

PUTNAM PROBLEMS

CALCULUS, ANALYSIS

2009-A-6. Let $f; [0, 1]^2 \rightarrow \mathbf{R}$ be a continuous function on the closed unit square such that $\partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous on the interior $(0, 1)^2$. Let $a = \int_0^1 f(0, y)dy$, $b = \int_0^1 f(1, y)dy$, $c = \int_0^1 f(x, 0)dx$, $d = \int_0^1 f(x, 1)dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0, 1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a$$

and

$$\frac{\partial f}{\partial y}(x_0, y_0) = d - c.$$

2008-A-4. Define $f : \mathbf{R} \rightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

Does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

2008-B-1. What is the maximum number of rational points that can lie on a circle in \mathbf{R}^2 whose centre is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

2008-B-2. Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)dt$. Evaluate

$$\lim_{n \rightarrow \infty} \frac{n!F_n(1)}{\ln n}.$$

2008-B-5. Find all continuously differentiable functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that for every rational number q , the number $f(q)$ is rational and has the same denominator as q . (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: \gcd means greatest common divisor.)

2007-B-2. Suppose that $f : [0, 1] \rightarrow \mathbf{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left| \int_0^\alpha f(x)dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

2006-A-1. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

2006-A-5. Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \dots, n$. Prove that

$$\frac{a_1 + a_2 + \dots + a_n}{a_1 a_2 \dots a_n}$$

is an integer and determine its value.

2006-B-2. Prove that, for every set $X = \{x_1, x_2, \dots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

2006-B-5. For each continuous function $f : [0, 1] \rightarrow \mathbf{R}$, let $I(f) = \int_0^1 x^2 f(x) dx$ and $J(f) = \int_0^1 x(f(x))^2 dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f .

2006-B-6. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n \geq 0$. Evaluate

$$\lim_{n \rightarrow \infty} \frac{a_n^{k+1}}{n^k}.$$

2005-A-5. Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} dx.$$

2005-B-3. Find all differentiable functions $f : (0, \infty) \rightarrow (0, \infty)$ for which there is a positive real number a such that

$$f' \left(\frac{a}{x} \right) = \frac{x}{f(x)}$$

for all $x > 0$.

2004-A-6. Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x, y) dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x, y) dy \right)^2 dx \leq \left(\int_0^1 \int_0^1 f(x, y) dx dy \right)^2 + \int_0^1 \int_0^1 [f(x, y)]^2 dx dy.$$

2004-B-3. Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq f(x)\}$$

has perimeter k units and area k square units for some real number k .

2004-B-5. Evaluate

$$\lim_{x \rightarrow 1^-} \prod_{n=0}^{\infty} \left(\frac{1+x^{n+1}}{1+x^n} \right)^{x^n}.$$

2003-A-3. Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x .

2003-B-6. Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy \geq \int_0^1 |f(x)| dx .$$

2002-A-1. Let k be a positive integer. The n th derivative of $1/(x^k - 1)$ has the form $(P_n(x))/(x^k - 1)^{n+1}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

2002-B-3. Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne} .$$

2001-B-5. Let a and b be real numbers in the interval $(0, \frac{1}{2})$ and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x . Prove that $g(x) = cx$ for some constant c .

2000-A-4. Show that the improper integral

$$\lim_{B \rightarrow \infty} \int_0^B \sin(x) \sin(x^2) dx$$

converges.

2000-B-3. Let $f(t) = \sum_{j=1}^N a_j \sin(2\pi jt)$, where each a_j is real and $a_n \neq 0$. Let N_k denote the number of zeros (including multiplicities) of $d^k f/dt^k$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \dots \quad \text{and} \quad \lim_{k \rightarrow \infty} N_k = 2N .$$

[Added note: Presumably one is to restrict t to the interval $[0, 1)$ when counting the zeros.]

2000-B-4. Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x . Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

1999-A-5. Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^1 |p(x)| dx .$$

1999-B-4. Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive for all x . Suppose that $f'''(x) \leq f(x)$ for all x . Show that $f'(x) < 2f(x)$ for all x .

1998-A-3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0 .$$

1997-A-3. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot \dots \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \dots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots \right) dx .$$

1996-A-6. Let $c \geq 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x) = f(x^2 + c)$ for all $x \in \mathbf{R}$. [Note: \mathbf{R} is the set of real numbers.]

1995-A-2. For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

converge?

1994-A-2. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x -axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y -axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

1994-B-3. Find the set of all real numbers k with the following property:

For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x , there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

1994-B-5. For any real number α , define the function f_α by $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

($\lfloor x \rfloor$ denotes the greatest integer $\leq x$, and $f_\alpha^k = f_\alpha \circ \cdots \circ f_\alpha$ is the k -fold composition of f_α .)

1993-A-1. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.

1993-A-5. Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{1}{100}}^{\frac{1}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx$$

is a rational number.

1993-B-4. The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1$, $0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all x , $0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y)dy = g(x) \quad \text{and} \quad \int_0^1 g(y)K(x, y)dy = f(x) .$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

1992-A-2. Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 C(-y-1) \left(\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} + \cdots + \frac{1}{y+1992} \right) dy .$$

1992-A-4. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \quad n = 1, 2, 3, \dots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \dots$.

1992-B-3. For any pair (x, y) of real numbers, a sequence $(a_n(x, y))_{n \geq 0}$ is defined as follows:

$$\begin{aligned} a_0(x, y) &= x \\ a_{n+1}(x, y) &= \frac{(a_n(x, y))^2 + y^2}{2}, \quad \text{for all } n \geq 0. \end{aligned}$$

Find the area of the region

$$\{(x, y) | (a_n(x, y))_{n \geq 0} \text{ converges}\}$$

1992-B-4. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}} \left(\frac{p(x)}{x^3 - x} \right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.