Canadian Mathematical Olympiad - 2008

Solutions may be sent to
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provided you have not consulted any solutions to these problems from another source.

QUESTION 1

$ABCD$ is a convex quadrilateral for which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

QUESTION 2

Determine all functions f that take rational values for which

$$f(2f(x)+f(y)) = 2x + y \ ,$$

for rational x and y.

QUESTION 3

Let a, b, c be positive real numbers for which $a+b+c=1$. Prove that

$$\frac{a-bc}{a+bc} + \frac{b-ca}{b+ca} + \frac{c-ab}{c+ab} \leq \frac{3}{2} \ .$$

QUESTION 4

Find all functions $f : \mathbf{N} \rightarrow \mathbf{N}$ such that

$$(f(n))^p \equiv n \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

QUESTION 5

A *self-avoiding rook walk* on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, *i.e.*, the rook's path is non-self-intersecting).

Let $R(m,n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, $R(m,1) = 1$ for all natural numbers m; $R(2,2) = 2$; $R(3,2) = 4$; $R(3,3) = 11$. Find a formula for $R(3,n)$ for each natural number n.