PUTNAM PROBLEMS

GEOMETRY

2016-A-4. Consider a $(2m-1)\times(2n-1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. This region is to be tiled using tiles of the two types:

A 3-omino, with vertices $(0,0)$, $(2,0)$, $(2,1)$, $(1,1)$, $(1,2)$, $(0,2)$,

and a 4-onimo, with vertices $(0,0)$, $(2,0)$, $(2,1)$, $(1,1)$, $(1,2)$, $(-1,2)$, $(-1,1)$, $(0,1)$

made up of three and four unit squares respectively. The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?

2016-B-3. Supppose that S is a finite set of points in the plane such that the area of triangle ΔABC is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.

2015-A-1. Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

2015-B-4. Let T be the set of all triples (a, b, c) of positive integers for which there exist triangles with side lengths a, b, c. Express

$$\sum_{(a,b,c)\in T} \frac{2^a}{3^b 5^c}$$

as a rational number in lowest terms.

2013-A-5. For $m \geq 3$, a list of $\binom{m}{3}$ numbers a_{ijk} $(1 \leq i < j < k \leq m)$ is said to be *area definite* for $\mathbf{R}^n$ if the inequality

$$\sum \{a_{ijk} \cdot \text{Area } (\Delta A_i A_j A_k) : 1 \leq i < j < k \leq m\} \geq 0$$

holds for every choice of m points $A_1, \cdots, A_m$ in $\mathbf{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbf{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbf{R}^2$, then it is area definite for $\mathbf{R}^3$.

2012-A-1. Let $d_1, d_2, \cdots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

2012-B-2. Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: if a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

2011-A-1. [Diagram provided with problem.] Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0,0)$, $P_1, P_2, \cdots, P_n$ such that $n \geq 2$ and

• The directed line segments P_0P_1, P_1P_2, $\cdots$, $P_{n-1}P_n$ are in successive directions east (for P_0P_1), north, west, south, east, *etc.*.
• The lengths of these line segments are positive and strictly increasing.

How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011$, $0 \leq y \leq 2011$ cannot be the last point P_n, of any growing spiral?

2010-B-2. Given A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

2008-B-1. What is the maximum number of rational points that can be on a circle in $\mathbf{R}^2$ whose centre is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

2007-A-6. A *triangulation* $\mathfrak{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side of P is a side of exactly one triangle in $\mathfrak{T}$. Say that $\mathfrak{T}$ is *admissible* if every internal vertex is shared by 6 or more triangles. [An example is given.] Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.

2006-B-3. Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

2004-A-2. For $i = 1, 2$, let T_i be a triangle with side lengths a_i, b_i, c_i and area A_i. Suppose that $a_1 \le a_2$, $b_1 \le b_2$, $c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$?

2003-B-5. Let A, B and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distances from P to A, B, C respectively. Show that there is a triangle with side lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

2002-A-2. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

2001-A-4. Triangle ABC has area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of triangle RST.

2001-A-6. Can an arc of a parabola inside a circle of radius 1 have length greater than 4?

2000-A-3. The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and that the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

2000-A-5. Three distinct points with integer coefficients lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

2000-B-6. Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \cdots, \pm 1)$ in $n-$dimensional space, with $n \ge 3$. Show that there are three distinct points in B which are vertices of an equilateral triangle.

1999-B-1. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$. [Here $|PQ|$ denotes the length of the segment PQ.]

1998-A-1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

1998-A-2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the $x-$axis and B be the area of the region lying to the right of the

$y-$axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position of s.

1998-A-5. Let $\mathfrak{F}$ be a finite collection of open discs in $\mathbf{R}^2$ whose union contains a set $E \subseteq \mathbf{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, D_2, \cdots, D_n$ in $\mathfrak{F}$ such that

$$\bigcup_{j=1}^{n} 3D_j \supseteq E \ .$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

1998-A-6. Let A, B, C denote distinct points with integer points with integer coordinates in $\mathbf{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1 \ ,$$

then A, B, C are three vertices of a square. Here $[XY]$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

1998-B-2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the $x-$axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

1998-B-3. Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P a regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin\alpha + B\cos\beta$, where A, B, α and β are real numbers.

1997-A-1. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

1997-B-6. The dissection of the $3-4-5$ triangle shown below has diameter $5/2$.

Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between pairs of points belonging to the same part.)

1996-A-1. Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed into that rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangles.

1996-A-2. Let C_1 and C_2 be circles whose centers are 10 units apart and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exist points x on C_1 and Y on C_2 such that M is the midpoint of the lines segment XY.

1996-B-6. Let $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \cdots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0) \quad .$$

1995-B-2. An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin(x/a)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?