Quadratic inverse example

Let \( f(x) = x^2 + 3x + 2 = (x - \frac{3}{2})^2 + \frac{1}{4} \). The equation \( x^2 + 3x + 2 = y \) has the solution \( x = \frac{1}{2}(-3 \pm \sqrt{1 + 4y}) \) so the (composition) inverse function \( g(x) = f^{-1}(x) \) is apparently given by

\[
g(x) = \frac{-3 \pm \sqrt{1 + 4x}}{2}.
\]

Let us check this and see what happens.

\[
f(g(x)) = (g(x))^2 + 3g(x) + 2 = \left[ \frac{5 \pm 3\sqrt{1 + 4x}}{2} \right] + x + \left[ \frac{-9 \pm 3\sqrt{1 + 4x}}{2} \right] + 2 = x.
\]

So far, so good. Now compose the two functions in the opposite order:

\[
g(f(x)) = \frac{-3 \pm \sqrt{1 + 4(x^2 + 3x + 2)}}{2}
\]

\[
= \frac{-3 \pm \sqrt{4x^2 + 12x + 9}}{2} = \frac{-3 \pm (2x + 3)}{2} = x \text{ or } -x - 3.
\]

We get \( x \) for the composite value, as expected, but where did the second answer come from?

Let us go back to the definition of inverse. In order for a function to have an inverse from its range to its domain, it must be one-one. This is not the case for the quadratic \( f(x) \), where each value exceeding \(-1/4 \) is assumed twice. In order to provide for an inverse, we restrict the domain of \( f(x) \) to a set upon which it is one-one.

Let

\[
f_1(x) = f(x) = x^2 + 3x + 2
\]

for \( x \geq -3/2 \). The inverse \( g_1(x) \) of \( f_1(x) \) is the function

\[
g_1(x) = \frac{-3 + \sqrt{1 + 4x}}{2}
\]

defined for \( x \geq -1/4 \). As \( x \) increases from \(-1/4 \), the value \( g_1(x) \) increases from \(-3/2 \). Note that the graphs of \( f_1(x) \) and \( g_1(x) \) are images of each other reflected in the line \( y = x \).

Now

\[
g_1(f_1(x)) = \frac{-3 + \sqrt{1 + 4(x^2 + 3x + 2)}}{2} = \frac{-3 + \sqrt{(2x + 3)^2}}{2}.
\]
Observe that, by definition of the radical, \( \sqrt{z} \) refers to the positive square root of \( z \). Since \( x \geq -\frac{3}{2} \), the positive square root of \((2x + 3)^2\) is equal to \(2x + 3\) and we find that

\[
g_1(f_1(x)) = \frac{-3 + (2x - 3)}{2} = x.
\]

Now define

\[
f_2(x) = f(x) = x^2 + 3x + 2\]

for \( x \leq -\frac{3}{2} \). The inverse of this function is

\[
g_2(x) = \frac{-3 - \sqrt{1 + 4x}}{2}
\]

defined for \( x \geq -\frac{1}{4} \). As \( x \) increases from \(-\frac{1}{4}\) the value of \( g_2(x) \) decreases from \(-\frac{3}{2}\). Again, observe that the graphs of \( f_2(x) \) and \( g_2(x) \) are reflections of each other in the line \( y = 3 \). We have that

\[
g_2(f_2(x)) = \frac{-3 - \sqrt{1 + 4(x^2 + 3x + 2)}}{2} = \frac{-3 - \sqrt{(2x + 3)^2}}{2}.
\]

This time, since \( x \leq -\frac{3}{2} \), we have that \( 2x + 3 \leq 0 \) and the positive square root of \((2x + 3)^2\) is \(-(2x + 3)\). Therefore

\[
g_2(f_2(x)) = \frac{-3 - (-(2x + 3))}{2} = \frac{-3 + 2x + 3}{2} = x.
\]