PREPARATION NOTES FOR THE PUTNAM COMPETITION

Note that the questions and solutions for the Putnam examinations are published in the *American Mathematical Monthly*. Generally, you may find them close to the end of each volume, in the October or November issue. Back issues of the *Monthly* can be found in the mathematics library (SS 622).

NUMBER THEORY

Putnam problems

1998-A-4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

1998-B-5. Let N be the positive integer with 1998 decimal digits, all of them 1; that is, $N = 1111\cdots 11$ (1998 digits). Find the thousandth digit after the decomal point of $\sqrt{N}$.

1998-B-6. Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

1997-A-5. Let N_n denote the number of ordered $n-$tuples of positive integers $(a_1, a_2, \cdots, a_n)$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

1997-B-3. For each positive integer n write the sum $\sum_{m=1}^{n} \frac{1}{m}$ in the form $\frac{p_n}{q_n}$ where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

1997-B-5. Prove that for $n \geq 2$,

$$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}n \equiv 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}n-1 \pmod{n} \ .$$

1996-A-5. If p is a prime number greater than 3, and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.
(For example, $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 7 + 21 + 35 + 35 = 2 \cdot 7^2$.)

1995-A-3. The number $d_1d_2\cdots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2\cdots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i in $d_1d_2\cdots d_9$ by the corresponding digit e_i $(1 \leq i \leq 9)$ is divisible by 7. The number $f_1f_2\cdots f_9$ is related to $e_1e_2\cdots e_9$ in the same way; that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1d_2\cdots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

1995-A-4. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels x_1, x_2, $\cdots$, x_n satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \qquad \text{for} \quad k = 1, 2, \cdots n \ .$$

1994-B-1. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is, a member of the set $\{0, 1, 4, 9, 16, \cdots, \}$. a is **within** n of b if $b - n \leq a \leq b + n$.)

1994-B-6. For any integer a, set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \leq a, b, c, d \leq 99$,

$$n_2 + n_b \equiv n_c + n_d \pmod{10100}$$

implies $\{a, b\} = \{c, d\}$.

1993-A-4. Let $x_1, x_2, \cdots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \cdots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

1993-B-1. Find the smallest positive integer n such that for every integer m, with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994} \quad .$$

1993-B-5. Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

1993-B-6. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$, and replace them with $2x$ and $y - x$.

1992-A-3. For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2 + y^2)^m = (xy)^n$.

1992-A-5. For each positive integer n, let

$$a_n = \begin{cases} 0 & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1 & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}, \quad \text{for} \quad 0 \leq j \leq m-1 \ .$$

1989-A-1. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

1988-B-1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \cdots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers.

1988-B-6. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at + b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \cdots\}$.

1987-A-2. The sequence of digits

$$1\,2\,3\,4\,5\,6\,7\,8\,9\,1\,0\,1\,1\,1\,2\,1\,3\,1\,4\,1\,5\,1\,6\,1\,7\,1\,8\,1\,9\,2\,0\,2\,1 \cdots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in hich the $m-$digit numbers are placed, define $f(n)$ to be m. For example $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two digit integer 55. Find, with proof, $f(1987)$.

Other problems

1. (a) Let m be a positive integer. Prove that the number of digits used in writing down the numbers from 1 up to 10^m using ordinary decimal digits is equal to the number of zeros required in writing down the numbers from 1 up to 10^{m+1}.

   (b) Suppose that the numbers from 1 to n are written down using ordinary base-10 digits. Let $h(n)$ be the number of zeros used. Thus, $h(5) = 0$, $h(11) = 1$, $h(87) = 8$ and $h(306) = 57$. Does there exists an integer k such that $h(n) > n$ for every integer n exceeding k? If not, provide a proof; if so, give a specific value.

REAL NUMBERS

Putnam problems

1998-B-5. Let N be a positive integer with 1998 decimal digits, all of them 1; that is, $N = 1111 \cdots 11$ (1998 digits). Find the thousandth digits after the decomal point of $\sqrt{N}$.

1997-B-1. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$S_n = \sum_{m=1}^{6n-1} \min \left(\left\{ \frac{m}{6n} \right\}, \left\{ \frac{m}{3n} \right\} \right) .$$

(Here, min (a, b) denotes the minimum of a and b.)

1995-A-1. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (notnecessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

1995-B-6. For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \cdots\}.$$

Prove that $\{1, 2, 3, \cdots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$. [As usual $\lfloor x \rfloor$ is the greatest integer $\leq x$.]

1994-A-5. Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{r_{1994}} , \qquad \text{with} \;\; i_1 < i_2 < \cdots < i_{1994} \;\; .$$

Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

1990-A-2. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n, m = 0, 1, 2, \cdots)$? Justify your answer.

1990-A-4. Consider a paper punch that can be centered at any point of the plane, and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

Other problems

1. Let τ be the "golden ratio", *i.e.*, τ is the positive real for which $\tau^2 = \tau + 1$. Prove that, for each positive integer n,

$$\lfloor \tau \lfloor \tau n \rfloor \rfloor + 1 = \lfloor \tau^2 n \rfloor \quad .$$

INEQUALITIES

Putnam problems

1998-B-1. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

1998-B-2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the $x-$axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

1996-B-2. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}} \quad .$$

1996-B-3. Given that $\{x_1, x_2, \cdots, x_n\} = \{1, 2, \cdots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1 \ .$$

1988-B-2. Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

Other problems

1. If $x > 1$, prove that

$$x > (1 + \frac{1}{x})^{x-1} \quad .$$

2. How many permutations $\{x_1, x_2, \cdots, x_n\}$ of $\{1, 2, \cdots, n\}$ are there such that the cyclic sum

$$\sum_{i=1}^{n} |x_i - x_{i+1}|$$

is (a) maximum? (b) minimum? (Note that $x_{n+1} = x_1$.) [CM 2018]

3. Let a, b, c, d be distinct real numbers for which

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \qquad \text{and} \qquad ac = bd \quad .$$

Find the maximum value of

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} \ .$$

[CM 2020]

4. Let $n \geq m \geq 1$ and $x \geq y \geq 0$. Suppose that $x^{n+1} + y^{n+1} \leq x^m - y^m$. Prove that $x^n + y^n \leq 1$. [CM 2044]

5. Show that for positive reals $a.\ b.\ c$,

$$3\max\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\} \geq (a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \ .$$

[CM 2064]

6. Find all values of λ for which

$$2(x^3+y^3+z^3)+3(1+3\lambda)xyz \geq (1+\lambda)(x+y+z)(yz+zx+xy)$$

holds for all positive real x, y, z. [CM 2105]

7. Prove the inequality

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left[\sum_{i=1}^{n}(a_i+b_i)\right]\left[\sum_{i=1}^{n}\frac{a_i b_i}{a_i+b_i}\right]$$

for any positive real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$. [CM 2113]

8. Let $a, b > 1$. Prove that

$$\prod_{k=1}^{n}(ak+b^{k-1}) \leq \prod_{k=1}^{n}(ak+b^{n-k}) \quad .$$

[CM 2145]

9. Suppose that a, b, c are real and that $|ax^2+bx+c| \leq 1$ for $-1 \leq x \leq 1$. Prove that $|cx^2+bx+a| \leq 2$ for $-1 \leq x \leq 1$. [CM 2153]

10. Let n be a positive integer and let t be a positive real. Suppose that $x_n = (1/n)(1+t+t^2+\cdots+t^{n-1})$. Show that, for each pair r, s of positive integers, there is a positive integer m for which $x_r x_s \leq x_m$. [CM 2159]

11. Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots b_n > 0$ and n be a positive integers. Prove that

$$\sqrt[n]{\prod_{k=1}^{n}(a_k+b_k)} \geq \sqrt[n]{\prod_{k=1}^{n} a_k} + \sqrt[n]{\prod_{k=1}^{n} b_k} \quad .$$

[CM 2176]

12. Prove that the inequality

$$\left(\frac{ab+ac+ad+bc+bd+cd}{6}\right)^{\frac{1}{2}} \geq \left(\frac{abc+abd+acd+bcd}{4}\right)^{\frac{1}{3}}$$

holds for any positive reals a, b, c, d.

13. Let $a > 0$, $0 \leq x_1, x_2, \cdots, x_n \leq a$ and n be an integer exceeding 1. Suppose that

$$x_1 x_2 \cdots x_n = (a-x_1)^2(a-x_2)^2\cdots(a-x_n)^2 \ .$$

Determine the maximum possible value of the product. [CM 1781]

14. Prove that, if n and m are positive integers for which $n \geq m^2 \geq 16$, then $2^n \geq n^m$. [CM 2163]

15. Let $x, y, z \geq 0$ with $x + y + z = 1$. For real numbers a, b, determine the maximum value of $c = c(a, b)$ for which $a + bxy \geq c(yz + zx + xy)$. [CM 2172]

16. Let $0 < x, y < 1$. Prove that the minimum of x^2+xy+y^2, $x^2+x(y-1)+(y-1)^2$, $(x-1)^2+(x-1)y+y^2$ and $(x-1)^2 + (x-1)(y-1) + (y-1)^2$ does not exceed 1/3.

17. Let $a, b, c > 0$, $a < bc$ and $1 + a^3 = b^3 + c^3$. Prove that $1 + a < b + c$.

SEQUENCES, SERIES AND RECURRENCES

Notes

1. $x_{n+1} = ax_n$ has the general solution $x_n = x_1 a^{n-1}$.
2. $x_{n+1} = x_n + b$ has the general solution $x_n = x_1 + (n-1)b$.
3. $x_{n+1} = ax_n + b$ (with $a \neq 1$) can be rewritten $x_{n+1} + k = a(x_n + k)$ where $(a-1)k = b$ and so reduces to the recurrence 1.
4. $x_{n+1} = ax_n + bx_{n-1}$ has different general solution depending on the discriminant of the characteristic polynomial $t^2 - at - b$.
(a) If $a^2 - 4b \neq 0$ and the distinct roots of the characteristic polynomial are r_1 and r_2, then the general solution of the recurrence is
$$x_n = c_1 r_1^n + c_2 r_2^n$$
where the constants c_1 and c_2 are chosen so that
$$x_1 = c_1 r_1 + c_2 r_2 \qquad \text{and} \qquad x_2 = c_1 r_1^2 + c_2 r_2^2 \ .$$
(b) If $a^2 - 4b = 0$ and r is the double root of the characteristic polynomial, then
$$x_n = (c_1 n + c_0) r^n$$
where c_1 and c_0 are chosen so that
$$x_1 = (c_1 + c_0) r \qquad \text{and} \qquad x_2 = (2c_1 + c_0) r^2 \ .$$
5. $x_{n+1} = (1-s)x_n + sx_{n-1} + r$ can be rewritten $x_{n+1} - x_n = -s(x_n - x_{n-1}) + r$ and solved by a previous method for $x_{n+1} - x_n$.
6. $x_{n+1} = ax_n + bx_{n-1} + c$ where $a + b \neq 1$ can be rewritten $(x_{n+1} + k) = a(x_n + k) + b(x_{n-1} + k)$ where $(a + b - 1)k = c$ and solved for $x_n + k$.
7. The general homogeneous linear recursion has the form
$$x_{n+k} = a_{k-1} x_{n+k-1} + \cdots + a_1 x_{n+1} + a_0 \ .$$
Its characteristic polynomials is
$$t^k - a_{k-1} t^{k-1} - \cdots - a_1 t - a_0 \ .$$
Let r be a root of this polynomial of multiplicity m; then the nth term of the recurrence is a linear combination of terms of the type
$$(c_{m-1} r^{m-1} + \cdots + c_1 r + c_0) r^n \ .$$

1998-A-4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

1998-B-4. Find necessaary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0 \ .$$

1997-A-6. For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1} \ .$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

1994-A-1. Suppose that a sequence $a_1, a_2, a_3, \cdots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

1994-A-5. Let $(r_n)_{n\geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}}$$

with $i_1 < i_2 < \cdots < i_{1994}$. Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

1993-A-2. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1$$

for $n = 1, 2, 3, \cdots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

1993-A-6. The infinite sequence of $2's$ and $3's$

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \cdots$$

has the property that, if one forms a second sequence that records the number of $3's$ between successive $2's$, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

1992-A-1. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions
(i) $f(f(n)) = n$, for all integers n;
(ii) $f(f(n+2)+2) = n$ for all integers n;
(iii) $f(0) = 1$.

1992-A-5. For each positive integer n, let

$$a_n = \begin{cases} 0, & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1, & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}$$

for $0 \le j \le m-1$.

1991-B-1. For each integer $n \ge 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \le n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \ge 0$. For what positive integers A is this sequence eventually constant?

1990-A-1. Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \ge 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3} \quad .$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

1988-B-4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is

$$\sum_{n=1}^{\infty} (a_n)^{n/(n+1)} \quad .$$

1985-A-4. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \ge 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

1979-A-3. Let x_1, x_2, x_3, $\cdots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-2}} \qquad \text{for} \quad n = 3, 4, 5, \cdots \quad .$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

1975-B-6. Show that, if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then
(a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and
(b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

Other problems

1. Let n be an even positive integer and let $x_1, x_2, \cdots, x_n$ be n positive reals. Define

$$f(x_1, \cdots, x_n) = \left(\frac{x_2+x_3}{2}, \frac{x_2+x_3}{2}, \frac{x_4+x_5}{2}, \frac{x_4+x_5}{2}, \cdots, \frac{x_n+x_1}{2}, \frac{x_n+x_1}{2} \right) \quad .$$

   Determine

$$\lim_{k\to\infty} f^k(x_1, \cdots, x_n)$$

   where f^k denotes the kth iterate of f, *i.e.*, $f^k = f \circ f^{k-1}$ for $k \ge 2$.

2. Let $0 < x_0 < 1$ and, for $n \ge 0$, $x_{n+1} = x_n(1 - x_n)$. Prove that $\sum x_n$ diverges while $\sum x_n^2$ converges. Discuss the behaviour of nx_n as $n \to \infty$.

3. (a) Consider a finite family of arithmetic progressions of integers, each extending infinitely in both directions. Each two of the progressions have a term in common. Prove that all progressions have a term in common.

(b) Can the assumption that all terms be integers be dropped?

4. A class of sequences is defined by

$$S_1 = \{1, 1\}$$
$$S_2 = \{1, 2, 1\}$$
$$S_3 = \{1, 3, 2, 3, 1\}$$
$$S_4 = \{1, 4, 3, 5, 2, 5, 3, 4, 1\}$$

and for integers $n \geq 3$, if

$$S_n = \{a_1, a_2, \cdots, a_{m-1}, a_m\} \quad ,$$

then

$$S_{n+1} = \{a_1, a_1 + a_2, a_2, a_2 + a_3, \cdots, a_{m-1}, a_{m-1} + a_m, a_m\} \quad .$$

How many terms in S_n are equal to n?

5. Prove of disprove, where $i^2 = -1$, that

$$\frac{1}{4i} \sum \{ i^k \tan(\frac{k\pi}{4n}) : 1 \leq k \leq 4n, gcd(n, k) = 1 \}$$

is an integer. [CM 2129]

CALCULUS, ANALYSIS

Putnam problems

1998-A-3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0 \ .$$

1997-A-3. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdots 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx \ .$$

1996-A-6. Let $c \geq 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = f(x^2 + c)$ for all $x \in \mathbf{R}$. [Note: $\mathbf{R}$ is the set of real numbers.]

1995-A-2. For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x + a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x - b}} \right) dx$$

converge?

1994-A-2. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the $x-$axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the $y-$axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

1994-B-3. Find the set of all real numbers k with the following property:

For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

1994-B-5. For any real number α, define the function f_α by $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2) \ .$$

($\lfloor x \rfloor$ denotes the greatest integer $\leq x$, and $f_\alpha^k = f_\alpha \circ \cdots \circ f_\alpha$ is the $k-$fold composition of f_α.)

1993-A-1. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.

1993-A-5. Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{1}{100}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx$$

is a rational number.

1993-B-4. The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1$, $0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all x, $0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y)dy = g(x) \qquad \text{and} \qquad \int_0^1 g(y)K(x, y)dy = f(x) \ .$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

1992-A-2. Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1 + x)^\alpha$. Evaluate

$$\int_0^1 C(-y - 1)\left(\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} + \cdots + \frac{1}{y+1992} \right) dy \ .$$

1992-A-4. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f(\frac{1}{n}) = \frac{n^2}{n^2 + 1}, \qquad n = 1, 2, 3, \cdots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \cdots$.

1992-B-3. For any pair (x, y) of real numbers, a sequence $(a_n(x, y))_{n \geq 0}$ is defined as follows:

$$a_0(x, y) = x$$

$$a_{n+1}(x, y) = \frac{(a_n(x, y))^2 + y^2}{2} \quad , \quad \text{for all} \ \ n \geq 0 \ .$$

Find the area of the region

$$\{(x,y)|(a_n(x,y))_{n\geq 0} \text{ converges}\}$$

.

1992-B-4. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with x^3-x. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x}{x^3-x}\right)=\frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

Other problems

1. Let $f(x)$ be a continuously differentiable real function defined on the closed interval $[0,1]$ for which

$$\int_0^1 f(x)dx = 0 \quad .$$

   Prove that

$$2\int_0^1 f(x)^2dx \leq \int_0^1 |f'(x)|dx \cdot \int_0^1 |f(x)|dx \quad .$$

2. Let $0<x,y<1$. Prove that the minimum of x^2+xy+y^2, $x^2+x(y-1)+(y-1)^2$, $(x-1)^2+(x-1)y+y^2$ and $(x-1)^2+(x-1)(y-1)+(y-1)^2$ does not exceed $1/3$.

3. Integrate

$$\int \tan^2(x-a)\tan^2(x-b)dx \quad .$$

4. Integrate

$$\int_0^{\frac{\pi}{2}} \frac{x\cos x\sin x}{\cos^4 x+\sin^4 x}dx \ .$$

5. Let $O=(0,0)$ and $Q=(1,0)$. Find the point P on the line with equation $y=x+1$ for which the angle OPQ is a maximum.

DIFFERENTIAL EQUATIONS

First Order Equations

1. Linear $y'+p(x)y=q(x)$
   Muliply through by the integrating factor $\exp(\int p(x)dx)$ to obtain

$$(y\exp(\int p(x)dx))' = q(x)\exp(\int p(x)dx) \quad .$$

2. Separation of variables $y'=f(x)g(y)$
   Put in the form $dy/g(y)=f(x)dx$ and integrate both sides.

3. Homogeneous $y'=f(x,y)$ where $f(tx,ty)=f(x,y)$.
   Let $y=ux$, $y'=u'x+u$ to get $u'x+u=f(1,u)$.

4. Fractional linear
$$y' = \frac{ax+by}{cx+dy} \quad .$$
Do as in case 3, or introduce an auxiliary variable t and convert to a system
$$\frac{dy}{dt} = ax + by \qquad\qquad \frac{dx}{dt} = cx + dy$$
and try $x = c_1 e^{\lambda t}$, $y = c_2 e^{\lambda t}$. If this yields two distinct values for λ, the ratio $c_1 : c_2$ can be found.If there is only one value of λ, try $x = (c_1 + c_2 t)e^{\lambda t}$, $y = (c_3 + c_4 t)e^{\lambda t}$.

5. Modified fractional linear
$$y' = \frac{ax+by+c}{hx+ky+r} \quad .$$
(i) If $ak - bh \neq 0$, choose p, q so that $ap + bq + c = 0$, $hp + kq + r = 0$ and make a change of variables: $X = x - p$, $Y = y - q$.
(ii) If $ak - bh = 0$, set $u = ax + by + c$ to obtain a separation of variables equation in y and u.

6. General linear exact equation
$$P(x,y)dx + Q(x,y)dy = 0$$
where
$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 \quad .$$
The equation has a solution of the form $f(x,y) = c$, where $\partial f/\partial x = P$ and $\partial f/\partial y = Q$. We have
$$f(x,y) = \int_{x_0}^{x} P(u,x)du + \int_{y_0}^{y} Q(x_0,v)dv$$
or
$$f(x,y) = \int_{y_0}^{y} Q(x,v)dv + \int_{x_0}^{x} P(u,y_0)du$$
where (x_0, y_0) is any point.

7. General linear inexact equation
$$P(x,y)dx + Q(x,y)dy = 0$$
where
$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \neq 0 \quad .$$
We need to find an "integrating factor" $h(x,y)$ to satisfy
$$h(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) + P\frac{\partial h}{\partial y} - Q\frac{\partial h}{\partial x} = 0 \quad .$$
There is no general method for equations of this type, but one can try assuming that h is a function of x alone, y alone, xy, x/y or y/x. If h can be found, multiply the equation through by h and proceed as in Case 6.

8. Riccati equation $y' = f_0(x) + f_1(x)y + f_2(x)y^2$ where $f_2(x) \neq 0$.
If a solution y_0 is known, set $y = y_0 + (1/u)$ to get a first order linear equation in u and x.

9. Bernoulli equation $y' + p(x)y = q(x)y^n$.
If $n = 0$, use Case 1. If $n = 1$, use Case 2. If $n = 2$, consider Case 8. If $n \neq 0, 1$, set $u = y^{1-n}$ to get a first order equation in u and x.

Linear equations of nth order with constant coefficients

A linear equation of the nth degree with constant coefficients has the form

$$c_n y^{(n)} + c_{n-1} y^{(n-1)} + \cdots + c_2 y'' + c_1 y' + c_0 y = q(x)$$

where the c_i are constants and y is an unknown function of x. The general solution of such an equation is the sum of two parts:

the complementary function (general solution of the homogeneous equation formed by taking $q(x) = 0$);

a particular integral (any solution of the given equation).

The complementary function is the sum of terms of the form

$$(a_{r-1} x^{r-1} + a_{r-2} x^{r-2} + \cdots + a_2 t^2 + a_1 t + a_0) e^{\lambda x}$$

where λ is a root of multiplicity r of the auxiliary polynomial

$$c_n t^n + c_{n-1} t^{n-1} + \cdots + c_2 t^2 + c_1 t + c_0$$

and a_i are arbitrary constants.

A particular integral can be found when $q(x)$ can be written as the sum of terms of the type $h(x)e^{\rho x}$ where $h(x)$ is a polynomial and ρ is complex. This includes the cases $q(x) = h(x)e^{\alpha x} \sin \beta x$ and $q(x) = h(x)e^{\alpha x} \cos \beta x$, with α and β real. When the c_i and the coefficients of $h(x)$ are real, solve with $q(x) = h(x)e^{(\alpha + i\beta)}x$ and take the real or imaginary parts, respectively, of the solution obtained.

Operational calculus: Let $Du = u'$. The left side of the equation can be written $p(D)y = q(x)$ where $p(t) = c_n t^n + \cdots + c_1 t + c_0$. For any polynomial $p(t)$, we have the operational rules

$$p(D)e^{rx} = p(r)e^{rx} \qquad \text{and} \qquad p(D)(ue^{rx}) = e^{rx} p(D + r)u \quad .$$

The following examples illustrate how a particular integral can be obtained without having to deal with special cases or undetermined coefficients:

(i) $y'' + 2y' + 2y = 2e^{-x} \sin x$.

The solution of this equation is the imaginary part of the solution of the following equation

$$(D^2 + 2D + 2)y = 2e^{(-1+i)x} \quad .$$

We try for a particular integral of the form $y = ue^{(-1+i)x}$, where the exponent agrees with the exponent on the right side of the equation. Then, factoring the polynomial in D and substituting for y, we obtain:

$$(D + 1 - i)(D + 1 + i)(ue^{(-1+i)x}) = 2e^{(-1+i)x} \quad .$$

Bringing the exponent through and cancelling it, we get

$$D(D + 2i)u = (D + 2i)Du = 2 \qquad (*)$$

Differentiate:

$$(D + 2i)D^2 u = 0 \qquad (**) \; .$$

Any u for which (*) and (**) hold will yields a particular integral. Choose u such that $u'' = 0$. Then (**) holds. To satisfy (*), we need $2iDu = 2$ or $Du = -i$. Hence take $u(x) = -ix$. A particular integral of the complex equation is $-ix \exp((-1 + i)x)$, and a particular integral of the original equation is

$$\text{Im } (-ixe^{(-1+i)x}) = -xe^{-x} \cos x \quad .$$

(ii) $y'' - 4y' + 4y = 8x^2 e^{2x} \sin 2x$

A particular integral of this equation is the imaginary part of the particular integral of the equation

$$(D-2)^2 y = 8x^2 e^{(2+2i)x}.$$

Let $y = ue^{(2+2i)x}$. Then, bring the exponential through the operator as before and cancelling, we get

$$(D+2i)^2 u = (D^2 + 4iD - 4)u = 8x^2 \quad (1)$$

$$(D^2 + 4iD - 4)Du = 16x \quad (2)$$

$$(D^2 + 4iD - 4)D^2 u = 16 \quad (3)$$

$$(D^2 + 4iD - 4)D^3 u = 0 \quad (4)\ .$$

Let $D^3 u = 0$ to satisfy (4). Then (3) requires $-4D^2 u = 16$ or $D^2 u = -4$. Now, (2) requires $-16i - 4Du = 16x$ or $Du = -16x - 16i$. Finally, to make (1) hold, we need $u = -2x^2 - 4ix + 3$. The solution to (ii) is thus

$$y = \mathrm{Im}(-2x^2 - 4ix + 3)e^{(2+2i)x} \quad .$$

Putnam questions

1997-B-2. Let f be a twice differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x)\ ,$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

1995-A-5. Let $x_1, x_2, \cdots, x_n$ be differentiable (real-valued) functions of a single variable t which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\cdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

for some constants $a_{ij} \geq 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \cdots, x_n$ necessarily linearly dependent?

1989-B-3. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a nonnegative integer, define

$$\mu_n = \int_0^\infty x^n f(x)dx$$

(sometimes called the nth moment of f).

a. Express μ_n in terms of μ_0.

b. Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that this limit is 0 only if $\mu_0 = 0$.

1988-A-2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

GEOMETRY

Putnam problems

1998-A-1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

1998-A-2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the $x-$axis and B be the area of the region lying to the right of the $y-$axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position of s.

1998-A-5. Let $\mathfrak{F}$ be a finite collection of open discs in $\mathbf{R}^2$ whose union contains a set $E \subseteq \mathbf{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, D_2, \cdots, D_n$ in $\mathfrak{F}$ such that

$$\bigcup_{j=1}^{n} 3D_j \supseteq E \ .$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

1998-A-6. Let A, B, C denote distinct points with integer points with integer coordinates in $\mathbf{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1 \ ,$$

then A, B, C are three vertices of a square. Here $[XY]$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

1998-B-2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the $x-$axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

1998-B-3. Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P a regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin\alpha + B\cos\beta$, where A, B, α and β are real numbers.

1997-A-1. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

1997-B-6. The dissection of the $3-4-5$ triangle shown below has diameter $5/2$.

Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between [airs of points belonging to the same part.)

1996-A-1. Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed into that rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangles.

1996-A-2. Let C_1 and C_2 be circles whose centers are 10 units apart and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exist points x on C_1 and Y on C_2 such that M is the midpoint of the lines segment XY.

1996-B-6. Let $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \cdots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0) \quad .$$

1995-B-2. An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin(x/a)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?

Other problems

1. A large sheet of paper is ruled by horizontal and vertical lines spaced a distance of 1 cm. A clear plastic sheet is free to slide on top of it. Some ink has been spilled on the sheet, making one or more blots. The total area of these blots is less than 1 sq. cm. Prove that the plastic sheet can be positioned so that none of the intersections of the horizontal and vertical lines is covered by any of the blots.

2. A rectangular sheet of paper is laid upon a second rectangular sheet of identical size as indicated in the diagram. Prove that the second sheet covers at least half the area of the first sheet.

3. Three plane mirrors that meet at a point are mutually perpendicular. A ray of light reflects off each mirror exactly once in succession. Prove that the initial and final directions of the ray are parallel and opposite.

4. Let $ABCD$ be a square with E the midpoint of CD. The vertex B is folded up to E and the page is flattened to produce a straight crease. If the fold along this crease also takes A to F, prove that EF intersects AD in a point that trisects the side AD.

5. A closed curve is drawn in the plane which may intersect itself any finite number of times. The curve passes through each point of self-intersection exactly twice. Suppose that the points of self-intersection

are labelled A, B, C, $\cdots$. Beginning at any (nonintersection) point on the curve, trace along the curve recording each intersection point in turn as you pass through it, until you return to the starting point. Each intersection point will be recorded exactly twice. Prove that between the two occurrences of any intersection point are *evenly* many points of intersection.

For the example below, starting at P and proceeding to the right, we encounter the points $ABCADFGEFGEDBC$ in order before returning to P.

6. Let A, B, C be three points in the plane, any pair of which are unit distance apart. For each point P, we can determine a triple of nonnegative real numbers (u, v, w), where u, v, w are the respective lengths of PA, PB, PC. Of course, not every triple of nonnegative reals arise in this way, and when two of the numbers are given, there are at most finitely many possibilities for the third. This suggests that there must be a relationship among them.

   (a) Find a polynomial equation that must be satisfied by u, v and w as described above.

   (b) If we take fixed values of u and v and regard the equation in (a) as a polynomial in w, analyze the character of its roots and relate this to the geometry of the situation.

7. A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

   and

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 \ .$$

GROUP THEORY AND AXIOMATICS

The following concepts should be reviewed: group, order of groups and elements, cyclic group, conjugate elements, commute, homomorphism, isomorphism, subgroup, factor group, right and left cosets.

Lagrange's Theorem: The order of a finite group is exactly divisible by the order of any subgroup and by the order of any element of the group.

A group of prime order is necessarily commutative and has no proper subgroups.

A subset S of a group G is a set of *generators* for G iff every element of G can be written as a product of elements in S and their inverses. A *relation* is an equation satisfied by one or more elements of the group. Many Putnam problems are based on the possibility that some relations along with the axioms will imply other relations.

Putnam problems

1997-A-4. Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element a in G such that $\psi(x) = a\phi(x)$ is a homomorphism (that is, $\psi(xy) = \psi(x)\psi(y)$ for all x and y in G).

1996-A-4. Let S be a set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that:

1. $(a, b, c) \in S$ if and only if $(b, c, a) \in S$,

2. $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$,

3. (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

Prove that there exists a one-to-one function $g : A \to \mathbf{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$. [Note: $\mathbf{R}$ is the set of real numbers.]

1989-B-2. Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \cdots\}$ is finite. Must S be a group?

1978-A-4. A "bypass" operation on a set S is a mapping from $S \times S$ to S with the property

$$B(B(w, x), B(y, z)) = B(w, z)$$

for all w, x, y, z in S.

(a) Prove that $B(a, b) = c$ implies $B(c, c) = c$ when B is a bypass.

(b) Prove that $B(a, b) = c$ implies $B(a, x) = B(c, x)$ for all x in S when B is a bypass.

(c) Construct a table for a bypass operation B on a finite set S with the following three properties: (i) $B(x, x) = x$ for all x in S. (ii) There exists d and e in S with $B(d, e) = d \neq e$. (iii) There exists f and g in S with $B(f, g) \neq f$.

1977-B-6. Let H be a subgroup with h elements in a group G.Suppose that G has an element a such that, for all x in H, $(xa)^3 = 1$, the identity. In G, let P be the subset of all products $x_1ax_2a \cdots x_na$, with n a positive integer and the x_i in H.
(a) Show that P is a finite set.
(b) Show that, in fact, P has no more than $3h^2$ elements.

1976-B-2. Suppose that G is a group generated by elements A and B, that is, every element of G can be written as a finite "word" $A^{n_1}B^{n_2}A^{n_3} \cdots B^{n_k}$, where $n_1, n_2, \cdots n_k$ are any integers, and $A^0 = B^0 = 1$, as usual. Also, suppose that

$$A^4 = B^7 = ABA^{-1}B = 1\ , \quad A^2 \neq 1\ , \quad \text{and} \quad B \neq 1\ .$$

(a) How many elements of G are of the form C^2 with C in G?
(b) Write each such square as a word in A and B.

1975-B-1. In the additive group of ordered pairs of integers (m, n) (with addition defined componentwise), consider the subgroup H generated by the three elements

$$(3, 8) \qquad (4, -1) \qquad (5, 4)\ .$$

Then H has another set of generators of the form

$$(1, b) \qquad (0, a)$$

for some integers a, b with $a > 0$. Find a.

1972-B-3. Let A and B be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$ and $B^{2n-1} = 1$ for some positive integer n. Prove $B = 1$.

1969-B-2. Show that a finite group can not be the union of two of its proper subgroups. Does the statement remain true if "two" is replaced by "three"?

1968-B-2. A is a subset of a finite group G, and A contains more than one half of the elements of G. Prove that each element of G is the product of two elements of A.

Other problems

1. A set S is nonnegative real numbers is said to be *closed under* $\pm$ iff, for each x, y in S, either $x + y$ or $|x - y|$ belongs to S. For instance, if $\alpha > 0$ and n is a positive integer, then the set

$$S(n, \alpha) \equiv \{0, \alpha, 2\alpha, \cdots, n\alpha\}$$

has the property. Show that every finite set closed under $\pm$ is either $\{0\}$, is of the form $S(n, \alpha)$, or has exactly four elements.

2. S is a set with a distinguished element u upon which an operation $+$ is defined that, for all $a, b, c \in S$, satisfies these axioms:
   (a) $a + u = a$;
   (b) $a + a = u$;
   (c) $(a + c) + (b + c) = a + b$.

   Define $a * b = a + (u + b)$. Prove that, for all $a, b, c \in S$,

$$(a * b) * c = a * (b * c) \quad .$$

3. Suppose that a and b are two elements of a group satisfying $ba = ab^2$, $b \neq 1$ and $a^{31} = 1$. Determine the order of b.

FIELDS

Putnam problems

1987-B-6. Let F be the field of p^2 elements where p is an odd prime. Suppose S is a set of $(p^2 - 1)/2$ distinct nonzero elements of F with the property that for each $\alpha \neq 0$ in F, exactly one of α and $-\alpha$ is in S. Let N be the number of elements in the intersection $S \cap \{2\alpha : \alpha \in S\}$. Prove that N is even.

1979-B-3. Let F be a finite field having an odd number m of elements. Let $p(x)$ be an irreducible (*i.e.*, nonfactorable) polynomial over F of the form

$$x^2 + bx + c \qquad\qquad b, c \in F \quad .$$

For how many elements k in F is $p(x) + k$ irreducible over F?

ALGEBRA

Putnam problems

1997-B-4. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1 \quad >$$

1995-B-4. Evaluate

$$\sqrt[8]{2207-\frac{1}{2207-\frac{1}{2207-\cdots}}}\quad .$$

Express your answer in the form $(a+b\sqrt{c})/d$, where a,b,c,d are integers.

1993-B-2. For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k)=\sum_{j=0}^{n}\binom{n}{j}\binom{n}{k-2j}\ ,$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a\geq 0$, $\binom{a}{b}=\frac{a!}{b!(a-b)!}$ for $0\leq b\leq a$ and $\binom{a}{b}=0$ otherwise.)

Other problems

1. Solve the equation

$$\sqrt{x+5}=5-x^2\quad .$$

2. The numbers $1,2,3,\cdots$ are placed in a triangular array and certain observations concerning row sums are made as indicated below:

					1					
				2		3				
			4		5		6			
		7		8		9		10		
	11		12		13		14		15	
16		17		18		19		20		21

$$\begin{aligned}1&=(0+1)(0^2+1^2)\\5&=1^2+2^2\\15&=(1+2)(1^2+2^2)\\34&=2\times(1^2+4^2)\\65&=(2+3)(2^2+3^2)\\111&=3\times(1^2+6^2)\end{aligned}$$

$$1=1^4\qquad\qquad 1+15=2^4\qquad\qquad 1+15+65=3^4$$

$$5=1+2^2=1(1^2+2^2)$$

$$5+34=3+6^2=(1+2)(2^2+3^2)$$

$$5+34+111=6+12^2=(1+2+3)(3^2+4^2)$$

Formulate and prove generalizations of these observations.

3. Let n be a positive integer and x a real number not equal to a positive integer. Prove that

$$\frac{n}{x}+\frac{n(n-1)}{x(x-1)}+\frac{n(n-1)(n-2)}{x(x-1)(x-2)}+\cdots+\frac{n(n-1)(n-2)\cdot\cdots\cdot 1}{x(x-1)(x-2)\cdots(x-n+1)}=\frac{n}{x-n+1}\quad .$$

4. Determine a value of the parameter θ so that

$$f(x) \equiv \cos^2 x + \cos^2(x+\theta) - \cos x \cos(x+\theta)$$

is a constant function of x.

MATRICES, DETERMINANTS AND LINEAR ALGEBRA

1996-B-4. For any square matrix A, we can define $\sin A$ by the usual power series

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1} \quad .$$

Prove or disprove: There exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix} \quad .$$

1995-B-3. To each positive integer with n^2 decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

1994-A-4. Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

1994-B-4. For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \qquad \text{and} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

Show that $\lim_{n\to\infty} d_n = \infty$.

1992-B-5. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{matrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ . & . & . & . & \cdots & . \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{matrix}$$

Is the set

$$\{\frac{D_n}{n!} : n \geq 2$$

bounded?

1992-B-6. Let M be a set of real $n \times n$ matrices such that
(i) $I \in M$, where I is the $n \times n$ identity matrix;
(ii) if $A \in M$ and $B \in M$, then either $AB \in M$ and $-AB \in M$, but not both;
(iii) if $A \in M$ and $B \in M$, then either $AB = BA$ or $AB = -BA$;
(iv) if $A \in M$ and $A \neq I$, then there is at least one $B \in M$ such that $AB = -BA$.

Prove that M contains at most n^2 matrices.

1991-A-2. Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

1990-A-5. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

1986-A-4. A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:
(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.
(b) The sum of the n entries of a transversal is the same for all transversals of A.
An example of such a matric A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad .$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4 \quad .$$

where the a_i's and b_i's are rational numbers.

1986-B-6. Suppose that A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n \times n$ identity matrix, M^t is the transpose of M. Prove that $A^tD - C^tB = I$.

1985-B-6. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \le i \le r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \text{tr } (M_i) = 0$, where tr (A) denotes the trace of the matrix A. Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.

COMBINATORICS

Putnam problems

1997-A-2. Players $1, 2, 3, \cdots, n$ are seated around a table and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who then passes two players to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

1996-A-3. Suppose that each of twenty students has made a choice of anywhere from zero to six courses from a total of six courses offered. Prove or disprove: There are five students and two courses such that all five have chosen both courses or all five have chosen neither.

1996-B-1. Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with a proof, the number of subsets of $\{1, 2, \cdots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets if selfish.

1996-B-5. Given a finite string S of symbol X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S **balanced** if every substring T of (consecutive symbols of) S has $-2 \le \Delta(T) \le 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

1995-A-4. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n - 1$. Prove that we can cut the necklace to form a string whose consecutive labels

$x_1, x_2, \cdots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \qquad \text{for} \;\; k = 1, 2, \cdots, n \; .$$

1995-B-1. For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

1995-B-5. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking **either**

a. one bean from a heap, provided at least two beans are left behing in that heap, **or**

b. a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

1994-A-3. Show that if the points of an isosceles right triangle of side length 1 are earch colored with one of four colors, then there must be two points of the same color which are at least a distance $2-\sqrt{2}$ apart.

1994-A-6. Let f_1, f_2, $\cdots$, f_{10} be bijections of the set of integers such that for each integer n, there is some composition $f_{1_1} \circ f_{1_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathfrak{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}} : e_i = 0 \;\; \text{or} \;\; 1 \;\; \text{for} \;\; 1 \leq i \leq 10\}$$

(f_i^0 is the identity function and $f_i^1 = f_i$). Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathfrak{F}$ map A to itself.

1993-A-3. Let $\mathfrak{P}_\mathfrak{n}$ be the set of subsets of $\{1,2,\cdots,n\}$. Let $c(n,m)$ be the number of functions $f : \mathfrak{P}_n \rightarrow \{1,2,\cdots.m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n,m) = \sum_{j=1}^{m} j^n \; .$$

1992-B-1. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

Other problems

1. Let n and k be positive integers. Determine the number of ways of choosing k numbers from $\{1,2,\cdots,n\}$ so that no three consecutive numbers appear in any choice.

2. There are n safes and n keys. Each key opens exactly one safe and each safe is opened by exactly one key. The keys are locked in the safes at random, with one key in each safe. k of the safes are broken open and the keys inside retrieved. What is the probability that the remaining safes can be opened with keys?

3. A class with at least 35 students goes on a cruise. Seven small boats are hired, each capable of carrying 300 kilograms. The combined weight of the students is 1800 kilograms. It is determined that any group of 35 students can fit into the boats without exceeding the individual capacity of any of them. Prove that it is unnecessary to prevent any student from taking the cruise.

4. Each can of K_9-Food costs 3 dollars and contains 10 units of protein and 10 units of carbohydrate. Each can of Pooch-Mooch costs 5 dollars and has 10 units of protein, 20 units of carbohydrate and 5 units of vitamins. Each puppy needs a daily feed with 45 units of protein, 60 units of carbohydrate and 5 units of vitamins. How would you feed adequately 10 puppies for 10 days in the cheapest way?

5. A rectangle is partitioned into smaller rectangles (not necessarily congruent to one another) with sides parallel to to those of the large rectangle. Each small rectangle has at least one side an integer number of units long. Prove that the large rectangle does also.

6. During a long speech, each member of the audience fell asleep exactly twice. For any pair of auditors, there was a moment when both of them were asleep. Prove that there must have been a moment during the speech when at least a third of the audience were asleep.

7. A group of students with ages ranging from 17 to 23, inclusive, with at least one student of each age, represents 11 universities. Prove that there are at least 5 students such that each has more members of the group of the same age than members from the same university.

8. On a $2n \times 2n$ chessboard, $3n$ squares are chosen at random. Prove that n rooks (castles) can be placed on the board so that each chosen square is either occupied by a rook or under attack from at least one rook. (Note that each rook can attack any square in the same row or column which is visible from the rook.)

9. At a party, every two people greet each other in exactly one of four ways (nodding, shaking hands, kissing, hugging). Candy kisses Randy, but not Sandy. For every three people, their three pairwise greetings are either all the same or all different. What is the maximum number of people at the party?

PROBABILITY

1995-A-6. Suppose that each of n people writes down the numbers 1, 2, 3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that, for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

1993-B-2. Consider the following game played with a deck fof $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing the number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. If we assume optimal strategy by both A and B, what is the probability that A wins?

1993-B-3. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

1992-A-6. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)