

# PUTNAM PROBLEMS

## NUMBER THEORY

**2006-A-3.** Let  $1, 2, 3, \dots, 2005, 2006, 2007, 2009, 2012, 2016, \dots$  be a sequence defined by  $x_k = k$  for  $k = 1, 2, \dots, 2006$  and  $x_{k+1} = x_k + x_{k-2005}$  for  $k \geq 2006$ . Show that the sequence has 2005 consecutive terms each divisible by 2006.

**2005-A-1.** Show that every positive integer is a sum of one or more numbers of the form  $2^r 3^s$ , where  $r$  and  $s$  are nonnegative integers and no summand divides another. (For example,  $23 = 9 + 8 + 6$ .)

**2005-B-2.** Find all positive integers  $n, k_1, k_2, \dots, k_n$  such that  $k_1 + k_2 + \dots + k_n = 5n - 4$  and

$$\frac{1}{k_1} + \dots + \frac{1}{k_n} = 1 .$$

**2005-B-4.** For positive integers  $m$  and  $n$ , let  $f(m, n)$  denote the number of  $n$ -tuples  $(x_1, x_2, \dots, x_n)$  of integers such that  $|x_1| + |x_2| + \dots + |x_n| \leq m$ . Show that  $f(m, n) = f(n, m)$ .

**2004-A-1.** Basketball star Shanille O'Keal's team statistician keeps track of the number,  $S(N)$ , of successful free throws she has made in her first  $N$  attempts of the season. Early in the season,  $S(N)$  is less than 80% of  $N$ , but by the end of the season,  $S(N)$  was more than 80% of  $N$ . Was there necessarily a moment in between when  $S(N)$  was exactly 80% of  $N$ ?

**2004-A-3.** Define a sequence  $\{u_n\}_{n=0}^{\infty}$  by  $u_0 = u_1 = u_2 = 1$ , and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all  $n \geq 0$ . Show that  $u_n$  is an integer for all  $n$ . (By convention,  $0! = 1$ .)

**2004-B-2.** Let  $m$  and  $n$  be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n} .$$

**2004-B-6.** Let  $\mathfrak{A}$  be a non-empty set of positive integers, and let  $N(x)$  denote the number of elements of  $\mathfrak{A}$  not exceeding  $x$ . Let  $\mathfrak{B}$  denote the set of positive integers  $b$  that can be written in the form  $b = a - a'$  with  $a \in \mathfrak{A}$  and  $a' \in \mathfrak{A}$ . Let  $b_1 < b_2 < \dots$  be the members of  $\mathfrak{B}$ , listed in increasing order. Show that if the sequence  $b_{i+1} - b_i$  is unbounded, then  $\lim_{x \rightarrow \infty} N(x)/x = 0$ .

**2003-A-1.** Let  $n$  be a fixed positive integer. How many ways are there to write  $n$  as a sum of positive integers,

$$n = a_1 + a_2 + \dots + a_k ,$$

with  $k$  an arbitrary positive integer, and  $a_1 \leq a_2 \leq \dots \leq a_k \leq a_1 + 1$ ? For example, with  $n = 4$ , there are four ways:  $4, 2+2, 1+1+2, 1+1+1+1$ .

**2003-A-6.** For a set  $S$  of nonnegative integers, let  $r_S(n)$  denote the number of ordered pairs  $(s_1, s_2)$  such that  $s_1 \in S, s_2 \in S, s_1 \neq s_2$ , and  $s_1 + s_2 = n$ . Is it possible to partition the nonnegative integers into two sets  $A$  and  $B$  in such a way that  $r_A(n) = r_B(n)$  for all  $n$ ?

**2003-B-2.** Let  $n$  be a positive integer. Starting with the sequence  $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$ , for a new sequence of  $n-1$  entries  $\frac{3}{4}, \frac{5}{12}, \dots, \frac{2n-1}{2n(n-1)}$ , by taking the averages of two consecutive entries in the first sequence.

Repeat the averaging of neighbours on the second sequence to obtain a sequence of  $n-2$  entries and continue until the final sequence consists of a single number  $x_n$ . Show that  $x_n < 2/n$ .

**2003-B-3.** Show that for each positive integer  $n$ ,

$$n! = \prod_{i=1}^n \text{lcm}\{1, 2, \dots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and  $\lfloor x \rfloor$  denotes the greatest integer  $\leq x$ .)

**2003-B-4.** Let  $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z-r_1)(z-r_2)(z-r_3)(z-r_4)$  where  $a, b, c, d, e$  are integers,  $a \neq 0$ . Show that if  $r_1 + r_2$  is a rational number, and if  $r_1 + r_2 \neq r_3 + r_4$ , then  $r_1 r_2$  is a rational number.

**2002-A-3.** Let  $n \geq 2$  be an integer and  $T_n$  be the number of non-empty subsets  $S$  of  $\{1, 2, 3, \dots, n\}$  with the property that the average of the elements of  $S$  is an integer. Prove that  $T_n - n$  is always even.

**2002-A-5.** Define a sequence by  $a_0 = 1$ , together with the rules  $a_{2n+1} = a_n$  and  $a_{2n+2} = a_n + a_{n+1}$  for each integer  $n \geq 0$ . Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{2}, \dots \right\}.$$

**2002-A-6.** Fix an integer  $b \geq 2$ . Let  $f(1) = 1$ , and  $f(2) = 2$ , and for each  $n \geq 3$ , define  $f(n) = nf(d)$ , where  $d$  is the number of base- $b$  digits of  $n$ . For which values of  $b$  does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

**2002-B-5.** A palindrome in base  $b$  is a positive integer whose base- $b$  digits read the same forwards and backward; for example, 2002 is a 4-digits palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base  $b$  for at least 2002 different values of  $b$ .

**2001-A-5.** Prove that there are unique positive integers  $a, n$  such that

$$a^{n+1} - (a+1)^n = 2001.$$

**2001-B-1.** Let  $n$  be an even positive integer. Write the numbers  $1, 2, \dots, n^2$  in the squares of an  $n \times n$  grid so that the  $k$ th row, from right to left is

$$(k-1)n+1, (k-1)n+2, \dots, (k-1)n+n.$$

Colour the squares of the grid so that half the squares in each row and in each column are red and the other half are black (a checkerboard colouring is one possibility). Prove that for each such colouring, the sum of the numbers on the red squares is equal to the sum of the numbers in the black square.

**2001-B-3.** For any positive integer  $n$  let  $\langle n \rangle$  denote the closest integer to  $\sqrt{n}$ . Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

**2001-B-4.** Let  $S$  denote the set of rational numbers different from  $-1$ ,  $0$  and  $1$ . Define  $f : S \rightarrow S$  by  $f(x) = x - (1/x)$ . Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset ,$$

where  $f^{(n)} = f \circ f \circ \dots \circ f$  ( $n$  times). (Note:  $f(S)$  denotes the set of all values  $f(s)$  for  $s \in S$ .)

**2000-A-2.** Prove that there exist infinitely many integers  $n$  such that  $n$ ,  $n + 1$ , and  $n + 2$  are each the sum of two squares of integers. (Example:  $0 = 0^2 + 0^2$ ,  $1 = 0^2 + 1^2$ , and  $2 = 1^2 + 1^2$ .)

**2000-B-1.** Let  $a_j$ ,  $b_j$ , and  $c_j$  be integers for  $1 \leq j \leq N$ . Assume, for each  $j$ , that at least one of  $a_j$ ,  $b_j$ ,  $c_j$  is odd. Show that there exist integers  $r, s, t$  such that  $ra_j + sb_j + tc_j$  is odd for at least  $4N/7$  values of  $j$ ,  $1 \leq j \leq N$ .

**2000-B-2.** Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers  $n \geq m \geq 1$ . [Here  $\binom{n}{m} = \frac{n!}{m!(n-m)!}$  and  $\gcd(m, n)$  is the greatest common divisor of  $m$  and  $n$ .]

**2000-B-5.** Let  $S_0$  be a finite set of positive integers. We define sets  $S_1, S_2, \dots$  of positive integers as follows:

*Integer  $a$  is in  $S_{n+1}$  if and only if exactly one of  $a - 1$  or  $a$  is in  $S_n$ .*

Show that there exists infinitely many integers  $N$  for which

$$S_N = S_0 \cup \{N + a : a \in S_0\} .$$

**1999-A-6.** The sequence  $\{a_n\}_{n \geq 1}$  is defined by  $a_1 = 1$ ,  $a_2 = 2$ ,  $a_3 = 24$ , and for  $n \geq 4$ ,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1} a_{n-2}^2}{a_{n-2} a_{n-3}} .$$

Show that, for all  $n$ ,  $a_n$  is an integer multiple of  $n$ .

**1999-B-6.** Let  $S$  be a finite set of integers, each greater than 1. Suppose that for each integer  $n$  there is some  $s \in S$  such that  $\gcd(s, n) = 1$  or  $\gcd(s, n) = s$ . Show that there exists  $s, t \in S$  such that  $\gcd(s, t)$  is prime. [Here  $\gcd(a, b)$  denotes the greatest common divisor of  $a$  and  $b$ .]

**1998-A-4.** Let  $A_1 = 0$  and  $A_2 = 1$ . For  $n > 2$ , the number  $A_n$  is defined by concatenating the decimal expansions of  $A_{n-1}$  and  $A_{n-2}$  from left to right. For example,  $A_3 = A_2 A_1 = 10$ ,  $A_4 = A_3 A_2 = 101$ ,  $A_5 = A_4 A_3 = 10110$ , and so forth. Determine all  $n$  such that 11 divides  $A_n$ .

**1998-B-5.** Let  $N$  be the positive integer with 1998 decimal digits, all of them 1; that is,  $N = 1111 \dots 11$  (1998 digits). Find the thousandth digit after the decimal point of  $\sqrt{N}$ .

**1998-B-6.** Prove that, for any integers  $a, b, c$ , there exists a positive integer  $n$  such that  $\sqrt{n^3 + an^2 + bn + c}$  is not an integer. ■

**1997-A-5.** Let  $N_n$  denote the number of ordered  $n$ -tuples of positive integers  $(a_1, a_2, \dots, a_n)$  such that  $1/a_1 + 1/a_2 + \dots + 1/a_n = 1$ . Determine whether  $N_{10}$  is even or odd.

**1997-B-3.** For each positive integer  $n$  write the sum  $\sum_{m=1}^n \frac{1}{m}$  in the form  $\frac{p_n}{q_n}$  where  $p_n$  and  $q_n$  are relatively prime positive integers. Determine all  $n$  such that 5 does not divide  $q_n$ .

**1997-B-5.** Prove that for  $n \geq 2$ ,

$$2^{\overbrace{\dots}^2} \} n \equiv 2^{\overbrace{\dots}^2} \} n - 1 \pmod{n}.$$

**1996-A-5.** If  $p$  is a prime number greater than 3, and  $k = \lfloor 2p/3 \rfloor$ , prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \dots + \binom{p}{k}$$

of binomial coefficients is divisible by  $p^2$ .

(For example,  $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 7 + 21 + 35 + 35 = 2 \cdot 7^2$ .)

**1995-A-3.** The number  $d_1 d_2 \dots d_9$  has nine (not necessarily distinct) decimal digits. The number  $e_1 e_2 \dots e_9$  is such that each of the nine 9-digit numbers formed by replacing just one of the digits  $d_i$  in  $d_1 d_2 \dots d_9$  by the corresponding digit  $e_i$  ( $1 \leq i \leq 9$ ) is divisible by 7. The number  $f_1 f_2 \dots f_9$  is related to  $e_1 e_2 \dots e_9$  in the same way; that is, each of the nine numbers formed by replacing one of the  $e_i$  by the corresponding  $f_i$  is divisible by 7. Show that, for each  $i$ ,  $d_i - f_i$  is divisible by 7. [For example, if  $d_1 d_2 \dots d_9 = 199501996$ , then  $e_6$  may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

**1995-A-4.** Suppose we have a necklace of  $n$  beads. Each bead is labelled with an integer and the sum of all these labels is  $n - 1$ . Prove that we can cut the necklace to form a string whose consecutive labels  $x_1, x_2, \dots, x_n$  satisfy

$$\sum_{i=1}^k x_i \leq k - 1 \quad \text{for } k = 1, 2, \dots, n.$$

**1994-B-1.** Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is, a member of the set  $\{0, 1, 4, 9, 16, \dots\}$ .  $a$  is **within**  $n$  of  $b$  if  $b - n \leq a \leq b + n$ .)

**1994-B-6.** For any integer  $a$ , set  $n_a = 101a - 100 \cdot 2^a$ . Show that for  $0 \leq a, b, c, d \leq 99$ ,

$$n_2 + n_b \equiv n_c + n_d \pmod{10100}$$

implies  $\{a, b\} = \{c, d\}$ .

**1993-A-4.** Let  $x_1, x_2, \dots, x_{19}$  be positive integers each of which is less than or equal to 93. Let  $y_1, y_2, \dots, y_{93}$  be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some  $x_i$ 's equal to a sum of some  $y_j$ 's.

**1993-B-1.** Find the smallest positive integer  $n$  such that for every integer  $m$ , with  $0 < m < 1993$ , there exists an integer  $k$  for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

**1993-B-5.** Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

**1993-B-6.** Let  $S$  be a set of three, not necessarily distinct, positive integers. Show that one can transform  $S$  into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say  $x$  and  $y$ , where  $x \leq y$ , and replace them with  $2x$  and  $y - x$ .

**1992-A-3.** For a given positive integer  $m$ , find all triples  $(n, x, y)$  of positive integers, with  $n$  relatively prime to  $m$ , which satisfy  $(x^2 + y^2)^m = (xy)^n$ .

**1992-A-5.** For each positive integer  $n$ , let

$$a_n = \begin{cases} 0 & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1 & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist positive integers  $k$  and  $m$  such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}, \quad \text{for } 0 \leq j \leq m-1.$$

**1989-A-1.** How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

**1988-B-1.** A *composite* (positive integer) is a product  $ab$  with  $a$  and  $b$  not necessarily distinct integers in  $\{2, 3, 4, \dots\}$ . Show that every composite is expressible as  $xy + xz + yz + 1$ , with  $x$ ,  $y$ , and  $z$  positive integers.

**1988-B-6.** Prove that there exist an infinite number of ordered pairs  $(a, b)$  of integers such that for every positive integer  $t$  the number  $at + b$  is a triangular number if and only if  $t$  is a triangular number. (The triangular numbers are the  $t_n = n(n+1)/2$  with  $n$  in  $\{0, 1, 2, \dots\}$ ).

**1987-A-2.** The sequence of digits

1 2 3 4 5 6 7 8 9 1 0 1 1 1 2 1 3 1 4 1 5 1 6 1 7 1 8 1 9 2 0 2 1  $\dots$

is obtained by writing the positive integers in order. If the  $10^n$ -th digit in this sequence occurs in the part of the sequence in which the  $m$ -digit numbers are placed, define  $f(n)$  to be  $m$ . For example  $f(2) = 2$  because the 100th digit enters the sequence in the placement of the two digit integer 55. Find, with proof,  $f(1987)$ .