# COMPOSING QUADRATIC POLYNOMIALS

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ABSTRACT. The sequence of oblong numbers (products of two consecutive integers) is the occasion for a simple conjecture that is readily established by algebra. This leads to conjectures about other sequences, and then about quadratic polynomials. Ultimately, the situation can be generalized to obtain an interesting result about conditions under which a polynomial can be expressed as the composition of two others.

This mathematical odyssey is presented as material that can be used with secondary students to highlight different aspects of algebra, including its role in proving conjectures and the different connotations of variables.

## 1. Introduction

Starting with a simple observation about oblong numbers, we arrive a general property of quadratic polynomials. This brings out a number of important aspects of algebra that should be kept in mind when teaching the subject. The discussion continues to obtain an interesting result about the composition of quadratic polynomials and finally we describe how an Ottawa high school student parlayed the situation into a publishable paper.

The purpose of this paper is not to provide a recipe for a presentation or activity that is to be slavishly followed, but rather to present a coherent body of ideas that may provide the basis for productive individual or classroom activity. The teacher must keep in mind the interests, background and abilities of the students, and tailor what she does to these factors. Some of it may be suitab le for the classroom, as there will be elements that support the curriculum. Some may be a group of students or for a project by one or two students. The teacher must also be prepared to take on board a question or observation from a student which may allow the discussion to move in a productive direction. The ultimate purpose is to enrich the algebraic experience so that the student gains judgment and flexibility in dealing with algebraic expressions and can maintain a substantive chain of analysis.

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## 2. Oblong and other interesting quadratic sequences

The identity $x^2y^2 = (xy)^2$ applied to integers $x$ and $y$ expresses the important observation that the product of two squares is itself a square.

Let us consider *oblong numbers*. These are the products $x(x+1)$ of two consecutive integers $x$ and $x+1$. The sequence of oblong numbers begins with

$$0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210.$$

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Unlike with squares, it is not the case that the product of two oblong numbers is always oblong. But it sometimes happens. In fact, it appears that the product of two *consecutive* oblong numbers is oblong.

How can we establish whether this is true? One possibility is to collect evidence and see whether we can make a conjecture. For this purpose, we need some notation: $f(x) = x(x+1)$. Then we can check that $f(0)f(1) = 0 = f(0)$, $f(1)f(2) = 12 = f(3)$, $f(2)f(3) = 72 = f(8)$, $f(3)f(4) = 240 = 15 \times 16 = f(15)$ and $f(4)f(5) = 600 = 24 \times 25 = f(24)$. In each case, we note that the argument for each function on the right side of the equation is one less than a perfect square, and are led to conjecture that $f(x)f(x+1) = f((x+1)^2 - 1) = f(x^2 + 2x)$. This approach involves recognizing a pattern, expressing it algebraically and then establishing that it actually holds.

A second approach is to work from the algebraic expression for $f(x)$, set a goal and then manipulate the algebra to achieve the goal. Thus $f(x)f(x+1) = x(x+1).(x+1(x+2)$. The goal is to express this as the product of two consecutives. To achieve this, the viewer has to be aware that this might depend on how we pair off the four terms. Noting that $x(x+2) = x^2 + 2x = (x+1)^2 - 1$, we see that

$$f(x)f(x+1) = ((x+1)^2 - 1)(x+1)^2) = f((x+1)^2 - 1) = f(x^2 + 2x).$$

This, of course, is the same as before, but notice that each path shines a light on a somewhat different set of skills.

If may be that some student may note that it is easier to present the product of two consecutive pairs in the form $(x-1)x$ rather than $x(x+1)$. In this case, the form of the product

$$(x-1)x \cdot x(x+1) = (x^2 - 1)x^2$$

is much easier to discern.

We can now look at other sequences. For example, the sequence of numbers of the form $f(x) = x^2 + 1$, namely

$$1, 2, 5, 10, 17, 26, 37, 50, 65, 82, 101, \ldots$$

has the same property that the product of two consecutives is again in the sequence. We can go through a similar process to verify that this is always true. For example,

$$\begin{aligned} f(x)f(x+1) &= (x^2+1)((x+1)^2+1) = x^2(x+1)^2 + x^2 + (x+1)^2 + 1 \\ &= (x^4 + 2x^3 + 3x^2 + 2x + 1) + 1 = (x^2 + x + 1)^1 + 1 \\ &= f(x^2 + x + 1). \end{aligned}$$

We can generalize this slighty by looking at $f(x) = x^2 - 1, x^2 + 2, x^2 + 3$ or more broadly $f(x) = x^2 + k$ where $k$ is any integers.

Another ineresting case is $f(x) = x^2 + x + 1$. If is interesting to note that $f(x-1) = x^2 - x = 1$ so that

$$f(x-1)f(x) = (x^2 - x + 1)(x^2 + x + 1) = (x^2 + 1)^2 - x^2 \quad = x^4 + x^2 + 1 = f(x^2),$$

so that $f(x)f(x+1) = f((x+1)^2) = f(x^2 + 2x + 1)$, a fact that can be checked directly.

## 3. Monic quadratics

After playing around with several such sequences for which the product of two consecutives is also in the sequence, we might note that in each case the general term is given by the sum of $x^2$ and a linear polynomial in $x$. This raise the question as to whether in any sequence for which the general term is a *monic* quadratic polynomial (*i.e.* one of the form $f(x) = x^2 + bx + c$ whose leading coefficient is 1) has the same property.

Playing around to find, for given $x$, a number $z$ for which $f(x)f(x+1) = f(z)$ is not an easy task. However, we can bring some structural insight into play. Think of the graph of the equation $y = f(x)$ in the Cartesian plane. It is a parabola. We want to the values of $f(x)$ at two consecutive integers. By shifting its position along the axis and integral distance, we can make these consecutive integers take the values 0 and 1. If we can find an abscissa at which $f$ takes the value $f(0)f(1)$, we can then shift back to solve our problem for the product $f(x)f(x+1)$.

Let us put this in algebraic terms. Define, for the variable $t$,

$$h(t) = f(x + t).$$

We are shifting our perspective on $x$; instead of being a variable in the representation of $f$, we are now regarding it as a parameter (*i.e.* fixed for the time being). The experienced student will recognize that $h(t)$ is a monic quadratic polynomial of the form $t^2 + pt + q$, where $p$ and $q$ are functions of $a, b, x$. This can be made explicit:

$$h(t) = (t + x)^2 + b(t + x) + c = t^2 + (2x + b)t + (x^2 + bx + c).$$

Thus $p = 2x+b$ and $q = x^2+bx+c = f(x)$. (The student who knows some calculus will recognize that $p = f'(x)$.) Since $f(x)f(x + 1) = h(0)h(1)$, we have simplified the situation enormously.

Let us compute.

$$h(0)h(1) = q(1 + p + q) = q + pq + q^2 = q^2 + pq + q = h(q) = h(h(0)).$$

While the calculation is straightforward, the equation is rich in significance. While $p$ and $q$ initially present as coefficients for the function $h$, we observe, by reversing the order of the terms, that we get an evaluation of $h$ at the variable $q$. This is a change of perspective on what is first a simple expression of the coefficients. Thus, we identify that product as an evaluation of $h$ at $q$, and finally recognize that $q$ is the value of $h$ at 0.

Return to our original function.

$$\begin{aligned} f(x)f(x+1) &= h(0)h(1) = h(q) = h(h(0)) = f(x + h(0)) \\ &= f(x + f(x)) \end{aligned}$$

This can be put in the form

$$f(x)f(x + 1) = f(g(x)),$$

where $g(x) = x + f(x)$.

Since we have looked at some specific examples for $f$, we can go back and check that this formula works (thus getting some manipulative practice). Here are the results:

When $f(x) = x(x + 1)$, $g(x) = x^2 + x + x = x^2 + 2x$.

When $f(x) = x^2 + 1$, $g(x) = x^2 + x + 1$.

When $f(x) = x^2 + x + 1$, $g(x) = x^2 + 2x + 1 = (x + 1)^2$.

## 4. The general quadratic

One way to interpret the last situation is that, for a monic quadratic $f(x)$, the polynomial $f(x)f(x + 1)$ is a polynomia of the fourth degree, and we have represented it as the composition of two quadratic. In this more general formulation, we can ask whether it is necessary for the leading coefficient for $f$ to be 1. The answer is **no**. This led to my posing the following problem to students writing a competition. I will present the solutions and comment on them later. Notice that they take advantage of the different representations of a quadratic polynomial.

*Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which*

$$f(x)f(x + 1) = g(h(x)).$$

*Solution 1.* [A. Remorov] Let $f(x) = a(x - r)(x - s)$. Then

$$\begin{aligned} f(x)f(x + 1) &= a^2(x - r)(x - s + 1)(x - r + 1)(x - s) \\ &= a^2(x^2 + x - rx - sx + rs - r)(x^2 + x - rx - sx + rs - s) \\ &= a^2[(x^2 - (r + s - 1)x + rs) - r][(x^2 - (r + s - 1)x + rs) - s] \\ &= g(h(x))\ , \end{aligned}$$

where $g(x) = a^2(x - r)(x - s) = af(x)$ and $h(x) = x^2 - (r + s - 1)x + rs$.

*Solution 2.* Let $f(x) = ax^2 + bx + c$, $g(x) = px^2 + qx + r$ and $h(x) = ux^2 + vx + w$. Then

$$f(x)f(x+1) = a^2x^4 + 2a(a+b)x^3 + (a^2+b^2+3ab+2ac)x^2 + (b+2c)(a+b)x + c(a+b-c)$$

$$\begin{aligned} g(h(x)) &= p(ux^2 + vx + w)^2 + q(ux + vx + w) + r \\ &= pu^2x^4 + 2puvx^3 + (2puw + pv^2 + qu)x^2 + (2pvw + qv)x + (pw^2 + qw + r). \end{aligned}$$

Equating coefficients, we find that $pu^2 = a^2$, $puv = a(a + b)$, $2puw + pv^2 + qu = a^2 + b^2 + 3ab + 2ac$, $(b + 2c)(a + b) = (2pw + q)v$ and $c(a + b + c) = pw^2 + qw + r$. We need to find just one solution of this system. Let $p = 1$ and $u = a$. Then $v = a + b$ and $b + 2c = 2pw + q$ from the second and fourth equations. This yields the third equation automatically. Let $q = b$ and $w = c$. Then from the fifth equation, we find that $r = ac$.

Thus, when $f(x) = ax^2 + bx + c$, we can take $g(x) = x^2 + bx + ac$ and $h(x) = ax^2 + (a + b)x + c$.

*Solution 3.* [S. Wang] Suppose that

$$if(x) = a(x + h)^2 + k = a(t - (1/2))^2 + k,$$

where $t = x + h + \frac{1}{2}$. Then $f(x + 1) = a(x + 1 + h)^2 + k = a(t + (1/2))^2 + k$, so that

$$\begin{aligned} f(x)f(x + 1) &= a^2(t^2 - (1/4))^2 + 2ak(t^2 + (1/4)) + k^2 \\ &= a^2t^4 + \left(-\frac{a^2}{2} + 2ak\right)t^2 + \left(\frac{a^2}{16} + \frac{ak}{2} + k^2\right). \end{aligned}$$

Thus, we can achieve the desired representation with $h(x) = t^2 = x^2 + (2h + 1)x + \frac{1}{4}$ and $g(x) = a^2x^2 + (\frac{-a^2}{2} + 2ak)x + (\frac{a^2}{16} + \frac{ak}{2} + k^2)$.

*Solution 4.* [V. Krakovna] Let $f(x) = ax^2 + bx + c = au(x)$ where $u(x) = x^2 + dx + e$, where $b = ad$ and $c = ae$. If we can find functions $v(x)$ and $w(x)$ for which $u(x)u(x+1) = v(w(x))$, then $f(x)f(x+1) = a^2v(w(x))$, and we can take $h(x) = w(x)$ and $g(x) = a^2v(x)$.

Define $p(t) = u(x+t)$, so that $p(t)$ is a monic quadratic in $t$. Then, noting that $p''(t) = u''(x+t) = 2$, we have that

$$p(t) = u(x+t) = u(x) + u'(x)t + \frac{u''(x)}{2}t^2 = t^2 + u'(x)t + u(x) \ ,$$

from which we find that

$$\begin{aligned} u(x)u(x+1) = p(0)p(1) &= u(x)[u(x) + u'(x) + 1] \\ &= u(x)^2 + u'(x)u(x) + u(x) = p(u(x)) = u(x + u(x)). \end{aligned}$$

Thus, $u(x)u(x+1) = v(w(x))$ where $w(x) = x + u(x)$ and $v(x) = u(x)$. Therefore, we get the desired representation with

$$h(x) = x + u(x) = x^2 + \left(1 + \frac{b}{a}\right)x + \frac{c}{a}$$

and

$$g(x) = a^2v(x) = a^2u(x) = af(x) = a^2x^2 + abx + ac.$$

*Comment.* The second solution can also be obtained by looking at special cases, such as when $a = 1$ or $b = 0$, getting the answer and then making a conjecture.

This quartet of solutions illustrates in particular how one can make use of different standard representations of a quadratic polynomial. A quadratic can be represented in factored form as in Solution 1, as a sum of terms in descending order of degree as in Solution 2, or by completing the square as in Solution 3. While these forms may be algebraically equivalent, each reveals different information about the quadratic and casts the mechanism underlying the solution in its own particular light.

In Solution 1, note how the four factors of $f(x)f(x+1)$ are arranged to bring out the special form of $g$ as a multiple of $f$ in its factored form. This involve an ability to see above the details of an expression to its over all form.

Solution 2 is the most pedestrian approach and mainly requires the intestinal fortitude to work through the algebraic manipulation. This involves the important idea that two polynomials are everywhere equal if and only if their corresponding coefficients agree. Equating coefficients entails finding a system of five equations in nine unknowns, and so it is underdetermined. We need to recognize in this solution that the representation we are looking for is probably not unique, and that to prove the result, we only need one possibility. This gives us a license to set some of the variables equal to numbers that will make the computations easy.

Solution 4 works in stages, and begins by reducing the problem to one about monic quadratic polynomials. In this solution, I have involved derivatives in the representation of $p(t)$, but for high school students this can be avoided as in the previous section.

## 5. A slight diversion.

Consider the pair $\{x_n\}$ and $\{y_n\}$ of bilateral sequences defined as follows. Let $x_0$ and $x_1$ be any pair of inteegers, and let $y_0 = x_1 - x_0$. For convenience, we will take $x_0 = r$, $y_0 = d$ and $x_1 = r + d$. For any integer $n$, define

$$x_n = x_{n-1} + y_{n-1}$$

and

$$y_n = y_{n-1} + 2.$$

Then it can be shown by induction that for all $n$,

$$x_n = n(n-1) + r + dn$$

and

$$y_n = d + 2n.$$

The sequence $\{x_n\}$ has the interesting property that the product $x_0 x_1$ also appeats in the sequence. Indeed,

$$x_r = r(r-1) + r + dr = r^2 + dr = r(r+d) = x_0 x_1.$$

Note that $x_n$ is a monic quadratic polynomial in $n$, and that we have just noted that

$$x_{x_0} = x_0 x_1.$$

## 6. When is a polynomial a composition of other polynomials?

One Ottawa, ON high school student, James Rickards, carried the result much further and actually got a note published in the *American Mathematical Monthly*, the flagship jounral of the *Mathematical Association of America.* When $f(x) = x^2 + bx + c$ is a monic, quadratic polynomial, it is straightforward to check that $f(0)f(1)) = f(c) = f(f(0))$. By a translation of the variable, it follows that $f(x)f(x+1) = f(x + f(x))$ identically in $x$. This can be generalized to general quadratic polynomials to obtain $f(x)f(x+1) = g(h(x))$ where $f(x) = ax^2 + bx + c$, $g(x) = x^2 + bx + ac$ and $g(x) = ax^2 + (a+b)x + c$.

There is more significance to this result than a chance computation. If the roots of the polynomial $f(x)$ are $r$ and $s$, then the roots of $f(x+1)$ are $r-1$ and $s-1$, and we note that $f(x)f(x+1)$ is a quartic polynomial for which the sum of two of its roots is equal to the sum of the other two. This is the critical observation. For, it turns out that a quartic polynomial can be expressed as the composite of two quadratics *if and only if* two of its roots have the same sum as the other two. We generalize this to polynomials of higher degree.

For a set $X = \{x_1, x_2, \cdots, x_n\}$, we denote by $\sigma_k(X)$ the $k$th symmetric function of the variables $x_i$, the sum of $\binom{n}{k}$ products of $k$ of the $x_i$:

$$\sigma_k(X) = \sum \{x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} : 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le n\}$$

for $1 \le k \le n$. We define $\sigma_0(X) = 1$. We recall that if $f(x) = g(h(x))$ for polynomials $f$, $g$, $h$, then the degree of $f$ is the product of the degrees of $g$ and $h$.

*The criterion for composition.* Suppose that $f(x)$ is a polynomial with complex coefficients of degree $mn$ where $m$ and $n$ are integers exceeding 1. We begin by assuming that the leading coefficient of $f(x)$ is 1 and generalize later.

**Proposition 1.** The monic polynomial $f(x)$ can be written as the composite $g(h(x))$ of a polynomial $h$ of degree $n$ and $g$ of degree $m$ if and only if $R$ can be partitioned into $m$ sets $S_1$, $S_2$, $\cdots$, $S_m$, each with $n$ elements (not necessarily distinct) such that, for each integer $j$ with $0 \le j \le n-1$,

$$\sigma_j(S_1) = \sigma_j(S_2) = \cdots = \sigma_j(S_m).$$

*Proof.* Suppose that the set $R$ of roots of $f$ can be partitioned as indicated. Let $R = \{r_1, r_2, \cdots, r_{mn}\}$ be the set of roots of $f$, each listed as often as its multiplicity and indexed so that $S_1 = \{r_1, r_2, \cdots, r_n\}$, $S_2 = \{r_{n+1}, r_{n+2}, \cdots, r_{2n}\}$, $\cdots$, $S_m = \{r_{nm-n+1}, r_{mn-n+2}, \cdots, r_{mn}\}$. For $1 \leq i \leq m$, let

$$y_i(x) = (x - r_{(i-1)n+1})(x - r_{(i-1)n+2}) \cdots (x - r_{in})$$

be the monic polynomial whose roots are the elements of $S_i$. Then, if $i$ and $j$ are distinct positive ingers not exceeding $m$, the condition that the corresponding symmetric functions of $S_i$ and $S_j$ are equal except for the $n$th implies that $y_i(x) - y_j(x)$ is a constant. Define $z_i = y_1(x) - y_i(x)$ for $1 \leq i \leq m$.

Let $h(x) = y_1(x)$ and

$$g(x) = (x - z_1)(x - z_2) \cdots (x - z_m).$$

Then

$$g(h(x)) = (y_1(x) - z_1)(y_1(x) - z_2) \cdots (y_1(x) - z_m) = y_1(x) y_2(x) \cdots y_m(x) = \prod_{i=1}^{mn} (x - r_i) = f(x).$$

Now we prove that the condition on the roots of $f$ is necessary. Suppose that we are given polynomials $g$ and $h$ of respective degrees $m$ and $n$ for which $f(x) = g(h(x))$. Let

$$g(x) = (x - t_1)(x - t_2) \cdots (x - t_m).$$

For each positive integer not exceeding $m$, let

$$u_i(x) = h(x) - t_i = (x - r_{(i-1)n+1})(x - r_{(i-1)n+2}) \cdots (x - r_{in}),$$

say, where each linear factor is listed as often as the multiplicity of the corresponding root of $u_i$. Then

$$\begin{aligned} f(x) = g(h(x)) &= (h(x) - t_1)(h(x) - t_2) \cdots (h(x) - t_m) \\ &= (x - r_1)(x - r_2) \cdots (x - r_{mn}), \end{aligned}$$

so that the $r_j$ are the roots of $f$.

For each index $i$ with $1 \leq i \leq m$, $u_i(x) = h(x) - t_i$ so that all the coefficients of $u_i$ except the constant are independent of $i$. It follows that all the symmetric functions of the roots of the polynomials $u_i$ agree except the $n$th. Thus, we obtain the desired partition where $s_i$ consists of the roots of $u_i$. □

Let us deal with polynomials in general. Suppose that $f(x)$ is a polynomial of degree $mn$ and leading coefficient $a$, so that $f(x) = au(x)$ for some monic polynomial $u(x)$. Then we show that $f(x)$ is a composite of polynomials of degrees $m$ and $n$ if and only if $u(x)$ is so. Suppose that $f(x) = g(h(x))$ where $g(x)$ is of degree $m$ with leading coefficient $b$ and $h(x)$ is of degree $n$ with leading coefficient $c$. Then, by comparison of leading coefficients, we have that $a = bc^m$. It can be checked that $u(x) = v(w(x))$ where $v(x) = (bc^m)^{-1} g(cx)$ and $w(x) = c^{-1} h(x)$.

On the other hand, suppose that $u(x) = v(w(x))$ for some monic polynomials $v(x)$ and $w(x)$ of respective degrees $m$ and $n$. Then $f(x) = g(h(x))$ with $g(x) = au(x)$ and $h(x) = v(x)$.

We note that, even for monic polynomials, the decomposition of $f(x)$ as a composite $g(h(x))$ is not unique. For example, for arbitrary values of $a$ and $d$, the pairs $(g(x), h(x)) = (x^2 + d, x^2 + ax + 1)$ and $(g(x), h(x)) = (x^2 + 2x + d + 1, x^2 + ax)$ both yield

$$f(x) = x^4 + 2ax^3 + (a^2 + 2)x^2 + 2ax + 1 + d = (x^2 + ax + 1)^2 + d\ .$$

## Reference

James Rickards, *When is a polynomial a composition of other polynomials? American Mathematical Monthy* **118:4** (April, 2011), 358-363

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