Department of Education, Ontario

Annual Examinations, 1960

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Note 1: Ten questions constitute a full paper.

Note 2: A supply of squared paper and a book of mathematical tables may be obtained from the Presiding Officer.

1. (a) Find the greatest common divisor d of 405 and 864.

   (b) Determine two integers m, n, such that $d = 405m + 864n$.

   (c) Are the integers m and n of (b) unique. If not, what other values of m, n are possible?

2. Find the number of ways of arranging the letters a a a a a b b b c c d e f in a row, if the letters b are separated from one another.

3. Successive coefficients in the expansion of $(1 + x)^n$ where n is a positive integer are denoted by a_0, a_1, a_2, $\cdots$, a_n.

   (a) Prove that (i) $a_0 + a_1 + a_2 + \cdots + a_n = 2^n$, (ii) $a_1 + 2a_2 + 3a_3 + \cdots + na_n = n2^{n-1}$.

   (b) If b_0, b_1, b_2, $\cdots$, b_n are $n + 1$ successive points of an arithmetic progression with sum S, prove that

$$(n + 1)(a_0b_0 + a_1b_1 + a_2b_2 + \cdots + a_mb_n) = 2^nS \ .$$

4. (a) A polynomial of degree n with real coefficients is expressed as the product of factors with real coefficients of smallest possible degrees. What are the degrees of the factors?

   (b) Express the rational function

$$f(x) = \frac{7x^2 - 2x + 3}{x^4 - 3x^3 + x^2 - 3x}$$

as a sum of partial fractions, one associated with each factor of the denominator.

5. Plot the graphs of the following equations:

   (a) $y = x - 2|x|$,

   (b) $|x| + |y| + |y - x| = 2$.

6. Prove, analytically, that the medians of a triangle are concurrent.

7. (a) Show that every circle $C : x^2 + y^2 - 2ax = 1$ passes through two fixed points U, V.

   (b) If C cuts the $x-$axis in P_1, P_2, prove that $OP_1 \cdot OP_2$ is a constant.

   (c) Find the equations of a family of circles C' each of which cuts every circle C orthogonally.

   (d) In what sense do the points U, V of (a) belong to the family of circles C'?

8. The normal and the tangent at P on the ellipse with equation $x^2/a^2 + y^2/b^2 = 1$ meet the $x-$axis at Q and R. For what positions of P is the triangle PQR isosceles?

9. (a) Construct a periodic trigonometric function of x which vanishes at the point $x = \alpha$, has a maximum value of m, and has a period of p.

   (b) Draw the graph, for values of x in the interval $0 \leq x \leq 2\pi$, of

$$2 \sin \frac{1}{2} x + \frac{1}{2} \sin 2x \ .$$

10. A, B, C, D are four points on a line and O is a point not on the line. Show that

$$\frac{AB \cdot CD}{AD \cdot BC} = \frac{\sin AOB \cdot \sin COD}{\sin AOD \cdot \sin BOC} \ .$$

11. The sides of a triangle have the ratios $3 : 7 : 8$. Show that the angles of the triangle are in arithmetic progression.

12. A metal block weighing 20 pounds rests on a table. A string attached to the bottom of the block passes over a smooth pulley at the edge of the table and from it a weight of 5 pounds is suspended. The coefficient of friction between the block and the table is $1/2$. Taking these data to be exact, find to the nearest degree, the least angle at which the table top should be inclined in order that the block should slide.