

PUTNAM PROBLEMS

MATRICES, DETERMINANTS AND LINEAR ALGEBRA

2006-B-4. Let Z denote the set of points in \mathbf{R}^n whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are vertices of a hypercube in \mathbf{R}^n .) Given a vector subspace V of \mathbf{R}^n , let $Z(V)$ denote the number of members of Z that lie in V . Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbf{R}^n$ of dimension k , of the number of points in $V \cap Z$.

2005-A-4. Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

2004-A-3. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n . (By convention, $0! = 1$.)

2002-A-4. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

1999-B-5. For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k .

1996-B-4. For any square matrix A , we can define $\sin A$ by the usual power series

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1} .$$

Prove or disprove: There exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix} .$$

1995-B-3. To each positive integer with n^2 decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find as a function of n , the sum of all the determinants associated with n^2 -digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

1994-A-4. Let A and B be 2×2 matrices with integer entries such that A , $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

1994-B-4. For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Show that $\lim_{n \rightarrow \infty} d_n = \infty$.

1992-B-5. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}$$

Is the set

$$\left\{ \frac{D_n}{n!} : n \geq 2 \right\}$$

bounded?

1992-B-6. Let M be a set of real $n \times n$ matrices such that

- (i) $I \in M$, where I is the $n \times n$ identity matrix;
- (ii) if $A \in M$ and $B \in M$, then either $AB \in M$ and $-AB \in M$, but not both;
- (iii) if $A \in M$ and $B \in M$, then either $AB = BA$ or $AB = -BA$;
- (iv) if $A \in M$ and $A \neq I$, then there is at least one $B \in M$ such that $AB = -BA$.

Prove that M contains at most n^2 matrices.

1991-A-2. Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

1990-A-5. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

1986-A-4. A *transversal* of an $n \times n$ matrix A consists of n entries of A , no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

- (a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.
- (b) The sum of the n entries of a transversal is the same for all transversals of A .

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4 .$$

where the a_i 's and b_i 's are rational numbers.

1986-B-6. Suppose that A, B, C, D are $n \times n$ matrices with entries in a field F , satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n \times n$ identity matrix, M^t is the transpose of M . Prove that $A^t D - C^t B = I$.

1985-B-6. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \text{tr}(M_i) = 0$, where $\text{tr}(A)$ denotes the trace of the matrix A . Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.