Junior Examinations, 1951

GRADE XIII

PROBLEMS

*To be taken only by candidates writing for certain University Scholarships involving Mathematics*

Ten questions constitute a full paper.

1. The side of a square and the radius of a circle vary in such a way that the sum of the perimeters of the square and the circle is a constant $k$. Find the radius of the circle when the sum of the areas of the square and the circle is a minimum.

2. Let $S_n$ be the sum of the first $n$ terms of a geometric progression. Express $S_{4n}$ as a fractional rational function of $S_{2n}$ and $S_n$.

3. For an ellipse $b^2 x^2 + a^2 y^2 = a^2 b^2$, the length of the subtangent corresponding to the point $(3, 12/5)$ is $16/3$. Find the eccentricity of the ellipse.

4. Find the equation of the locus of the middle points of the chords of the ellipse $b^2 x^2 + a^2 y^2 = a^2 b^2$ which are drawn from the positive end of the minor axis.

5. From the top of a hill the angle of depression of a point $P$ on the level plain below is $30$ degrees, and from a point three-quarters of the way down the hill the angle of the depression of $P$ is $15$ degrees. Assuming the data to be exact, find, to the nearest minute, the inclination of the hill.

6. A body of weight $W$ rests on a rough plane which is inclined at a constant angle to the horizontal. Two separate experiments show the following results:

(i) the least horizontal force that will cause the body to move up the plane is $P$ pounds;

(ii) the least force acting up the plane that will cause the body to move up the plane is $Q$ pounds.

If $\theta$ is the angle of friction, prove that

$$\cos \theta = \frac{PW}{Q\sqrt{P^2 + W^2}}.$$
7. Show that the equation
\[ \frac{1}{x + 2} + \frac{1}{y + 2} = \frac{1}{2} + \frac{1}{z + 2} \]
is not satisfied by any set of positive integers \(x, y, z\) in which \(x\) is equal or greater than four. Hence find all the sets of positive integers \(x, y, z\) which satisfy the given equation.

8. Show that
\[ \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{3^2}\right) \left(1 + \frac{1}{3^4}\right) \left(1 + \frac{1}{3^8}\right) \left(1 + \frac{1}{3^{16}}\right) \left(1 + \frac{1}{3^{32}}\right) \]
differs from 1.6 by less than \(10^{-30}\).

9. Given that the equation \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\) represents two parallel straight lines, show that \(h^2 = ab\), \(bg^2 = af^2\).

10. A hyperbola is met by a diameter in a point \(A\) and the conjugate hyperbola is met by a conjugate diameter in a point \(B\). Show that the line \(AB\) is parallel to one asymptote and is bisected by the other.

11. If \(A, B,\) and \(C\) are three acute angles such that \(\cos A = \tan B\), \(\cos B = \tan C\), \(\cos C = \tan A\), prove that
\[ A = B = C = \sin^{-1} \left(2 \sin \frac{\pi}{10}\right) . \]

12. Find all the values of \(x\) in the range \(0^\circ \leq x \leq 180^\circ\) which satisfy the equation
\[ 2 \cos x - \sqrt{3} \sin x = \sin 2x . \]