Department of Education, Ontario

Annual Examinations, 1955

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. Prove that the equation of any parabola situated anywhere in the plane with axis in any direction is an equation of the second degree of which the quadratic terms form a perfect square.

2. Two equilateral hyperbolas are so situated that the asymptotes of one are the axes of the other. Prove that they intersect at right angles at all their common points.

3. Find the equation of the tangent to the curve $y^2 = x^3$ at the point (x_1, y_1) on the curve.

4. The equation $-y+kx = k^2$ represents a family of straight lines of which $-y+ux = u^2$ and $-y+vx = v^2$ are two members. Find the point of intersection of these two member lines and the limiting position of this point of intersection as v approaches u. Find the locus of this limiting position as u varies.

5. For a triangle ABC prove that
$$\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C} = 0 \ .$$

6. The angles x and y are allowed to vary subject only to the restriction $x + y = z$, where z is a fixed positive acute angle. Find the least and greatest values taken by the product $\sin x \sin y$.

7. For any triangle ABC show that
$$a\cos^2\frac{A}{2} + b\cos^2\frac{B}{2} + c\cos^2\frac{C}{2} = \frac{\Delta}{R} + s \ ,$$
where Δ is the area of the triangle, R is the radius of the circumscribing circle and $2s$ is the perimeter.

8. The ends of a light string are fastened one to each end of a uniform bar AB. The ratio of the length of the string to that of the bar is $5:3$. The weight of the bar is W and an equal weight W is fastened to the bar at B. The whole is suspended under gravity by passing the string over a smooth peg and allowing the system to come to rest with the bar inclined at an angle θ to the horizontal. Show that

$$\sin\theta = \frac{4\sqrt{3}}{9} \ .$$

9. A charitable organization plan to bestow the following:

   \$5000 five years from now,

   \$10000 ten years from now,

   \$15000 fifteen years from now,

   and so on, with perpetually increasing amounts. Find the total present value of the gifts, assuming an interest rate of 4% per annum.

10. Find the cubic equation whose roots are the squares of the roots of the equation $x^3 - x^2 + 3x - 10 = 0$.

11. An athletic committee of five is to be chosen from a class of 8 boys and 6 girls. It has been agreed that there must be at least one boy and one girl on the committee and that of John and Mary, who are twins, neither one or either one but not both may be chosen. Find the number of possible committees.

12. Show that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be expressed as the product of two linear factors, real or imaginary, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \ .$$