = f(-1) + f(1)$, $f(-1) = -f(1)$. By induction, it can be established that for each intger n and rational x, $f(nx) = nf(x)$. If $k = f(1)$, we can establish from this that $f(n) = nk$, $f(1/n) = k/n$ and $f(m/n) = mk/n$ for each integer pair (m, n). Thus $f(x) = kx$ for all rational x. Since $f(f(x)) = x$, we must have $k^2 = 1$. Hence $f(x) = x$ or $f(x) = -x$. These check out.

Solution 2. In the functional equation, let

$$x = y = 2f(z) + f(w)$$

to obtain $f(x) = f(y) = 2z + w$ and

$$f(6z + 3w) = 6f(z) + 3f(w)$$

for all rational pairs (z, w). Set $(z, w) = (0, 0)$ to obtain $f(0) = 0$, $w = 0$ to obtain $f(6z) = 6f(z)$ and $z = 0$ to obtain $f(3w) = 3f(w)$ for all rationals z and w. Hence $f(6z + 3w) = f(6z) + f(3w)$. Replacing $(6z, 3w)$ by (u, v) yields

$$f(u + v) = f(u) + f(v)$$

for all rational pairs (u, v). Hence $f(x) = kx$ where $k = f(1)$ for all rational x. Substitution of this into the functional equation with $(x, y) = (1, 1)$ leads to $3 = f(3f(1)) = f(3k) = 3k^2$, so that $k = \pm 1$. It can be checked that both $f(x) \equiv 1$ and $f(x) \equiv -1$ satisfy the equation.

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3. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \leq \frac{3}{2} \; .$$

Solution 1. Note that

$$1 - \frac{a - bc}{a + bc} = \frac{2bc}{1 - b - c + bc} = \frac{2bc}{(1 - b)(1 - c)} \; .$$

The inequality is equivalent to

$$\frac{2bc}{(1 - b)(1 - c)} + \frac{2ca}{(1 - c)(1 - a)} + \frac{2ab}{(1 - a)(1 - b)} \geq \frac{3}{2} \; .$$

Manipulation yields the equivalent

$$4(bc + ca + ab - 3abc) \geq 3(bc + ca + ab + 1 - a - b - c - abc) \; .$$

This simplifies to $ab + bc + ca \geq 9abc$ or

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9 \; .$$

This is a consequence of the harmonic-arithmetic means inequality.

Solution 2. Observe that

$$a + bc = a(a + b + c) + bc = (a + b)(a + c)$$

and that $a + b = 1 - c$, with analogous relations for other permutations of the variables. Then

$$(b + c)(c + a)(a + b) = (1 - a)(1 - b)(1 - c) = (ab + bc + ca) - abc \; .$$

Putting the left side of the desired inequality over a common denominator, we find that it is equal to

$$\begin{aligned}\frac{(a - bc)(1 - a) + (b - ac)(1 - b) + (c - ab)(1 - c)}{(b + c)(c + a)(a + b)} &= \frac{(a + b + c) - (a^2 + b^2 + c^2) - (bc + ca + ab) + 3abc}{(b + c)(c + a)(a + b)} \\ &= \frac{1 - (a + b + c)^2 + (bc + ca + ab) + 3abc}{(ab + bc + ca) - abc} \\ &= \frac{(bc + ca + ab) + 3abc}{(bc + bc + ab) - abc} \\ &= 1 + \frac{4abc}{(a + b)(b + c)(c + a)} \; .\end{aligned}$$

Using the arithmetic-geometric means inequality, we obtain that

$$\begin{aligned}(a + b)(b + c)(c + a) &= (a^2 b + b^2 c + c^2 a) + (ab^2 + bc^2 + ca^2) + 2abc \\ &\geq 3abc + 3abc + 2abc = 8abc \; ,\end{aligned}$$

whence $4abc/[(a+b)(b+c)(c+a)] \leq \frac{1}{2}$. The desired result follows. Equality occurs exactly when $a = b = c = \frac{1}{3}$.

4. Find all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

Solution. The substitution $n = p$, a prime, yields $p \equiv (f(p))^p \equiv 0 \pmod{f(p)}$, so that p is divisible by $f(p)$. Hence, for each prime p, $f(p) = 1$ or $f(p) = p$.

Let $S = \{p : p \text{ is prime and } f(p) = p\}$. If S is infinite, then $f(n)^p \equiv n \pmod p$ for infinitely many primes p. By the little Fermat theorem, $n \equiv f(n)^p \equiv f(n)$, so that $f(n) - n$ is a multiple of p for infinitely many primes p. This can happen only if $f(n) = n$ for all values of n, and it can be verified that this is a solution.

If S is empty, then $f(p) = 1$ for all primes p, and any function satisfying this condition is a solution.

Now suppose that S is finite and non-empty. Let q be the largest prime in S. Suppose, if possible, that $q \geq 3$. Therefore, for any prime p exceeding q, $p \equiv 1 \pmod q$. However, this is not true. Let Q be the product of all the odd primes up to q. Then $Q + 2$ must have a prime factor exceeding q and at least one of them must be incongruent to $1 \pmod q$. (An alternative argument notes that Bertrand's postulate can turn up a prime p between q and $2q$ which fails to satisfy $p \equiv 1 \mod q$.)

The only remaining case is that $S = \{2\}$. Then $f(2) = 2$ and $f(p) = 1$ for every odd prime p. Since $f(n)^2 \equiv n \pmod 2$, $f(n)$ and n must have the same parity. Conversely, any function f for which $f(n) \equiv n \pmod 2$ for all n, $f(2) = 2$ and $f(p) = 1$ for all odd primes p satisfies the condition.

Therefore the only solutions are
• $f(n) = n$ for all $n \in \mathbf{N}$;
• any function f with $f(p) = 1$ for all primes p;
• any function for which $f(2) = 2$, $f(p) = 1$ for primes p exceeding 2 and $f(n)$ and n have the same parity.

5. A *self-avoiding rook walk* on a chessboard (a rectangular grid of squares) is a path traced by a sequence of rook moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, *i.e.*, the rook's path is non-self-intersecting.

Let $R(m, n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, $R(m, 1) = 1$ for all natural numbers m; $R(2, 2) = 2$; $R(3, 2) = 4$; $R(3, 3) = 11$. Find a formula for $R(3, n)$ for each natural number n.

Solution 1. Let $r_n = R(3, n)$. It can be checked directly that $r_1 = 1$ and $r_2 = 4$. Let $1 \leq i \leq 3$ and $1 \leq j$; let (i, j) denote the cell in the ith row from the bottom and the jth column from the left, so that the paths in question go from $(1, 1)$ to $(3, 1)$.

Suppose that $n \geq 3$. The rook walks fall into exactly one of the following six categories:

(1) One walk given by $(1, 1) \to (2, 1) \to (3, 1)$.

(2) Walks that avoid the cell $(2, 1)$: Any such walk must start with $(1, 1) \to (1, 2)$ and finish with $(3, 2) \to (3, 1)$; there are r_{n-1} such walks.

(3) Walks that begin with $(1, 1) \to (2, 1) \to (2, 2)$ and never return to the first row: Such walks enter the third row from $(2, k)$ for some k with $2 \leq k \leq n$ and then go along the third row leftwards to $(3, 1)$; there are $n - 1$ such walks.

(4) Walks that begin with $(1,1) \rightarrow (2,1) \rightarrow \cdots \rightarrow (2,k) \rightarrow (1,k) \rightarrow (1,k+1)$ and end with $(3,k+1) \rightarrow (3,k) \rightarrow (3,k-1) \rightarrow \cdots \rightarrow (3,2) \rightarrow (3,1)$ for some k with $2 \leq k \leq n-1$; there are $r_{n-2} + r_{n-3} + \cdots + r_1$ such walks.

(5) Walks that are the horizontal reflected images of walks in (3) that begin with $(1,1) \rightarrow (2,1)$ and never enter the third row until the final cell; there are $n-1$ such walks.

(6) Walks that are horizontal reflected images of walks in (5); there are $r_{n-2} + r_{n-3} + \cdots + r_1$ such walks.

Thus, $r_3 = 1 + r_2 + 2(2 + r_1) = 11$ and, for $n \geq 3$,

$$\begin{aligned} r_n &= 1 + r_{n-1} + 2[(n-1) + r_{n-2} + r_{n-3} + \cdots + r_1] \\ &= 2n - 1 + r_{n-1} + 2(r_{n-2} + \cdots + r_1) \ , \end{aligned}$$

and

$$r_{n+1} = 2n + 1 + r_n + 2(r_{n-1} + r_{n-2} + \cdots + r_1) \ .$$

Therefore

$$r_{n+1} - r_n = 2 + r_n + r_{n-1} \Longrightarrow r_{n+1} = 2 + 2r_n + r_{n-1} \ .$$

Thus

$$r_{n+1} + 1 = 2(r_n + 1) + (r_{n-1} + 1) \ ,$$

whence

$$r_n + 1 = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n+1} - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n+1} \ ,$$

and

$$r_n = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n+1} - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n+1} - 1 \ .$$

Solution 2. Employ the same notation as in Solution 1. We have that $r_1 = 1$, $r_2 = 4$ and $r_3 = 11$. Let $n \geq 3$. Consider the situation that there are r_{n+1} columns. There are basically three types of rook walks.

Type 1. There are four rook walks that enter only the first two columns.

Type 2. There are $3r_{n-1}$ rooks walks that do not pass between the second and third columns in the middle row (in either direction), *viz.* r_{n-1} of each of the types:

$$(1,1) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (3,1) \ ;$$

$$(1,1) \longrightarrow (2,1) \longrightarrow (2,2) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (3,1) \ ;$$

$$(1,1) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (2,2) \longrightarrow (2,1) \longrightarrow (3,1) \ .$$

Type 3. Consider the rook walks that pass between the second and third column along the middle row. They are of Type 3a:

$$(1,1) \longrightarrow * \longrightarrow (2,2) \longrightarrow (2,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (3,1) \ ,$$

or Type 3b:

$$(1,1) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (2,3) \longrightarrow (2,2) \longrightarrow * \longrightarrow (3,1) \ ,$$

where in each case the asterisk stands for one of two possible options.

We can associate in a two-one way the walks of Type 3a to a rook walk on the last n columns, namely

$$(1,2) \longrightarrow (2,2) \longrightarrow (2,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2)$$

and the walks of Type 3b to a rook walk on the last n columns, namely

$$(1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (2,3) \longrightarrow (2,2) \longrightarrow (3,2) \ .$$

The number of rook walks of the latter two types together is $r_n - 1 - r_{n-1}$. From the number of rook walks on the last n columns, we subtract one for $(1,2) \to (2,2) \to (3,2)$ and r_{n-1} for those of the type

$$(1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (2,3) \ .$$

Therefore, the number of rook walks of Type 3 is $2(r_n - 1 - r_{n-1})$ and we find that

$$r_{n+1} = 4 + 3r_{n-1} + 2(r_n - 1 - r_{n-1}) = 2 + 2r_n + r_{n-1} \ .$$

We can now complete the solution as in Solution 1.

Solution 3. Let $S(3,n)$ be the set of self-avoiding rook walks in which the rook occupies column n but does not occupy column $n+1$. Then $R(3,n) = |S(3,1)| + |S(3,2)| + \cdots + |S(3,n)|$. Furthermore, topological considerations allow us to break $S(3,n)$ into three disjoint subsets $S_1(3,n)$, the set of paths in which corner $(1,n)$ is not occupied, but there is a path segment $(2,n) \longrightarrow (3,n)$; $S_2(3,n)$, the set of paths in which corners $(1,n)$ and $(3,n)$ are both occupied by a path $(1,n) \longrightarrow (2,n) \longrightarrow (3,n)$; and $S_3(3,n)$, the set of paths in which corner $(3,n)$ is not occupied but there is a path segment $(1,n) \longrightarrow (2,n)$. Let $s_i(n) = |S_i(3,n)|$ for $i = 1,2,3$. Note that $s_1(1) = 0$, $s_2(1) = 1$ and $s_3(1) = 0$. By symmetry, $s_1(n) = s_3(n)$ for every positive n. Furthermore, we can construct paths in $S(3,n+1)$ by "bulging" paths in $S(3,n)$, from which we obtain

$$\begin{aligned} s_1(n+1) &= s_1(n) + s_2(n) \ ; \\ s_2(n+1) &= s_1(n) + s_2(n) + s_3(n) \ ; \\ s_3(n+1) &= s_2(n) + s_3(n) \ ; \end{aligned}$$

or, upon simplification,

$$\begin{aligned} s_1(n+1) &= s_1(n) + s_2(n) \ ; \\ s_2(n+1) &= 2s_1(n) + s_2(n) \ . \end{aligned}$$

Hence, for $n \geq 2$,

$$\begin{aligned} s_1(n+1) &= s_1(n) + 2s_1(n-1) + s_2(n-1) \\ &= s_1(n) + 2s_1(n-1) + s_1(n) - s_1(n-1) \\ &= 2s_1(n) + s_1(n-1) \ . \end{aligned}$$

and

$$\begin{aligned} s_2(n+1) &= 2s_1(n) + s_2(n) = 2s_1(n-1) + 2s_2(n-1) + s_2(n) \\ &= s_2(n) - s_2(n-1) + 2s_2(n-1) + s_2(n) \\ &= 2s_2(n) + s_2(n-1) \ . \end{aligned}$$

We find that

$$\begin{aligned} s_1(n) &= \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n-1} - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n-1} \ ; \\ s_2(n) &= \frac{1}{2}(1+\sqrt{2})^{n-1} + \frac{1}{2}(1-\sqrt{2})^{n-1} \ . \end{aligned}$$

Summing a geometric series yields that

$$\begin{aligned} R(3,n) &= (s_2(1) + \cdots + s_2(n)) + 2(s_1(1) + \cdots + s_1(n)) \\ &= \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\left(\frac{(1+\sqrt{2})^n - 1}{\sqrt{2}}\right) + \left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right)\left(\frac{(1-\sqrt{2})^n - 1}{-\sqrt{2}}\right) \\ &= \left(\frac{1}{2\sqrt{2}}\right)[(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}] - 1 \ . \end{aligned}$$

The formula agrees with $R(3,1) = 1$, $R(3,2) = 4$ and $R(3,3) = 11$.

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