PUTNAM PROBLEMS

NUMBER THEORY

2016-A-1. Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \frac{d^j}{dx^j} p(x)|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

2016-B-2. Define a positive integer n to be *squarish* if either n is itself a perfect square or the distance from n to the nearest perfect square is a perfect square. For example, 2016 is squarish, because the nearest perfect square to 2016 is $45^2 = 2025$ and $2025 - 2016 = 9$ is a perfect square. (Of the positive integers between 1 and 10, only 6 and 7 are not squarish.)

For a positive integer N, let $S(N)$ be the number of squarish integers between 1 and N, inclusive. Find positive constants α and β such that

$$\lim_{N\to\infty} \frac{S(N)}{N^\alpha} = \beta,$$

or show that no such constants exist.

2015-A-2. Let $a_0 = 1$, $a_1 = 2$, and

$$a_n = 4a_{n-1} - a_{n-2}$$

for $n \geq 2$. Find an odd prime factor of a_{2015}.

2015-A-5. Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

2015-B-2. Given a list of the positive integers $1, 2, 3, 4, \ldots$, take the first three numbers 1, 2, 3 and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers 4, 5, 7 and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and consider the sequence of sums produced $6, 16, 27, 36, \ldots$. Prove or disprove that there is some number in this sequence whose base 10 representation ends with 2015.

2015-B-4. Let T be the set of all triples (a, b, c) of positive integers for which there exist triangles with side lengths a, b, c. Express

$$\sum_{(a,b,c)\in T} \frac{2^a}{3^b 5^c}$$

as a rational number in lowest terms.

2015-B-6. For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $[1, \sqrt{2k})$. Evaluate

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{A(k)}{k}.$$

2014-B-1. A *base* 10 *over-expansion* of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \ldots, 10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ amd the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

2014-B-3. Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m + n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

2013-A-2. Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S onto the integers is one-one.

2012-A-4. Let q and r be integers with $q > 0$, and let A and B be intervals on the real line. Let T be the set of all $b + mq$ where b and m are integers with b in B, and let S be the set of all integers a in A such that ra is in T. Show that if the product of the lengths of A and B is less than q, then S is the intersection of A with some arithmetic progression.

2012-A-5. Let $\mathbf{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a field vector in $\mathbf{F}_p^n$ and let M be an $n \times n$ matrix with entries in $\mathbf{F}_p$, and define $G : \mathbf{F}_p^n \to \mathbf{F}_p^n$ by $G(x) = v + Mx$. Let $G^{(k)}$ denote the $k-$fold composition of G with itself, that is $G^{(1)}(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p, n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1, 2, \cdots, p^n$ are distinct.

2012-B-6. Let p be an odd prime such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

2011-A-4. For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

2011-B-1. Let h and k be positive integers. Prove that for every $\epsilon > 0$, there are positive integers m and n such that

$$\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon.$$

2011-B-2. Let S be the set of ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

2011-B-6. Let p be an odd prime. Show that for at least $(p + 1)/2$ values of n in $\{0, 1, 2, \cdots, p - 1\}$,

$$\sum_{k=0}^{p-1} k! n^k \quad \text{is not divisible by} \quad p.$$

2010-A-1. Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same. [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.

2010-A-4. Prove that for each positive integer n, the number

$$10^{10^{10^n}} + 10^{10^n} + 10^n - 1$$

is not prime.

2009-A-4. Let S be a set of rational numbers such that

(a)) $\in S$;

(b) If $x \in S$, then $x+1 \in S$ and $x-1 \in S$; and

(c) If $x \in S$ and $x \notin (0,1)$, then $1/(x(x-1) \in S$.

Must S contain all rational numbers?

2009-B-1. Show that every positive rational number can be written as a quotient of products of fctorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!} \ .$$

2009-B-3. Call a subset S of $\{1,2,\ldots,n\}$ *mediocre* if it has the following property: Whenever a and b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1,2,\ldots,n\}$. [For instance, every subset of $\{1,2,3\}$ except $\{1,3\}$ is mediocre, so $A(3)=7$.] Find all positive integers n such that $A(n+2)-2A(n+1)+A(n)=1$.

2009-B-6. Prove that for every positive integer n, there is a sequence of integers $a_0, a_1, \ldots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonnegative integer k, or of the form b mod c for some earlier terms b and c. [Here $b$mod$c$ denotes the remainder when b is divided by c, so $0 \le (b\text{mod}c) < c$.

2008-A-3. Start with a finite sequence $a_1, a_2, \ldots, a_n$ of integers . If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_i, a_j)$ and $\text{lcm}(a_j, a_k)$ respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

2008-B-4. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0)$, $h(1)$, ... , $h(p^2-1)$ are distinct modulo p^2. Show that $h(0)$, $h(1)$, ... , $h(p^3-1)$ are distinct modulo p^3.

2007-A-4. A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.

2007-B-1. Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n=1$.

2006-A-3. Let $1, 2, 3, \cdots, 2005, 2006, 2007, 2009, 2012, 2016, \cdots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \cdots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \ge 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

2005-A-1. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

2005-B-2. Find all positive integers $n, k_1, k_2, \cdots, k_n$ such that $k_1 + k_2 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1 \ .$$

2005-B-4. For positive integers m and n, let $f(m,n)$ denote the number of $n-$tuples $(x_1, x_2, \cdots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \le m$. Show that $f(m,n) = f(n,m)$.

2004-A-1. Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ as less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

2004-A-3. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

2004-B-2. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n} \ .$$

2004-B-6. Let $\mathfrak{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathfrak{A}$ not exceeding x. Let $\mathfrak{B}$ denote the set of positive integers b that can be written in the form $b = a - a'$ with $a \in \mathfrak{A}$ and $a' \in \mathfrak{A}$. Let $b_1 < b_2 < \cdots$ be the members of $\mathfrak{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then $\lim_{x\to\infty} N(x)/x = 0$.

2003-A-1. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k \ ,$$

with k an arbitrary positive integer, and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$, there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

2003-A-6. For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

2003-B-2. Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}$, for a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \cdots, \frac{2n-1}{2n(n-1)}$, by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbours on the second sequence to obtain a sequence of $n-2$ entries and continue until the final sequence consists of a single number x_n. Show that $x_n < 2/n$.

2003-B-3. Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \text{lcm}\{1, 2, \cdots, \lfloor n/i \rfloor\} \ .$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

2003-B-4. Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

2002-A-3. Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \cdots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

2002-A-5. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \cdots \right\} .$$

2002-A-6. Fix an integer $b \geq 2$. Let $f(1) = 1$, and $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base$-b$ digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

2002-B-5. A palindrome in base b is a positive integer whose base$-b$ digits read the same forwards and backward; for example, 2002 is a 4-digits palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base b for at least 2002 different values of b.

2001-A-5. Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001 \ .$$

2001-B-1. Let n be an even positive integer. Write the numbers $1, 2, \cdots, n^2$ in the squares of an $n \times n$ grid so that the kth row, from right to left is

$$(k-1)n + 1, (k-1)n + 2, \cdots, (k-1)n + n \ .$$

Colour the squares of the grid so that half the squares in each row and in each column are red and the other half are black (a checkerboard colouring is one possibility). Prove that for each such colouring, the sum of the numbers on the red squares is equal to the sum of the numbers in the black square.

2001-B-3. For any positive integer n let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} \ .$$

2001-B-4. Let S denote the set of rational numbers different from -1, 0 and 1. Define $f : S \to S$ by $f(x) = x - (1/x)$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset \ ,$$

where $f^{(n)} = f \circ f \circ \cdots \circ f$ (n times). (Note: $f(S)$ denotes the set of all values $f(s)$ for $s \in S$,)

2000-A-2. Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of two squares of integers. (Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.)

2000-B-1. Let a_j, b_j, and c_j be integers for $1 \leq j \leq N$. Assume, for each j, that at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

2000-B-2. Prove that the expression

$$\frac{\text{gcd }(m,n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$. [Here $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ and gcd (m, n) is the greatest common divisor of m and n.]

2000-B-5. Let S_0 be a finite set of positive integers. We define sets $S_1, S_2, \cdots$ of positive integers as follows:

Integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n.

Show that there exists infinitely many integers N for which

$$S_N = S_0 \cup \{N + a : a \in S_0\} \ .$$

1999-A-6. The sequence $\{a_n\}_{n\geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}} \ .$$

Show that, for all n, a_n is an integer multiple of n.

1999-B-6. Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that gcd $(s, n) = 1$ or gcd $(s, n) = s$. Show that there exists $s, t \in S$ such that gcd (s, t) is prime. [Here gcd (a, b) denotes the greatest common divisor of a and b.]

1998-A-4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

1998-B-5. Let N be the positive integer with 1998 decimal digits, all of them 1; that is, $N = 1111\cdots 11$ (1998 digits). Find the thousandth digit after the decomal point of $\sqrt{N}$.

1998-B-6. Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

1997-A-5. Let N_n denote the number of ordered $n-$tuples of positive integers $(a_1, a_2, \cdots, a_n)$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

1997-B-3. For each positive integer n write the sum $\sum_{m=1}^{n} \frac{1}{m}$ in the form $\frac{p_n}{q_n}$ where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

1997-B-5. Prove that for $n \geq 2$,

$$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n \equiv \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n-1 \pmod{n} \ .$$

1996-A-5. If p is a prime number greater than 3, and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.
(For example, $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 7 + 21 + 35 + 35 = 2 \cdot 7^2$.)

1995-A-3. The number $d_1d_2\cdots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2\cdots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i in $d_1d_2\cdots d_9$ by the corresponding digit e_i $(1 \leq i \leq 9)$ is divisible by 7. The number $f_1f_2\cdots f_9$ is related to $e_1e_2\cdots e_9$ in the same way; that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1d_2\cdots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

1995-A-4. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels x_1, x_2, $\cdots$, x_n satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \qquad \text{for} \quad k = 1, 2, \cdots n\ .$$

1994-B-1. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is, a member of the set $\{0, 1, 4, 9, 16, \cdots, \}$. a is **within** n of b if $b-n \leq a \leq b+n$.)

1994-B-6. For any integer a, set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \leq a, b, c, d \leq 99$,

$$n_2 + n_b \equiv n_c + n_d \pmod{10100}$$

implies $\{a, b\} = \{c, d\}$.

1993-A-4. Let $x_1, x_2, \cdots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \cdots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

1993-B-1. Find the smallest positive integer n such that for every integer m, with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}\ .$$

1993-B-5. Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

1993-B-6. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$, and replace them with $2x$ and $y-x$.

1992-A-3. For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2+y^2)^m = (xy)^n$.

1992-A-5. For each positive integer n, let

$$a_n = \begin{cases} 0 & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1 & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}, \qquad \text{for} \quad 0 \leq j \leq m-1\ .$$

1989-A-1. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

1988-B-1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \cdots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers.

1988-B-6. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at + b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \cdots\}$.

1987-A-2. The sequence of digits

$$1\,2\,3\,4\,5\,6\,7\,8\,9\,1\,0\,1\,1\,1\,2\,1\,3\,1\,4\,1\,5\,1\,6\,1\,7\,1\,8\,1\,9\,2\,0\,2\,1\cdots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in hich the $m-$digit numbers are placed, define $f(n)$ to be m. For example $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two digit integer 55. Find, with proof, $f(1987)$.