Department of Education, Ontario

Annual Examinations, 1943

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. If \( n \) is a positive integer, prove that

\[
(1 - x)^{3n} + 3nx(1 - x)^{3n-2} + \frac{3n(3n - 3)}{1 \cdot 2} x^2 (1 - x)^{3n-4} + \cdots = (1 - x^3)^n.
\]

2. Solve the system:

\[
\begin{align*}
31x^2y^2 - 7y^4 - 112xy + 64 &= 0 \\
x^2 - 7xy + 4y^2 + 8 &= 0.
\end{align*}
\]

3. If the quadratic function \( 3x^2 + 2pxy + 2y^2 + 2ax - 4y + 1 \) can be resolved into factors linear in \( x \) and \( y \), prove that \( p \) must be a root of the equation \( p^2 + 4ap + 2a^2 + 6 = 0 \).

4. Sum to \( n \) terms the series whose \( n \)th term is

\[
\frac{n^4 + 2n^3 + n^2 - 1}{n^2 + n}.
\]

5. If \( C \) and \( D \) are two points on one branch of a hyperbola whose foci are \( A \) and \( B \), prove that, in general, \( A \) and \( B \) are on one branch of a hyperbola whose foci are \( C \) and \( D \).

Examine the case where \( C \) and \( D \) are on opposite branches of the hyperbola.

6. Starting at the vertex \( A \) of the parabola \( y^2 = 4px \), whose axis points east from \( A \), successive chords \( AB, BC, CD, \ldots \), are drawn pointing alternately north-east (\( AB, CD, \ldots \)) and south-east (\( BC, \ldots \)). Find the co-ordinates of the end of the \( n \)th chord.

7. Show that the locus of intersections of tangents to an ellipse \( ax^2 + by^2 = 1 \) at the ends of perpendicular diameters is the ellipse \( a^2 x^2 + b^2 y^2 = a + b \).
8. On a fixed line are three fixed points \( A, B, \) and \( C \). A line \( AQR \) turns about \( A \), while \( Q \) and \( R \) remain each at a fixed distance from \( A \). Find the locus of the intersection of \( BQ \) and \( CR \).

9. From a point \( c \) feet above the surface of a lake the angle of elevation of a cloud is \( \alpha \), and the angle of depression of its reflection in the lake is \( \beta \). Show that the height of the cloud above the lake is \( c \sin(\beta + \alpha) \cos(\beta - \alpha) \) feet.

10. If \( O_1, O_2, O_3 \) are the centres of the three escribed circles of a triangle \( ABC \), prove that the area of the triangle \( O_1O_2O_3 \) is

\[
\Delta \left( 1 + \frac{a}{b + c - a} + \frac{b}{c + a - b} + \frac{c}{a + b - c} \right)
\]

where \( \Delta \) is the area of triangle \( ABC \).

11. A triangle \( ABC \) is such that \( 3AB = 2AC \). Also a point \( D \) on \( BC \) is such that \( BD = 2DC \) and \( AD = BC \). Show that

\[
\tan \frac{\angle ADB}{2} = \sqrt{\frac{5}{19}}.
\]

12. If \( A, B, C \) are the angles of a triangle, prove that

\[
1 < \cos A + \cos B + \cos C \leq \frac{3}{2}.
\]