

## PUTNAM PROBLEMS

### INEQUALITIES

**2004-A-6.** Suppose that  $f(x, y)$  is a continuous real-valued function on the unit square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ . Show that

$$\int_0^1 \left( \int_0^1 f(x, y) dx \right)^2 dy + \int_0^1 \left( \int_0^1 f(x, y) dy \right)^2 dx \leq \left( \int_0^1 \int_0^1 f(x, y) dx dy \right)^2 + \int_0^1 \int_0^1 [f(x, y)]^2 dx dy .$$

**2003-A-2.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n} .$$

**2003-A-3.** Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers  $x$ .

**2003-A-4.** Suppose that  $a, b, c, A, B, C$  are real numbers,  $a \neq 0$  and  $A \neq 0$ , such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + c|$$

for all real numbers  $x$ . Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC| .$$

**2003-B-2.** Let  $n$  be a positive integer. Starting with the sequence  $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$ , form a new sequence of  $n - 1$  entries  $3/4, 5/12, \dots, (2n - 1)/2n(n - 1)$ , by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbours of the second sequence to obtain a third sequence of  $n - 2$  entries and continue until the final sequence consists of a single number  $x_n$ . Show that  $x_n < 2/n$ .

**2003-B-6.** Let  $f(x)$  be a continuous real-valued function defined on the interval  $[0, 1]$ . Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy \geq \int_0^1 |f(x)| dx .$$

**2002-B-3.** Show that, for all integers  $n > 1$ ,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne} .$$

**1999-B-4.** Let  $f$  be a real function with a continuous third derivative such that  $f(x), f'(x), f''(x), f'''(x)$  are positive for all  $x$ . Suppose that  $f'''(x) \leq f(x)$  for all  $x$ . Show that  $f'(x) < 2f(x)$  for all  $x$ .

**1998-B-1.** Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for  $x > 0$ .

**1998-B-2.** Given a point  $(a, b)$  with  $0 < b < a$ , determine the minimum perimeter of a triangle with one vertex at  $(a, b)$ , one on the  $x$ -axis, and one on the line  $y = x$ . You may assume that a triangle of minimum perimeter exists.

**1996-B-2.** Show that for every positive integer  $n$ ,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}} .$$

**1996-B-3.** Given that  $\{x_1, x_2, \dots, x_n\} = \{1, 2, \dots, n\}$ , find, with proof, the largest possible value, as a function of  $n$  (with  $n \geq 2$ ), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 .$$

**1988-B-2.** Prove or disprove: If  $x$  and  $y$  are real numbers with  $y \geq 0$  and  $y(y+1) \leq (x+1)^2$ , then  $y(y-1) \leq x^2$ .