

## PUTNAM PROBLEMS

### MATRICES, DETERMINANTS AND LINEAR ALGEBRA

**2009-A-3.** Let  $d_n$  be the determinant of the  $n \times n$  matrix whose entries, from left to right and then from top to bottom, are  $\cos 1, \cos 2, \dots, \cos n^2$ . (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of  $\cos$  is always in radians, not degrees.) Evaluate  $\lim_{n \rightarrow \infty} d_n$ .

**2009-B-4.** Say that a polynomial with real coefficients in two variables,  $x, y$ , is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space  $V$  over  $\mathbf{R}$ . Find the dimension of  $V$ .

**2008-A-2.** Alan and Barbara play a game in which they take turns filling entries of an initially empty  $2008 \times 2008$  array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if its is zero. Which player has a winning strategy?

**2006-B-4.** Let  $Z$  denote the set of points in  $\mathbf{R}^n$  whose coordinates are 0 or 1. (Thus  $Z$  has  $2^n$  elements, which are vertices of a hypercube in  $\mathbf{R}^n$ .) Given a vector subspace  $V$  of  $\mathbf{R}^n$ , let  $Z(V)$  denote the number of members of  $Z$  that lie in  $V$ . Let  $k$  be given,  $0 \leq k \leq n$ . Find the maximum, over all vector subspaces  $V \subseteq \mathbf{R}^n$  of dimension  $k$ , of the number of points in  $V \cap Z$ .

**2005-A-4.** Let  $H$  be an  $n \times n$  matrix all of whose entries are  $\pm 1$  and whose rows are mutually orthogonal. Suppose  $H$  has an  $a \times b$  submatrix whose entries are all 1. Show that  $ab \leq n$ .

**2004-A-3.** Define a sequence  $\{u_n\}_{n=0}^{\infty}$  by  $u_0 = u_1 = u_2 = 1$ , and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all  $n \geq 0$ . Show that  $u_n$  is an integer for all  $n$ . (By convention,  $0! = 1$ .)

**2003-B-1.** Do there exist polynomials  $a(x), b(x), c(y), d(y)$  such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

**2002-A-4.** In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty  $3 \times 3$  matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the  $3 \times 3$  matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

**1999-B-5.** For an integer  $n \geq 3$ , let  $\theta = 2\pi/n$ . Evaluate the determinant of the  $n \times n$  matrix  $I + A$ , where  $I$  is the  $n \times n$  identity matrix and  $A = (a_{jk})$  has entries  $a_{jk} = \cos(j\theta + k\theta)$  for all  $j, k$ .

**1996-B-4.** For any square matrix  $A$ , we can define  $\sin A$  by the usual power series

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: There exists a  $2 \times 2$  matrix  $A$  with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix} .$$

**1995-B-3.** To each positive integer with  $n^2$  decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for  $n = 2$ , to the integer 8617 we associate  $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$ . Find as a function of  $n$ , the sum of all the determinants associated with  $n^2$ -digit integers. (Leading digits are assumed to be nonzero; for example, for  $n = 2$ , there are 9000 determinants.)

**1994-A-4.** Let  $A$  and  $B$  be  $2 \times 2$  matrices with integer entries such that  $A$ ,  $A + B$ ,  $A + 2B$ ,  $A + 3B$ , and  $A + 4B$  are all invertible matrices whose inverses have integer entries. Show that  $A + 5B$  is invertible and that its inverse has integer entries.

**1994-B-4.** For  $n \geq 1$ , let  $d_n$  be the greatest common divisor of the entries of  $A^n - I$ , where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Show that  $\lim_{n \rightarrow \infty} d_n = \infty$ .

**1992-B-5.** Let  $D_n$  denote the value of the  $(n - 1) \times (n - 1)$  determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & 1 & 1 & \cdots & n + 1 \end{vmatrix}$$

Is the set

$$\left\{ \frac{D_n}{n!} : n \geq 2 \right\}$$

bounded?

**1992-B-6.** Let  $M$  be a set of real  $n \times n$  matrices such that

- (i)  $I \in M$ , where  $I$  is the  $n \times n$  identity matrix;
- (ii) if  $A \in M$  and  $B \in M$ , then either  $AB \in M$  and  $-AB \in M$ , but not both;
- (iii) if  $A \in M$  and  $B \in M$ , then either  $AB = BA$  or  $AB = -BA$ ;
- (iv) if  $A \in M$  and  $A \neq I$ , then there is at least one  $B \in M$  such that  $AB = -BA$ .

Prove that  $M$  contains at most  $n^2$  matrices.

**1991-A-2.** Let  $A$  and  $B$  be different  $n \times n$  matrices with real entries. If  $A^3 = B^3$  and  $A^2B = B^2A$ , can  $A^2 + B^2$  be invertible?

**1990-A-5.** If  $A$  and  $B$  are square matrices of the same size such that  $ABAB = 0$ , does it follow that  $BABA = 0$ ?

**1986-A-4.** A *transversal* of an  $n \times n$  matrix  $A$  consists of  $n$  entries of  $A$ , no two in the same row or column. Let  $f(n)$  be the number of  $n \times n$  matrices  $A$  satisfying the following two conditions:

- (a) Each entry  $\alpha_{i,j}$  of  $A$  is in the set  $\{-1, 0, 1\}$ .
- (b) The sum of the  $n$  entries of a transversal is the same for all transversals of  $A$ .

An example of such a matrix  $A$  is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

Determine with proof a formula for  $f(n)$  of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4 .$$

where the  $a_i$ 's and  $b_i$ 's are rational numbers.

**1986-B-6.** Suppose that  $A, B, C, D$  are  $n \times n$  matrices with entries in a field  $F$ , satisfying the conditions that  $AB^t$  and  $CD^t$  are symmetric and  $AD^t - BC^t = I$ . Here  $I$  is the  $n \times n$  identity matrix,  $M^t$  is the transpose of  $M$ . Prove that  $A^t D - C^t B = I$ .

**1985-B-6.** Let  $G$  be a finite set of real  $n \times n$  matrices  $\{M_i\}$ ,  $1 \leq i \leq r$ , which form a group under matrix multiplication. Suppose that  $\sum_{i=1}^r \text{tr}(M_i) = 0$ , where  $\text{tr}(A)$  denotes the trace of the matrix  $A$ . Prove that  $\sum_{i=1}^r M_i$  is the  $n \times n$  zero matrix.