Department of Education, Ontario

Annual Examinations, 1954

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. If \( P \) and \( Q \) are two points on an ellipse whose centre is \( C \), show that the area of the triangle \( CPQ \) is greatest when \( P \) and \( Q \) are the extremities of a pair of conjugate diameters.

2. The following lines form the sides of a quadrilateral: \( L_1 \equiv x - 2y = 0; \ L_2 \equiv 2x - 3y + 4 = 0; \ L_3 \equiv 2x - y - 4 = 0; \ L_4 \equiv x + y = 0. \)
   (a) Prove that \( L_1 L_2 + kL_3 L_4 = 0 \) is a family of curves passing through the vertices of the quadrilateral.
   (b) Find the member of this family in whose equation the coefficient of \( xy \) is 0.

3. A straight line cuts a hyperbola at the points \( P \) and \( P' \) and its asymptotes at the points \( Q \) and \( Q' \). Prove that the midpoint of \( PP' \) is also the midpoint of \( QQ' \).

4. (a) The base \( BC \) of a triangle \( ABC \) is fixed and the angle \( B \) is double the angle \( C \). Find the locus of \( A \).
   (b) On the base of the triangle of (a) a segment of a circle containing the angle \( 180^\circ - A \) is drawn. Show that the angle \( A \) can be trisected by the use of this segment and the locus referred to in (a).

5. It is given that \( \sin(y + z - x), \ \sin(z + x - y) \) and \( \sin(x + y - z) \) are in arithmetic progression. Show that, when \( \tan x, \ \tan y, \) and \( \tan z \) exist, they are in arithmetic progression.

6. Given an acute angle \( A \), find the value of \( \theta \) in the range \([0, \pi]\) for which \( \sin \theta \cos(A - \theta) \) is greatest.

7. A semi-circle is described on the diameter \( AB \) of length \( 2a \), and from the centre \( O \) a
radius $OC$ is drawn, making an angle $2\theta$ with $OA$. Circles are inscribed in the triangles $OBC$ and $OAC$. Show that the distance between the centres of the inscribed circles is

$$\sqrt{\frac{2 - \sin 2\theta}{(1 + \sin \theta)(1 + \cos \theta)}}.$$ 

8. A uniform plank of length $2a$ and weight $W$ is balanced on a fixed circular cylinder whose axis is horizontal and perpendicular to the length of the plank. A weight $W'$ is attached to one end of the plank, which now seeks a new position of equilibrium. Show that the plank will not slide off the cylinder, provided $W' < W\frac{b\theta}{(a - b\theta)}$, where $b$ is the radius of the cylinder and $\tan \theta = \mu$ is the coefficient of friction between the plank and the cylinder.

9. Show that the sum of $n$ terms of the geometric progression $a + ar + ar^2 + \cdots$ and the sum of $n$ terms of the geometric progression $a - ar + ar^2 - \cdots$ have as product the sum of $n$ terms of the geometric progression $a^2 + a^2r^2 + a^2r^4 + \cdots$, provided $n$ is odd.

10. Given that $x$ is positive but different from 1, and also that $n$ is a positive integer, show that

$$\frac{x^n - 1}{n} < \frac{x^{n+1} - 1}{n+1}.$$ 

11. (a) Given that $x + a + \sqrt{a^2 - b} = 0$, where $x$ is not 0, verify that

$$x + \frac{b}{x} + 2a = 0.$$ 

(b) Given that $y = px + q$, where $x + a + \sqrt{a^2 - b} = 0$, verify that

$$y + (ap - q) + \sqrt{(ap - q)^2 - (bp^2 - 2apq + q^2)} = 0.$$ 

12. (a) Given that

$$T_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

for $n = 1, 2, 3, \cdots$, show that $T_n + T_{n+1} = T_{n+2}$.

(b) Verify that $T_1 = 1, T_2 = 1, T_3 = 2$, and deduce that $T_4 = 3$ and $T_5 = 5$. 

2