

## PUTNAM PROBLEMS

### DIFFERENTIAL EQUATIONS

#### First Order Equations

1. Linear  $y' + p(x)y = q(x)$

Multiply through by the integrating factor  $\exp(\int p(x)dx)$  to obtain

$$(y \exp(\int p(x)dx))' = q(x) \exp(\int p(x)dx) .$$

2. Separation of variables  $y' = f(x)g(y)$

Put in the form  $dy/g(y) = f(x)dx$  and integrate both sides.

3. Homogeneous  $y' = f(x, y)$  where  $f(tx, ty) = f(x, y)$ .

Let  $y = ux$ ,  $y' = u'x + u$  to get  $u'x + u = f(1, u)$ .

4. Fractional linear

$$y' = \frac{ax + by}{cx + dy} .$$

Do as in case 3, or introduce an auxiliary variable  $t$  and convert to a system

$$\frac{dy}{dt} = ax + by \quad \frac{dx}{dt} = cx + dy$$

and try  $x = c_1 e^{\lambda t}$ ,  $y = c_2 e^{\lambda t}$ . If this yields two distinct values for  $\lambda$ , the ratio  $c_1 : c_2$  can be found. If there is only one value of  $\lambda$ , try  $x = (c_1 + c_2 t)e^{\lambda t}$ ,  $y = (c_3 + c_4 t)e^{\lambda t}$ .

5. Modified fractional linear

$$y' = \frac{ax + by + c}{hx + ky + r} .$$

(i) If  $ak - bh \neq 0$ , choose  $p, q$  so that  $ap + bq + c = 0$ ,  $hp + kq + r = 0$  and make a change of variables:  $X = x - p$ ,  $Y = y - q$ .

(ii) If  $ak - bh = 0$ , set  $u = ax + by + c$  to obtain a separation of variables equation in  $y$  and  $u$ .

6. General linear exact equation

$$P(x, y)dx + Q(x, y)dy = 0$$

where

$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 .$$

The equation has a solution of the form  $f(x, y) = c$ , where  $\partial f/\partial x = P$  and  $\partial f/\partial y = Q$ . We have

$$f(x, y) = \int_{x_0}^x P(u, x)du + \int_{y_0}^y Q(x_0, v)dv$$

or

$$f(x, y) = \int_{y_0}^y Q(x, v)dv + \int_{x_0}^x P(u, y_0)du$$

where  $(x_0, y_0)$  is any point.

7. General linear inexact equation

$$P(x, y)dx + Q(x, y)dy = 0$$

where

$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \neq 0 .$$

We need to find an “integrating factor”  $h(x, y)$  to satisfy

$$h\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\frac{\partial h}{\partial y} - Q\frac{\partial h}{\partial x} = 0 .$$

There is no general method for equations of this type, but one can try assuming that  $h$  is a function of  $x$  alone,  $y$  alone,  $xy$ ,  $x/y$  or  $y/x$ . If  $h$  can be found, multiply the equation through by  $h$  and proceed as in Case 6.

8. Riccati equation  $y' = f_0(x) + f_1(x)y + f_2(x)y^2$  where  $f_2(x) \neq 0$ .  
If a solution  $y_0$  is known, set  $y = y_0 + (1/u)$  to get a first order linear equation in  $u$  and  $x$ .
9. Bernoulli equation  $y' + p(x)y = q(x)y^n$ .  
If  $n = 0$ , use Case 1. If  $n = 1$ , use Case 2. If  $n = 2$ , consider Case 8. If  $n \neq 0, 1$ , set  $u = y^{1-n}$  to get a first order equation in  $u$  and  $x$ .

### Linear equations of $n$ th order with constant coefficients

A linear equation of the  $n$ th degree with constant coefficients has the form

$$c_n y^{(n)} + c_{n-1} y^{(n-1)} + \cdots + c_2 y'' + c_1 y' + c_0 y = q(x)$$

where the  $c_i$  are constants and  $y$  is an unknown function of  $x$ . The general solution of such an equation is the sum of two parts:

the complementary function (general solution of the homogeneous equation formed by taking  $q(x) = 0$ );  
a particular integral (any solution of the given equation).

The complementary function is the sum of terms of the form

$$(a_{r-1}x^{r-1} + a_{r-2}x^{r-2} + \cdots + a_2t^2 + a_1t + a_0)e^{\lambda x}$$

where  $\lambda$  is a root of multiplicity  $r$  of the auxiliary polynomial

$$c_n t^n + c_{n-1} t^{n-1} + \cdots + c_2 t^2 + c_1 t + c_0$$

and  $a_i$  are arbitrary constants.

A particular integral can be found when  $q(x)$  can be written as the sum of terms of the type  $h(x)e^{\rho x}$  where  $h(x)$  is a polynomial and  $\rho$  is complex. This includes the cases  $q(x) = h(x)e^{\alpha x} \sin \beta x$  and  $q(x) = h(x)e^{\alpha x} \cos \beta x$ , with  $\alpha$  and  $\beta$  real. When the  $c_i$  and the coefficients of  $h(x)$  are real, solve with  $q(x) = h(x)e^{(\alpha+i\beta)x}$  and take the real or imaginary parts, respectively, of the solution obtained.

*Operational calculus:* Let  $Du = u'$ . The left side of the equation can be written  $p(D)y = q(x)$  where  $p(t) = c_n t^n + \cdots + c_1 t + c_0$ . For any polynomial  $p(t)$ , we have the operational rules

$$p(D)e^{rx} = p(r)e^{rx} \quad \text{and} \quad p(D)(ue^{rx}) = e^{rx}p(D+r)u .$$

The following examples illustrate how a particular integral can be obtained without having to deal with special cases or undetermined coefficients:

- (i)  $y'' + 2y' + 2y = 2e^{-x} \sin x$  .

The solution of this equation is the imaginary part of the solution of the following equation

$$(D^2 + 2D + 2)y = 2e^{(-1+i)x} .$$

We try for a particular integral of the form  $y = ue^{(-1+i)x}$ , where the exponent agrees with the exponent on the right side of the equation. Then, factoring the polynomial in  $D$  and substituting for  $y$ , we obtain:

$$(D + 1 - i)(D + 1 + i)(ue^{(-1+i)x}) = 2e^{(-1+i)x} .$$

Bringing the exponent through and cancelling it, we get

$$D(D + 2i)u = (D + 2i)Du = 2 \quad (*)$$

Differentiate:

$$(D + 2i)D^2u = 0 \quad (**) .$$

Any  $u$  for which (\*) and (\*\*) hold will yields a particular integral. Choose  $u$  such that  $u'' = 0$ . Then (\*\*) holds. To satisfy (\*), we need  $2iDu = 2$  or  $Du = -i$ . Hence take  $u(x) = -ix$ . A particular integral of the complex equation is  $-ix \exp((-1 + i)x)$ , and a particular integral of the original equation is

$$\text{Im}(-ixe^{(-1+i)x}) = -xe^{-x} \cos x .$$

(ii)  $y'' - 4y' + 4y = 8x^2e^{2x} \sin 2x$

A particular integral of this equation is the imaginary part of the particular integral of the equation

$$(D - 2)^2y = 8x^2e^{(2+2i)x} .$$

Let  $y = ue^{(2+2i)x}$ . Then, bring the exponential through the operator as before and cancelling, we get

$$(D + 2i)^2u = (D^2 + 4iD - 4)u = 8x^2 \quad (1)$$

$$(D^2 + 4iD - 4)Du = 16x \quad (2)$$

$$(D^2 + 4iD - 4)D^2u = 16 \quad (3)$$

$$(D^2 + 4iD - 4)D^3u = 0 \quad (4) .$$

Let  $D^3u = 0$  to satisfy (4). Then (3) requires  $-4D^2u = 16$  or  $D^2u = -4$ . Now, (2) requires  $-16i - 4Du = 16x$  or  $Du = -16x - 16i$ . Finally, to make (1) hold, we need  $u = -2x^2 - 4ix + 3$ . The solution to (ii) is thus

$$y = \text{Im}(-2x^2 - 4ix + 3)e^{(2+2i)x} .$$

### Putnam questions

**2009-A-2.** Functions  $f, g, h$  are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f' = 2f^2gh + \frac{1}{gh} , \quad f(0) = 1 ,$$

$$g' = fg^2h + \frac{4}{fh} . \quad g(0) = 1 ,$$

$$h' = 3fgh^2 + \frac{1}{fg} , \quad h(0) = 1 .$$

Find an explicit formula for  $f(x)$  valid in some open interval around 0.

**2009-B-5.** Let  $f : (1, \infty) \rightarrow \mathbf{R}$  be a differentiable function such that

$$f'(x) = \frac{x^2 - (f(x))^2}{x^2((f(x))^2 + 1)}$$

for all  $x > 1$ . Prove that  $\lim_{x \rightarrow \infty} f(x) = \infty$ .

**2005-B-3.** Find all differentiable functions  $f : (0, \infty) \rightarrow (0, \infty)$  for which there is a positive real number  $a$  such that

$$f' \left( \frac{a}{x} \right) = \frac{x}{f(x)}$$

for all  $x > 0$ .

**1997-B-2.** Let  $f$  be a twice differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where  $g(x) \geq 0$  for all real  $x$ . Prove that  $|f(x)|$  is bounded.

**1995-A-5.** Let  $x_1, x_2, \dots, x_n$  be differentiable (real-valued) functions of a single variable  $t$  which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$

...

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n$$

for some constants  $a_{ij} \geq 0$ . Suppose that for all  $i$ ,  $x_i(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Are the functions  $x_1, x_2, \dots, x_n$  necessarily linearly dependent?

**1989-B-3.** Let  $f$  be a function on  $[0, \infty)$ , differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for  $x > 0$ . Assume that  $|f(x)| \leq e^{-\sqrt{x}}$  for  $x \geq 0$  (so that  $f(x)$  tends rapidly to 0 as  $x$  increases). For  $n$  a nonnegative integer, define

$$\mu_n = \int_0^\infty x^n f(x) dx$$

(sometimes called the  $n$ th moment of  $f$ ).

a. Express  $\mu_n$  in terms of  $\mu_0$ .

b. Prove that the sequence  $\{\mu_n \frac{3^n}{n!}\}$  always converges, and that this limit is 0 only if  $\mu_0 = 0$ .

**1988-A-2.** A not uncommon calculus mistake is to believe that the product rule for derivatives says that  $(fg)' = f'g'$ . If  $f(x) = e^{x^2}$ , determine, with proof, whether there exists an open interval  $(a, b)$  and a nonzero function  $g$  defined on  $(a, b)$  such that this wrong product rule is true for  $x$  in  $(a, b)$ .