

PUTNAM PROBLEMS

SEQUENCES, SERIES AND RECURRENCES

Notes

1. $x_{n+1} = ax_n$ has the general solution $x_n = x_1 a^{n-1}$.
2. $x_{n+1} = x_n + b$ has the general solution $x_n = x_1 + (n-1)b$.
3. $x_{n+1} = ax_n + b$ (with $a \neq 1$) can be rewritten $x_{n+1} + k = a(x_n + k)$ where $(a-1)k = b$ and so reduces to the recurrence 1.
4. $x_{n+1} = ax_n + bx_{n-1}$ has different general solution depending on the discriminant of the characteristic polynomial $t^2 - at - b$.
 - (a) If $a^2 - 4b \neq 0$ and the distinct roots of the characteristic polynomial are r_1 and r_2 , then the general solution of the recurrence is

$$x_n = c_1 r_1^n + c_2 r_2^n$$

where the constants c_1 and c_2 are chosen so that

$$x_1 = c_1 r_1 + c_2 r_2 \quad \text{and} \quad x_2 = c_1 r_1^2 + c_2 r_2^2 .$$

(b) If $a^2 - 4b = 0$ and r is the double root of the characteristic polynomial, then

$$x_n = (c_1 n + c_0) r^n$$

where c_1 and c_0 are chosen so that

$$x_1 = (c_1 + c_0) r \quad \text{and} \quad x_2 = (2c_1 + c_0) r^2 .$$

5. $x_{n+1} = (1-s)x_n + sx_{n-1} + r$ can be rewritten $x_{n+1} - x_n = -s(x_n - x_{n-1}) + r$ and solved by a previous method for $x_{n+1} - x_n$.
6. $x_{n+1} = ax_n + bx_{n-1} + c$ where $a + b \neq 1$ can be rewritten $(x_{n+1} + k) = a(x_n + k) + b(x_{n-1} + k)$ where $(a + b - 1)k = c$ and solved for $x_n + k$.
7. The general homogeneous linear recursion has the form

$$x_{n+k} = a_{k-1} x_{n+k-1} + \cdots + a_1 x_{n+1} + a_0 .$$

Its characteristic polynomials is

$$t^k - a_{k-1} t^{k-1} - \cdots - a_1 t - a_0 .$$

Let r be a root of this polynomial of multiplicity m ; then the n th term of the recurrence is a linear combination of terms of the type

$$(c_{m-1} r^{m-1} + \cdots + c_1 r + c_0) r^n .$$

Putnam questions

2016-B-1. Let x_0, x_1, x_2, \dots be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function \ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

2016-B-6. Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}.$$

2015-A-2. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015} .

2015-B-5. Let P_n be the number of permutations π of $\{1, 2, \dots, n\}$ such that

$$|i - j| = 1 \quad \text{implies} \quad |\pi(i) - \pi(j)| \leq 2$$

for all i, j in $\{1, 2, \dots, n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

does not depend on n , and find its value.

2014-A-1. Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

2014-A-3. Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

2013-B-1. For positive integers n , let the number $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n + 1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

2010-B-1. Is there an infinite sequence of real numbers a_1, a_2, a_3, \dots such that $a_1^m + a_2^m + a_3^m + \dots = m$ for every positive integer m ?

2009-B-6. Prove that for every positive integer n , there is a sequence $a_0, a_1, \dots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonnegative integer k or of the form $b \bmod c$ for some earlier positive terms b and c . (Here $b \bmod c$ denotes the remainder when b is divided by c , so $0 \leq (b \bmod c) < c$.)

2007-B-3. Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n \sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007} . ($\lfloor a \rfloor$ means the largest integer $\leq a$.)

2006-A-3. Let $1, 2, 3, \dots, 2005, 2006, 2007, 2009, 2012, 2016, \dots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \dots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

2006-B-6. Let k be an integer greater than 1. Suppose $a_k > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n \geq 0$. Evaluate

$$\lim_{n \rightarrow \infty} \frac{a_n^{k+1}}{n^k} .$$

2004-A-3. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n . (By convention, $0! = 1$.)

2002-A-5. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots \right\} .$$

2001-B-3. For any positive integer n let $\langle n \rangle$ denote the closest integer to \sqrt{n} . Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} .$$

2001-B-6. Assume that $\{a_n\}_{n \geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that

$$a_{n-1} + a_{n+i} < 2a_n$$

for $i = 1, 2, \dots, n-1$?

2000-A-1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that x_0, x_1, x_2, \dots are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

2000-A-6. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence a_0, a_1, \dots of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$, then either $a_1 = 0$ or $a_2 = 0$.

1999-A-3. Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n .$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m .$$

1999-A-4. Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n3^m + m3^n)} .$$

1999-A-6. The sequence $\{a_n\}_{n \geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1} a_{n-2}^2}{a_{n-2} a_{n-3}}.$$

Show that, for all n , a_n is an integer multiple of n .

1999-B-3. Let $A = \{(x, y) : 0 \leq x, y \leq 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim\{(1 - xy^2)(1 - x^2y)S(x, y) : (x, y) \rightarrow (1, 1), (x, y) \in A\}.$$

1998-A-4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n .

1998-B-4. Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

1997-A-6. For a positive integer n and any real number c , define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k , $1 \leq k \leq n$.

1994-A-1. Suppose that a sequence a_1, a_2, a_3, \dots satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

1994-A-5. Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n \rightarrow \infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \dots + r_{i_{1994}}$$

with $i_1 < i_2 < \dots < i_{1994}$. Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S .

1993-A-2. Let $(x_n)_{n \geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1$$

for $n = 1, 2, 3, \dots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

1993-A-6. The infinite sequence of 2's and 3's

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \dots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n ,

the n th term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m . (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .)

1992-A-1. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions

- (i) $f(f(n)) = n$, for all integers n ;
- (ii) $f(f(n+2) + 2) = n$ for all integers n ;
- (iii) $f(0) = 1$.

1992-A-5. For each positive integer n , let

$$a_n = \begin{cases} 0, & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1, & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}$$

for $0 \leq j \leq m - 1$.

1991-B-1. For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

1990-A-1. Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3} \quad .$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

1988-B-4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is

$$\sum_{n=1}^{\infty} (a_n)^{n/(n+1)} \quad .$$

1985-A-4. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i ?

1979-A-3. Let x_1, x_2, x_3, \dots be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}} \quad \text{for } n = 3, 4, 5, \dots \quad .$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n .

1975-B-6. Show that, if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, then

- (a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and
- (b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.