PUTNAM PROBLEMS

INEQUALITIES

2016-A-2. Given a positive integer \(n\), let \(M(n)\) be the largest integer \(m\) such that
\[
\binom{m}{n-1} > \binom{m-1}{n}.
\]
Evaluate
\[
\lim_{n \to \infty} \frac{M(n)}{n}.
\]

2007-B-2. Suppose that \(f : [0, 1] \to \mathbb{R}\) has a continuous derivative and that \(\int_0^1 f(x)dx = 0\). Prove that for every \(\alpha \in (0, 1)\),
\[
\left| \int_0^\alpha f(x)dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.
\]

2004-A-6. Suppose that \(f(x,y)\) is a continuous real-valued function on the unit square \(0 \leq x \leq 1, 0 \leq y \leq 1\). Show that
\[
\int_0^1 \left( \int_0^1 f(x,y)dx \right)^2 dy + \int_0^1 \left( \int_0^1 f(x,y)dy \right)^2 dx \leq \left( \int_0^1 \int_0^1 f(x,y)dxdy \right)^2 + \int_0^1 \int_0^1 |f(x,y)|^2 dxdy.
\]

2003-A-2. Let \(a_1, a_2, \cdots, a_n\) and \(b_1, b_2, \cdots, b_n\) be nonnegative real numbers. Show that
\[
(a_1a_2\cdots a_n)^{1/n} + (b_1b_2\cdots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2)\cdots(a_n + b_n))^{1/n}.
\]

2003-A-3. Find the minimum value of
\[
|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|
\]
for real numbers \(x\).

2003-A-4. Suppose that \(a, b, c, A, B, C\) are real numbers, \(a \neq 0\) and \(A \neq 0\), such that
\[
|ax^2 + bx + c| \leq |Ax^2 + Bx + c|
\]
for all real numbers \(x\). Show that
\[
|b^2 - 4ac| \leq |B^2 - 4AC|.
\]

2003-B-2. Let \(n\) be a positive integer. Starting with the sequence \(1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}\), form a new sequence of \(n - 1\) entries \(3/4, 5/12, \cdots, (2n - 1)/2n(n - 1)\), by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbours of the second sequence to obtain a third sequence of \(n - 2\) entries and continue until the final sequence consists of a single number \(x_n\). Show that \(x_n < 2/n\).

2003-B-6. Let \(f(x)\) be a continuous real-valued function defined on the interval \([0, 1]\). Show that
\[
\int_0^1 \int_0^1 |f(x) + f(y)|dxdy \geq \int_0^1 |f(x)|dx.
\]
2002-B-3. Show that, for all integers \( n > 1 \),
\[
\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.
\]

1999-B-4. Let \( f \) be a real function with a continuous third derivative such that \( f(x), f'(x), f''(x), f'''(x) \) are positive for all \( x \). Suppose that \( f'''(x) \leq f(x) \) for all \( x \). Show that \( f'(x) < 2f(x) \) for all \( x \).

1998-B-1. Find the minimum value of
\[
\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}
\]
for \( x > 0 \).

1998-B-2. Given a point \((a, b)\) with \( 0 < b < a \), determine the minimum perimeter of a triangle with one vertex at \((a, b)\), one on the \( x-\)axis, and one on the line \( y = x \). You may assume that a triangle of minimum perimeter exists.

1996-B-2. Show that for every positive integer \( n \),
\[
\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.
\]

1996-B-3. Given that \( \{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\} \), find, with proof, the largest possible value, as a function of \( n \) (with \( n \geq 2 \)), of
\[
x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.
\]

1988-B-2. Prove or disprove: If \( x \) and \( y \) are real numbers with \( y \geq 0 \) and \( y(y+1) \leq (x+1)^2 \), then \( y(y-1) \leq x^2 \).