

PUTNAM PROBLEMS

DIFFERENTIAL EQUATIONS

First Order Equations

1. Linear $y' + p(x)y = q(x)$

Multiply through by the integrating factor $\exp(\int p(x)dx)$ to obtain

$$(y \exp(\int p(x)dx))' = q(x) \exp(\int p(x)dx) .$$

2. Separation of variables $y' = f(x)g(y)$

Put in the form $dy/g(y) = f(x)dx$ and integrate both sides.

3. Homogeneous $y' = f(x, y)$ where $f(tx, ty) = f(x, y)$.

Let $y = ux$, $y' = u'x + u$ to get $u'x + u = f(1, u)$.

4. Fractional linear

$$y' = \frac{ax + by}{cx + dy} .$$

Do as in case 3, or introduce an auxiliary variable t and convert to a system

$$\frac{dy}{dt} = ax + by \quad \frac{dx}{dt} = cx + dy$$

and try $x = c_1 e^{\lambda t}$, $y = c_2 e^{\lambda t}$. If this yields two distinct values for λ , the ratio $c_1 : c_2$ can be found. If there is only one value of λ , try $x = (c_1 + c_2 t)e^{\lambda t}$, $y = (c_3 + c_4 t)e^{\lambda t}$.

5. Modified fractional linear

$$y' = \frac{ax + by + c}{hx + ky + r} .$$

(i) If $ak - bh \neq 0$, choose p, q so that $ap + bq + c = 0$, $hp + kq + r = 0$ and make a change of variables: $X = x - p$, $Y = y - q$.

(ii) If $ak - bh = 0$, set $u = ax + by + c$ to obtain a separation of variables equation in y and u .

6. General linear exact equation

$$P(x, y)dx + Q(x, y)dy = 0$$

where

$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 .$$

The equation has a solution of the form $f(x, y) = c$, where $\partial f/\partial x = P$ and $\partial f/\partial y = Q$. We have

$$f(x, y) = \int_{x_0}^x P(u, x)du + \int_{y_0}^y Q(x_0, v)dv$$

or

$$f(x, y) = \int_{y_0}^y Q(x, v)dv + \int_{x_0}^x P(u, y_0)du$$

where (x_0, y_0) is any point.

7. General linear inexact equation

$$P(x, y)dx + Q(x, y)dy = 0$$

where

$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \neq 0 .$$

We need to find an “integrating factor” $h(x, y)$ to satisfy

$$h\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\frac{\partial h}{\partial y} - Q\frac{\partial h}{\partial x} = 0 .$$

There is no general method for equations of this type, but one can try assuming that h is a function of x alone, y alone, xy , x/y or y/x . If h can be found, multiply the equation through by h and proceed as in Case 6.

8. Riccati equation $y' = f_0(x) + f_1(x)y + f_2(x)y^2$ where $f_2(x) \neq 0$.
If a solution y_0 is known, set $y = y_0 + (1/u)$ to get a first order linear equation in u and x .
9. Bernoulli equation $y' + p(x)y = q(x)y^n$.
If $n = 0$, use Case 1. If $n = 1$, use Case 2. If $n = 2$, consider Case 8. If $n \neq 0, 1$, set $u = y^{1-n}$ to get a first order equation in u and x .

Linear equations of n th order with constant coefficients

A linear equation of the n th degree with constant coefficients has the form

$$c_n y^{(n)} + c_{n-1} y^{(n-1)} + \cdots + c_2 y'' + c_1 y' + c_0 y = q(x)$$

where the c_i are constants and y is an unknown function of x . The general solution of such an equation is the sum of two parts:

the complementary function (general solution of the homogeneous equation formed by taking $q(x) = 0$);
a particular integral (any solution of the given equation).

The complementary function is the sum of terms of the form

$$(a_{r-1}x^{r-1} + a_{r-2}x^{r-2} + \cdots + a_2t^2 + a_1t + a_0)e^{\lambda x}$$

where λ is a root of multiplicity r of the auxiliary polynomial

$$c_n t^n + c_{n-1} t^{n-1} + \cdots + c_2 t^2 + c_1 t + c_0$$

and a_i are arbitrary constants.

A particular integral can be found when $q(x)$ can be written as the sum of terms of the type $h(x)e^{\rho x}$ where $h(x)$ is a polynomial and ρ is complex. This includes the cases $q(x) = h(x)e^{\alpha x} \sin \beta x$ and $q(x) = h(x)e^{\alpha x} \cos \beta x$, with α and β real. When the c_i and the coefficients of $h(x)$ are real, solve with $q(x) = h(x)e^{(\alpha+i\beta)x}$ and take the real or imaginary parts, respectively, of the solution obtained.

Operational calculus: Let $Du = u'$. The left side of the equation can be written $p(D)y = q(x)$ where $p(t) = c_n t^n + \cdots + c_1 t + c_0$. For any polynomial $p(t)$, we have the operational rules

$$p(D)e^{rx} = p(r)e^{rx} \quad \text{and} \quad p(D)(ue^{rx}) = e^{rx}p(D+r)u .$$

The following examples illustrate how a particular integral can be obtained without having to deal with special cases or undetermined coefficients:

- (i) $y'' + 2y' + 2y = 2e^{-x} \sin x$.

The solution of this equation is the imaginary part of the solution of the following equation

$$(D^2 + 2D + 2)y = 2e^{(-1+i)x} .$$

We try for a particular integral of the form $y = ue^{(-1+i)x}$, where the exponent agrees with the exponent on the right side of the equation. Then, factoring the polynomial in D and substituting for y , we obtain:

$$(D + 1 - i)(D + 1 + i)(ue^{(-1+i)x}) = 2e^{(-1+i)x} .$$

Bringing the exponent through and cancelling it, we get

$$D(D + 2i)u = (D + 2i)Du = 2 \quad (*)$$

Differentiate:

$$(D + 2i)D^2u = 0 \quad (**) .$$

Any u for which (*) and (**) hold will yield a particular integral. Choose u such that $u'' = 0$. Then (**) holds. To satisfy (*), we need $2iDu = 2$ or $Du = -i$. Hence take $u(x) = -ix$. A particular integral of the complex equation is $-ix \exp((-1 + i)x)$, and a particular integral of the original equation is

$$\text{Im}(-ixe^{(-1+i)x}) = -xe^{-x} \cos x .$$

(ii) $y'' - 4y' + 4y = 8x^2e^{2x} \sin 2x$

A particular integral of this equation is the imaginary part of the particular integral of the equation

$$(D - 2)^2y = 8x^2e^{(2+2i)x} .$$

Let $y = ue^{(2+2i)x}$. Then, bring the exponential through the operator as before and cancelling, we get

$$(D + 2i)^2u = (D^2 + 4iD - 4)u = 8x^2 \quad (1)$$

$$(D^2 + 4iD - 4)Du = 16x \quad (2)$$

$$(D^2 + 4iD - 4)D^2u = 16 \quad (3)$$

$$(D^2 + 4iD - 4)D^3u = 0 \quad (4) .$$

Let $D^3u = 0$ to satisfy (4). Then (3) requires $-4D^2u = 16$ or $D^2u = -4$. Now, (2) requires $-16i - 4Du = 16x$ or $Du = -16x - 16i$. Finally, to make (1) hold, we need $u = -2x^2 - 4ix + 3$. The solution to (ii) is thus

$$y = \text{Im}(-2x^2 - 4ix + 3)e^{(2+2i)x} .$$

Putnam questions

1997-B-2. Let f be a twice differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x) ,$$

where $g(x) \geq 0$ for all real x . Prove that $|f(x)|$ is bounded.

1995-A-5. Let x_1, x_2, \dots, x_n be differentiable (real-valued) functions of a single variable t which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$

...

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n$$

for some constants $a_{ij} \geq 0$. Suppose that for all i , $x_i(t) \rightarrow 0$ as $t \rightarrow \infty$. Are the functions x_1, x_2, \dots, x_n necessarily linearly dependent?

1989-B-3. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a nonnegative integer, define

$$\mu_n = \int_0^{\infty} x^n f(x) dx$$

(sometimes called the n th moment of f).

- Express μ_n in terms of μ_0 .
- Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that this limit is 0 only if $\mu_0 = 0$.

1988-A-2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b) .