Problems.

668. The nonisosceles right triangle $ABC$ has $\angle CAB = 90^\circ$. The inscribed circle with centre $T$ touches the sides $AB$ and $AC$ at $U$ and $V$ respectively. The tangent through $A$ of the circumscribed circle meets $UV$ produced in $S$. Prove that
(a) $ST \parallel BC$;
(b) $|d_1 - d_2| = r$, where $r$ is the radius of the inscribed circle and $d_1$ and $d_2$ are the respective distances from $S$ to $AC$ and $AB$.

669. Let $n \geq 3$ be a natural number. Prove that
\[1989 | n^n - n^n,\]
i.e., the number on the right is a multiple of 1989.

670. Consider the sequence of positive integers $\{1, 12, 123, 1234, 12345, \cdots\}$ where the next term is constructed by lengthening the previous term at the right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with “carrying” occurring as in addition. Thus, the ninth and tenth terms of the sequence are 123456789 and 1234567900 respectively. Determine which terms of the sequence are divisible by 7.

671. Each point in the plane is coloured with one of three distinct colours. Prove that there are two points that are unit distant apart with the same colour.

672. The Fibonacci sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. The real number $\tau$ is the positive solution of the quadratic equation $x^2 = x + 1$.
(a) Prove that, for each positive integer $n$, $F_{-n} = (-1)^{n+1}F_n$.
(b) Prove that, for each integer $n$, $\tau^n = F_n\tau + F_{n-1}$.
(c) Let $G_n$ be any one of the functions $F_{n+1}F_n$, $F_{n+1}F_{n-1}$ and $F_n^2$. In each case, prove that $G_{n+2} + G_n = 2(G_{n+1} + G_{n+1})$. 

673. ABC is an isosceles triangle with \( AB = AC \). Let \( D \) be the point on the side \( AC \) for which \( CD = 2AD \). Let \( P \) be the point on the segment \( BD \) such that \( \angle APC = 90^\circ \). Prove that \( \angle ABP = \angle PCB \).

674. The sides \( BC, CA, AB \) of triangle \( ABC \) are produced to the points \( R, P, Q \) respectively, so that \( CR = AP = BQ \). Prove that triangle \( PQR \) is equilateral if and only if triangle \( ABC \) is equilateral.

**Solutions.**

654. Let \( ABC \) be an arbitrary triangle with the points \( D, E, F \) on the sides \( BC, CA, AB \) respectively, so that
\[
\frac{BD}{DC} \leq \frac{BF}{FA} \leq 1
\]
and
\[
\frac{AE}{EC} \leq \frac{AF}{FB}.
\]
Prove that \( [DEF] \leq \frac{1}{4}[ABC] \), with equality if and only if two at least of the three points \( D, E, F \) are midpoints of the corresponding sides.

(Note: \([XYZ]\) denotes the area of triangle \( XYZ \).)

**Solution 1.** Let \( BF = \mu BA \), \( BD = \lambda BC \) and \( CE = \nu CA \).

The conditions are that
\[
\lambda \leq \mu \leq \frac{1}{2} \quad \text{and} \quad 1 - \nu \leq 1 - \mu \quad \text{or} \quad \mu \leq \nu.
\]
We observe that \([BDF] = \lambda \mu [ABC] \).

To see this, let \( BG = \lambda BA \). Then
\[
[BDF] = \frac{\mu}{\lambda} [BDG] = \frac{\mu}{\lambda} \lambda^2 [ABC] = \mu \lambda [ABC].
\]
Similarly \([AFE] = (1 - \mu)(1 - \lambda)[ABC] \) and \([DEC] = \nu (1 - \lambda)[ABC] \).

Hence
\[
[DEF] = (1 - \lambda \mu - (1 - \mu)(1 - \nu) - \nu(1 - \lambda))[ABC]
= (\mu - \mu \nu - \mu \lambda + \nu \lambda)[ABC]
= \left(\frac{1}{4} - (\frac{1}{2} - \mu)^2 - (\mu - \lambda)(\nu - \mu)\right)[ABC] \leq \frac{1}{4}[ABC]
\]
with equality if and only if \( \mu = 1/2 \) and either \( \lambda = \mu = 1/2 \) or \( \nu = \mu = 1/2 \). The result follows.

**Solution 2.** Let \( G \) be on \( AC \) so that \( FG \parallel BC \). Then, since \( \frac{AE}{EC} \leq \frac{AF}{FB} \), \( E \) lies in the segment \( AG \).

Since \( \frac{BD}{DC} \leq \frac{BF}{FA} \), \( DF \) produced is either parallel to \( AC \) or meets \( CA \) produced at a point \( X \) beyond \( A \). Hence the distance from \( G \) to \( FD \) is not less than the distance from \( E \) to \( FD \), so that \([DEF] \leq [FGD]\).

The area of \([FGD]\) does not change as \( D \) varies along \( BC \). To maximize \([DEF]\) is suffices to consider the special case of triangle \([FGD]\). Let \( AF = xAB \). Then \( FG = xBC \) and the heights of \( \Delta DFG \) and \( \Delta ABC \) are in the ratio \( 1 - x \). Hence
\[
\frac{[DFG]}{[ABC]} = x(1 - x)
\]
which is maximized when \( x = \frac{1}{2} \). The result follows from this, with \([DEF]\) being exactly one quarter of \([ABC]\) when \( F \) and \( G \) are the midpoints of \( AB \) and \( AC \) respectively.
Solution 3. Set up the situation as in the second solution. Let \( BF = tFA \). Then \( AB = (1 + t)FA \), and the height of the triangle \( FGD \) is \( t/(1 + t) \) times the height of the triangle \( ABC \). Hence

\[
[DEF] \leq [FGD] = \frac{t}{(1 + t)^2}[ABC].
\]

Now

\[
\frac{1}{4} - \frac{t}{(1 + t)^2} = \frac{(1 - t)^2}{4(1 + t)^2} \geq 0
\]

so that \( t(1 + t)^{-2} \leq 1/4 \) and the result follows. Equality occurs if and only if \( t = 1 \) and \( E = G \), i.e., \( F \) and \( E \) are both midpoints of their sides.

655. (a) Three ants crawl along the sides of a fixed triangle in such a way that the centroid (intersection of the medians) of the triangle they form at any moment remains constant. Show that this centroid coincides with the centroid of the fixed triangle if one of the ants travels along the entire perimeter of the triangle.

(b) Is it indeed always possible for a given fixed triangle with one ant at any point on the perimeter of the triangle to place the remaining two ants somewhere on the perimeter so that the centroid of their triangle coincides with the centroid of the fixed triangle?

(a) Solution. Recall that the centroid lies two-thirds of the way along the median from a vertex of the triangle to its opposite side. Let \( ABC \) be the fixed triangle and let \( RS \parallel AC \) and \( TU \parallel BA \) with \( PQ \parallel BC \) and \( TS \parallel BA \) with \( MN \parallel BC \) respectively; \( PQ \) and \( BC \) are both midpoints of their sides.

Observe, for example, that if \( A, X, Y \) are collinear and \( X \) and \( Y \) lie on \( PQ \) and \( BC \) respectively, then \( AX : XY = 2 : 3 \). It follows from this that, if one ant is at \( A \), then the centroid of the triangle formed by the three ants lies inside \( \Delta APQ \) (otherwise the midpoint of the side opposite the ant at \( A \) would not be in \( \Delta ABC \)). Similarly, if one ant is at \( B \) (respectively \( C \)) then the centroid of the ants’ triangle lies within \( \Delta BRS \) (respectively \( \Delta CTU \)). Thus, if one ant traverses the entire perimeter, the centroid of the ants’ triangle must lie inside the intersection of these three triangles, the singleton \( \{G\} \). The result follows.

(b) Solution 1. Suppose the vertices of the triangle are given by the planar vectors \( a, b \) and \( c \); the centroid of the triangle is at \( 1/3(a + b + c) \). Suppose that one ant is placed at \( ta + (1 - t)b \) for \( 0 \leq t \leq 1 \). Place the other two ants at \( tb + (1 - t)c \) and \( tc + (1 - t)a \). The centroid of the ants’ triangle is at

\[
\frac{1}{3}[(ta + (1 - t)b) + (tb + (1 - t)c) + (tc + (1 - t)a)] = \frac{1}{3}(a + b + c).
\]

(b) Solution 2. If one ant is at a vertex, then we can replace the remaining ants at the other vertices of the fixed triangle. Suppose, wolog, the ant is at \( X \) in the side \( BC \).

Let \( MN \) be the line joining the midpoints \( M \) and \( N \) of \( AB \) and \( AC \) respectively; \( MN \parallel BC \). Let \( XG \) meet \( MN \) at \( W \). Since \( BG : BN = CG : CM = 2 : 3 \), it follows, by considering the similar triangles \( BGX \) and \( NGW \), that \( XG : XW = 2 : 3 \). Hence the midpoint of the segment joining the other two ants’ positions must be at \( W \). Thus, the problem now is to find points \( Y \) and \( Z \) on the perimeter of \( \Delta ABC \) such that \( W \) is the midpoint of \(YZ\). We use a continuity argument.

Let \( UV \) be any segment containing \( W \) whose endpoints lie on the perimeter of \( \Delta ABC \). Let \( Y \) travel counterclockwise around the perimeter from \( U \) to \( V \), and let \( Z \) be a point on the perimeter such that \( W \) lies on \( YZ \). When \( Y \) is at \( U \), \( YW : WZ = VW : WV \), while when \( Y \) is at \( V \), \( YW : WZ = VW : WU \). Hence \( YW : WZ \) varies continuously from a certain ratio to its reciprocal, so there must be a position for which \( YW = WZ \).

(b) Solution 3. [A. Panayotov] Suppose that the triangle has vertices at \( (0,0), (1,0) \) and \( (u,v) \), so that its centroid is at \( (\frac{1}{3}(1 + u), \frac{1}{3}) \). Wolog, let one ant be at \( (a,0) \) where \( 0 \leq a \leq 1 \). Put the second ant at \((u,v)\).
Then we will place the third ant at a point \((b, 0)\) on the \(x\)-axis. We require that \(\frac{1}{3}(a + b + u) = \frac{1}{3}(1 + u)\), so that \(b = 1 - a\). Clearly, \(0 \leq b \leq 1\) and the result follows.

656. Let \(ABC\) be a triangle and \(k\) be a real constant. Determine the locus of a point \(M\) in the plane of the triangle for which

\[|MA|^2 \sin 2A + |MB|^2 \sin 2B + |MC|^2 \sin 2C = k.\]

**Solution.** Let \(O\) and \(R\) be the circumcentre and circumradius, respectively, of triangle \(ABC\). We have that

\[|MA|^2 = |MA|^2 = |MO + OA|^2 = |MO|^2 + |OA|^2 + 2MO \cdot OA\]

with similar expressions for \(MB\) and \(MC\). Therefore, we have that

\[|MA|^2 \sin 2A + |MB|^2 \sin 2B + |MC|^2 \sin 2C = (|MO|^2 + R^2)(\sin 2A + \sin 2B + \sin 2C) + 2MO \cdot (OA \sin 2A + OB \sin 2B + OC \sin 2C).\]

Now

\[
\sin 2A + \sin 2B + \sin 2C = \sin 2A + \sin 2B - \sin(2A + 2B)
\]

\[= \sin 2A(1 - \cos 2B) + \sin 2B(1 - \cos 2A)
\]

\[= 2 \sin A \cos A(2 \sin^2 B) + 2 \sin B \cos B(2 \sin^2 A)
\]

\[= 4 \sin A \sin B \sin(A + B) = 4 \sin A \sin B \sin C
\]

\[= 2[ABC] = 2R^2 \sin A \sin B \sin C,\]

since \([ABC] = \frac{1}{2}ab \sin C = 2R^2 \sin A \sin B \sin C\).

Also, we have that

\[\overline{OA} \sin 2A + \overline{OB} \sin 2B + \overline{OC} \sin 2C = \overline{O}.\]

To see this, let \(P\) be the intersection of the line \(AO\) with the side \(BC\) of the triangle. Observe that \(\angle BOP = 180^\circ - 2\angle ACB\), \(\angle COP = 180^\circ - 2\angle ABC\), \(\angle OBC = \angle OCB = 90^\circ - \angle BAC\). Applying the Law of Sines to triangle \(OPC\) yields that

\[
\frac{|OP|}{\sin(90^\circ - A)} = \frac{|OC|}{\sin(2C + A - 90^\circ)}.
\]

Since \(|OC| = R\), we find that

\[
|OA| = \frac{\cos(2C + A)}{\cos A}, |OP| = \frac{-2 \sin A \cos(2C + A)}{2 \sin A \cos A} |OP|
\]

\[= \frac{\sin 2B + \sin 2C}{\sin 2A} |OP|,
\]

so that

\[\overline{OA} = -\frac{\sin 2B + \sin 2C}{\sin 2A} \overline{OP}.\]

Applying the Law of Sines in triangle \(BOP\) and \(COP\), we obtain that

\[
\frac{|OP|}{\sin(90^\circ - A)} = \frac{|BP|}{\sin 2C}.
\]

4
and 
\[
\frac{|OP|}{\sin(90^\circ - A)} = \frac{|CP|}{\sin 2B}.
\]
Therefore \(|BP| \sin B = |CP| \sin 2C\), so that
\[
\sin 2BPB = -\sin 2CPC.
\]
and
\[
\overrightarrow{OA} \sin 2A + \overrightarrow{OB} \sin 2B + \overrightarrow{OC} \sin 2C = -(\sin 2B + \sin 2C)\overrightarrow{OP} + \sin 2BOB + \sin 2COC \overrightarrow{OP} = \sin 2BPB + \sin 2CPC \overrightarrow{OP} = \overrightarrow{O}.
\]
Therefore \((|MO|^2 + R^2)(2[ABC]/R^2) = k\) so that
\[
|MO|^2 = \frac{k - 2[ABC]}{2[ABC]} R^2.
\]
Therefore, when \(k < 2[ABC]\), the locus is the empty set. When \(k = 2[ABC]\), the locus consists solely of the circumcentre. When \(k > 2[ABC]\), the locus is a circle concentric with the circumcircle.

657. Let \(a, b, c\) be positive real numbers for which \(a + b + c = abc\). Find the minimum value of
\[
\sqrt{1 + \frac{1}{a^2}} + \sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{c^2}}.
\]

**Solution 1.** By repeated squaring it can be shown that
\[
\sqrt{x^2 + u^2} + \sqrt{y^2 + b^2} \geq \sqrt{(x + u)^2 + (y + v)^2},
\]
for \(x, y, u, v \geq 0\). Applying this inequality yields that
\[
\sqrt{1 + \frac{1}{a^2}} + \sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{c^2}} \geq \sqrt{(1 + 1)^2 + (\frac{1}{a} + \frac{1}{b})^2} + \sqrt{1 + \frac{1}{c^2}} \\
\geq \sqrt{(2 + 1)^2 + (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}.
\]
The given condition implies that \(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\), whereupon
\[
\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \geq 2 + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 2 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 3.
\]
It follows that the given expression is not less than \(2\sqrt{3}\), with equality occurring if and only if \(a = b = c = \sqrt{3}\).

**Solution 2.** [S. Sun] Using the inequality \(x^2 + y^2 + z^2 \geq xy + yz + zx\) for real \(x, y, z\), we find that the square of the quantity in question is not less than
\[
3 \left(\sqrt{1 + \frac{1}{a^2}} \sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{b^2}} \sqrt{1 + \frac{1}{c^2}} + \sqrt{1 + \frac{1}{c^2}} \sqrt{1 + \frac{1}{a^2}}\right).
\]
From the Arithmetic-Geometric Means Inequality, we find that
\[
\sqrt{1 + \frac{1}{a^2}} \sqrt{1 + \frac{1}{b^2}} = \sqrt{1 + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{a^2b^2}} \geq \sqrt{1 + \frac{2}{ab} + \frac{1}{a^2b^2}} = 1 + \frac{1}{ab},
\]

with similar inequalities for the other products. Since
\[
\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{a + b + c}{abc} = 1,
\]
we find that the square of the quantity in question is not less than \(3 \times 4 = 12\), so that the quantity has the minimum value \(2\sqrt{3}\), attainable if and only is \(a = b = c = \sqrt{3}\).

**Solution 3.** Let \(A, B, C\) be acute angles for which \(a = \tan A, b = \tan B\) and \(c = \tan C\). Then
\[
c = \frac{a + b}{1 - ab} = \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan(A + B) = \tan(\pi - A - B),
\]
so that \(C = \pi - A - B\). Substituting these values for \(a, b, c\) into the given expression yields
\[
\csc A + \csc B + \csc C.
\]
Since the cosecant function is convex in the interval \((0, \pi/2)\), by Jensen’s inequality, we deduce that
\[
\csc A + \csc B + \csc C \geq 3 \csc \left( \frac{A + B + C}{3} \right) = 3 \csc \frac{\pi}{3} = 2\sqrt{3},
\]
with equality if and only if \(A = B = C = \frac{\pi}{3}\). Thus, the minimum of the given expression is equal to \(2\sqrt{3}\) with equality if and only is \(a = b = c = \sqrt{3}\).

**658.** Prove that \(\tan 20^\circ + 4 \sin 20^\circ = \sqrt{3}\).

**Solution 1.** [C.J. Bao] Since
\[
(\sqrt{3}/2) \cos 20^\circ - (1/2) \sin 20^\circ = \sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ = \sin 40^\circ = 2 \sin 20^\circ \cos 20^\circ,
\]
it follows that
\[
\sqrt{3} \cos 20^\circ = \sin 20^\circ + 4 \sin 20^\circ \cos 20^\circ a.
\]
Division by \(\cos 20^\circ\) yields the desired result.

**Solution 2.** Let \(ABC\) be a triangle with \(\angle ABC = 60^\circ\) and \(\angle CAB = 30^\circ\). Let \(ABD\) be a triangle on the same side of \(AB\) with \(\angle ABD = 40^\circ\) and \(\angle DAB = 50^\circ\). Suppose that \(AC\) and \(BD\) intersect at \(E\), and that the length of \(BC\) is 1, so that the respective lengths of \(CA\) and \(AB\) are \(\sqrt{3}\) and 2. Then
\[
|AD| = |AB| \sin 40^\circ = 4 \sin 20^\circ \cos 20^\circ
\]
and
\[
|AE| = |AD| \sec 20^\circ = |AB| \cos 50^\circ \sec 20^\circ = 2 \sin 40^\circ \sec 20^\circ = 4 \sin 20^\circ.
\]
However, \(|CE| = |BC| \tan 20^\circ = \tan 20^\circ\). Therefore
\[
\tan 20^\circ + 4 \sin 20^\circ = |CE| + |AE| = |AC| = \sqrt{3}.
\]

**Solution 3.** [M. Essafty]
\[
\tan 20^\circ + 4 \sin 20^\circ = \frac{\sin 20^\circ + 4 \sin 20^\circ \cos 20^\circ}{\cos 20^\circ}
\]
\[
= \frac{\sin 20^\circ + 2 \sin 40^\circ}{\cos 20^\circ}
\]
\[
= \frac{\sin(30^\circ - 10^\circ) + 2 \sin(30^\circ + 10^\circ)}{\cos(30^\circ - 10^\circ)}
\]
\[
= \frac{3 \sin 30^\circ \cos 10^\circ + \sin 10^\circ \cos 30^\circ}{\cos 30^\circ \cos 10^\circ + \sin 30^\circ \sin 10^\circ}
\]
\[
= \frac{3 \cos 10^\circ + \sqrt{3} \sin 10^\circ}{\sqrt{3} \cos 10^\circ + \sin 10^\circ} = \sqrt{3}.
\]
Solution 4.

\[
\tan 20^\circ + 4 \sin 20^\circ = \frac{\sin 20^\circ + 4 \sin 20^\circ \cos 20^\circ}{\cos 20^\circ} = \frac{\sin 20^\circ + 2 \sin 40^\circ}{\cos 20^\circ}
= \frac{\sin 40^\circ + 2 \sin 30^\circ \cos 10^\circ}{\cos 20^\circ}
= \frac{2 \sin 60^\circ \cos 20^\circ}{\cos 20^\circ} = \sqrt{3}.
\]

Solution 5.

\[
\tan 20^\circ + 4 \sin 20^\circ = \frac{\sin 20^\circ + 4 \sin 20^\circ \cos 20^\circ}{\cos 20^\circ} = \frac{\sin 20^\circ + 2 \sin 40^\circ}{\cos 20^\circ}
= \frac{\sin 50^\circ \cos 30^\circ - (1/2) \cos 50^\circ + 2 \sin 40^\circ}{\cos 20^\circ}
= \frac{\sin 50^\circ \cos 30^\circ + (1/2) \cos 50^\circ + \cos 50^\circ}{\cos 20^\circ}
= \frac{\sin 80^\circ + \cos 50^\circ}{\cos 20^\circ}
= \frac{2 \sin 30^\circ \cos 20^\circ}{\cos 20^\circ} = \sqrt{3}.
\]

Solution 6. Let \(a = \cos 20^\circ\). Then, using the de Moivre formula \(\cos 3 \theta + i \sin 3 \theta = (\cos \theta + i \sin \theta)^3\) with \(\theta = 20^\circ\), we find that
\[
\frac{1}{2} = \cos 60^\circ = 4a^3 - 3a
\]
and
\[
\frac{\sqrt{3}}{2} = 3 \sin 20^\circ - 4 \sin^3 20^\circ = \sin 20^\circ (3 - 4(1 - a^2)) = \sin 20^\circ (4a^2 - 1).
\]
Therefore
\[
\tan 20^\circ + 4 \sin 20^\circ - \sqrt{3} = \sin 20^\circ [(1/a) + 4 - 8a^2 + 2] = a^{-1} \sin 20^\circ (1 + 6a - 8a^3) = 0.
\]

Solution 7. [B. Wu]

\[
\tan 60^\circ - \tan 20^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} - \frac{\sin 20^\circ}{\cos 20^\circ}
= \frac{\sin 40^\circ}{\cos 60^\circ \cos 20^\circ} = 4 \sin 20^\circ \cos 40^\circ \over \cos 20^\circ = 4 \sin 20^\circ,
\]
whence \(\tan 20^\circ + 4 \sin 20^\circ = \sqrt{3}\).

659. (a) Give an example of a pair \(a, b\) of positive integers, not both prime, for which \(2a - 1, 2b - 1\) and \(a + b\) are all primes. Determine all possibilities for which \(a\) and \(b\) are themselves prime.

(b) Suppose \(a\) and \(b\) are positive integers such that \(2a - 1, 2b - 1\) and \(a + b\) are all primes. Prove that neither \(a^b + b^a\) nor \(a^a + b^b\) are multiples of \(a + b\).

(a) First solution. \((a, b) = (3, 2)\) yields \(2a - 1 = 5, 2b - 1 = 3\) and \(a + b = 5\); \((a, b) = (3, 4)\) yields \(2a - 1 = 5, 2b - 1 = 7\) and \(a + b = 7\). Suppose that \(a\) and \(b\) are primes. Then for \(a + b\) to be prime, \(a + b\) must be odd, so that one of \(a\) and \(b\), say \(b\), is equal to 2. Thus, we require the \(a + 2\) and \(2a - 1\), along with \(a\), to be prime. This is true when \(a = 3\).
Now suppose $a$ is an odd prime exceeding 3. Then $a \equiv \pm 1 \pmod{6}$, so the only way $a$ and $a+2$ can both be prime is for $a \equiv -1 \pmod{6}$, whence $2a - 1 \equiv -3 \pmod{6}$. Thus, 3 divides $2a - 1$, and since $2a - 1 \geq 9$, $2a - 1$ must be composite.

(b) Solution 1. We first recall a bit of theory. Let $p$ be a prime. By Fermat’s Little Theorem, $a^{p-1} \equiv 1 \pmod{p}$ whenever $\gcd(a, p) = 1$. Let $d$ be the smallest positive integer for which $a^d \equiv \pm 1 \pmod{p}$. Then $d$ divides $p - 1$, and indeed divides any positive integer $k$ for which $a^k \equiv \pm 1 \pmod{p}$. Now to the problem.

Since $a + b$ is prime, $a \neq b$. Wolog, let $a > b$ and let $p = a + b$. Then $a \equiv -b \pmod{p}$, so that

$$a^b + b^a \equiv (-b)^b + b^a \equiv b^b((-1)^b + b^{a-b}) .$$

Suppose, if possible, that $p$ divides $a^b + b^a$. Then, since $b < p$, $\gcd(b, p) = 1$ and so $b^{a-b} \equiv (-1)^{b+1} \pmod{p}$. It follows that

$$b^{2b-1} = b^{(p-1) - (a-b)} \equiv (-1)^{b+1} \pmod{p} .$$

Now $2b - 1$ is prime, so that $2b - 1$ must be the smallest exponent $d$ for which $b^d \equiv \pm 1 \pmod{p}$. Hence $2b - 1$ divides $a - b$, so that for some positive integer $c$, $a - b = c(2b - 1)$, whence $a = b + 2bc - c$ and so

$$2a - 1 = 2b - 1 + (2b - 1)2c = (2b - 1)(2c + 1) .$$

But $2a - 1$ is prime and $2b - 1 > 1$, so $2c + 1 = 1$ and $c = 0$. This is a contradiction. Hence $p$ does not divide $a^b + b^a$.

Similarly, using the fact that $a^b + b^a \equiv (-b)^a + b^b \equiv b^b((-1)^a b^{a-b} + 1)$, we can show that $p$ does not divide $a^a + b^b$.

(b) Solution 2. [M. Boase] Suppose that $a$ and $b$ exist as specified. Exactly one of $a$ and $b$ is odd, since $a + b$ is prime. Let it be $a$. Modulo $a + b$, we have that

$$0 \equiv a^b + b^a = a^b + (-a)^a \equiv a^b - a^a \equiv a^{a(b-a)} \text{ or } a^b(1 - a^{a-b})$$

according as $a < b$ or $a > b$. Hence $a^{b-a} \equiv \equiv 0 \pmod{a+b}$. Now $a + b - 1 \equiv |b-a| = 2a - 1$ or $2b - 1$, and $a^{a+b-1} \equiv 1 \pmod{a+b}$ (by Fermat’s Little Theorem). Hence $a^{2a-1} \equiv a^{2b-1} \equiv 1 \pmod{a+b}$. Both $2a - 1$ and $2b - 1$ exceed 1 and are divisible by the smallest value of $m$ for which $a^m \equiv 1 \pmod{a+b}$. Since both are prime, $2a - 1 = 2b - 1 = m$, whence $a = b$, a contradiction. A similar argument can be applied to $a^a + b^b$.

(c) Solution 3. Suppose, if possible, that one of $a^b + b^a$ and $a^a + b^b$ is divisible by $a + b$. Then $a + b$ divides their product $a^a + b^b + (ab)^a + (ab)^b + a^{a+b}$. By Fermat’s Little Theorem, $a^{a+b} + b^{a+b} \equiv a+b \equiv 0 \pmod{a+b}$, so that $(ab)^a + (ab)^b \equiv 0 \pmod{a+b}$. Since $a + b$ is prime, it is odd and so $a \neq b$. Wolog, let $a > b$. Then

$$(ab)^a + (ab)^b \equiv (ab)^b[(ab)^{a-b} + 1]$$

and $\gcd(a, a+b) = \gcd(b, a+b) = 1$, so that $(ab)^{a-b} + 1 \equiv 0 \pmod{a+b}$. Since $(ab)^{a+b-1} \equiv 1 \pmod{a+b}$, it follows that $a^{a+b-1} \equiv (ab)^{2b-1} \equiv -1 \pmod{a+b}$. As in the foregoing solution, it follows that $a = b$, and we get a contradiction.

660. $ABC$ is a triangle and $D$ is a point on $AB$ produced beyond $B$ such that $BD = AC$, and $E$ is a point on $AC$ produced beyond $C$ such that $CE = AB$. The right bisector of $BC$ meets $DE$ at $P$. Prove that $\angle DBP = \angle BAC$.

Solution 1. Let the lengths $a, b, c, u$ and the angles $\alpha, \beta, \gamma, \lambda, \mu, \nu$ be as indicated in the diagram.

In the solution, we make use of the fact that if $p/q = r/s$, then both fractions are equal to $(p+r)/(q+s)$. Since $\angle DBP = 90^\circ + \lambda - 2\beta$, it follows that

$$2\mu = 180^\circ - (90^\circ - \alpha) - (90^\circ + \lambda - 2\beta) = \alpha + 2\beta - \lambda .$$
Similarly, \( 2\nu = \alpha + 2\gamma - \lambda \). Using the Law of Sines, we find that
\[
\frac{a}{\sin 2\alpha} = \frac{b}{\sin 2\beta} = \frac{c}{\sin 2\gamma} = \frac{b + c}{\sin(\beta + \sin 2\gamma)} = \frac{b + c}{2\cos \alpha \cos(\beta - \gamma)}.
\]
Hence
\[
\frac{a}{\sin \alpha} = \frac{b + c}{\cos(\beta - \gamma)}.
\]
Since \( a = 2u \sin \lambda \) and, by the Law of Sines,
\[
\frac{u}{\sin(90^\circ - \alpha)} = \frac{b}{\sin 2\mu} \quad \text{and} \quad \frac{u}{\sin(90^\circ - \alpha)} = \frac{c}{\sin 2\nu},
\]
we have that
\[
\frac{a}{2 \sin \lambda \cos \alpha} = \frac{u}{\cos \alpha} = \frac{b}{\sin 2\mu} = \frac{c}{\sin 2\nu} = \frac{b + c}{\sin 2\mu + \sin 2\nu} = \frac{b + c}{2 \sin(\mu + \nu) \cos(\mu - \nu)} = \frac{a}{2 \cos \lambda \cos(\beta - \gamma)} = \frac{a}{2 \cos \lambda \sin \alpha}.
\]
Hence \( \tan \alpha = \tan \lambda \) and so \( \alpha = \lambda \).

Solution 2. Let \( M \) be the midpoint of \( BC \). A rotation of \( 180^\circ \) about \( M \) interchanges \( B \) and \( C \) and takes \( E \) to \( G \), \( D \) to \( F \) and \( P \) to \( Q \). Then \( AB = CE = BG \) and \( AC = BD = CF \). Join \( GA \) and \( FA \). Let \( 2\alpha = \angle BAC \). Since \( A E \parallel BG \) and \( AB \) is a transversal, \( \angle GBA = \angle BAC = 2\alpha \). Since \( AB = BG \), \( \angle BGA = 90^\circ - \alpha \). But \( \angle BGF = \angle CED = 90^\circ - \alpha \). Thus, \( G, A, F \) are collinear.

Since \( GF \) and \( DE \) are equidistant from \( M \), we can use Cartesian coordinates with the origin at \( M \), the line \( y = 1 \) as \( GF \) and the line \( y = -1 \) as \( DE \). Let \( A \sim (a, 1) \), \( B \sim (-u, -mu) \), \( C \sim (u, mu) \). Then \( P \sim (m, -1) \), \( Q \sim (-m, 1) \),
\[
D \sim (a - \frac{2(a + u)}{1 + mu}, -1), \quad E \sim (a + \frac{2(a + u)}{1 + mu}, -1).
\]
Since \( |AC| = |BD| \), we find that \( u - a = -u - a + \frac{2(a + u)}{1 + mu} \), or \( a = mu^2 \). (We can check this by equating the slopes of \( AC \) and \( AE \).)

The slope of \( AE \) is \( -1/u \) and of \( AD \) is \( 1/u \), so that
\[
\tan \angle BAC = \frac{-2/u}{1 - (1/u^2)} = -\frac{2u}{u^2 - 1}.
\]
The slope of \( CQ \) is \((mu - 1)/(m + u)\) and of \( BQ \) is \((1 + mu)/(u - m)\), so that
\[
\tan \angle BPC = \tan \angle BQC = \frac{(mu - 1)(u - m) - (mu + 1)(u + m)}{(u - m)(u + m) + (mu - 1)(mu + 1)} = \frac{-2(m^2u + u)}{u^2 - m^2 + m^2u^2 - 1} = \frac{-2(m^2 + 1)u}{(1 + m^2)(u^2 - 1)} = \frac{-2u}{u^2 - 1}.
\]
The result follows.

Solution 3. [M. Boase] Let \( XAY \) be drawn parallel to \( DE \).
Since $M$ is the midpoint of $BC$, the distance from $M$ to $DE$ is the average of the distances from $B$ and $C$ to $DE$. Similarly, the distance from $M$ to $XY$ is the average of the distances from $B$ and $C$ to $XY$. The distance of $B$ (resp. $C$) to $DE$ equals the distance of $C$ (resp. $B$) to $XY$. Hence, $M$ is equidistant from $DE$ and $XY$. If $PM$ produced meets $XY$ in $Q$, then $PM = MQ$ and so $\angle BQC = \angle BPC$.

Select $R$ on $MQ$ (possibly produced) so that $\angle BAC = \angle BRC$. Since $\triangle ADE \parallel \triangle RBC$, $\angle RBC = \angle ADE$. Since $BARC$ is a concyclic quadrilateral, $\angle BAR = 180^\circ - \angle RCB = 180^\circ - \angle ADE = 180^\circ - \angle XAD = \angle BAQ$ from which it follows that $R = Q$ and so $\triangle BPC = \triangle BQC = \angle BRC = \angle BAC$.

Solution 4. [Jimmy Chui] Set coordinates: $A \sim (0, (m + n)b)$, $B \sim (-ma, nb)$, $C \sim (na, mb)$ $D \sim (-m + n)a, 0)$ and $E \sim ((m + n)a, 0)$ where $m = |AB|$, $n = |AC|$ and $a^2 + b^2 = 1$. Then the line $BC$ has the equation

$$\frac{m - n}{a}x - \frac{m + n}{b}y + m^2 + n^2 = 0$$

and the right bisector of $BC$ has equation

$$\frac{m + n}{b}x + \frac{m - n}{a}y + \frac{(a^2 - b^2)(m^2 - n^2)}{2ab} = 0$$

Thus $P \sim \left(\frac{(b^2 - a^2)(m - n)}{2a}, 0\right)$.

Now

$$|BC|^2 = m^2 + n^2 + 2mn(a^2 - b^2)$$

and

$$|BP|^2 = \frac{m^2 + n^2 + 2mn(a^2 - b^2)}{4a^2}$$

so that $|BC|/|BP| = 2a$. Also $|DE|/|AD| = 2(m + n)a / (m + n) = 2a$ so that $\triangle BPC$ is similar to $\triangle ADE$ and the result follows.

Solution 5. Determine points $L$ and $N$ on $DE$ such that $BL \parallel AE$ and $LN = NE$. Now

$$\frac{LE}{LD} = \frac{AB}{BD} = \frac{CE}{CA}$$

so that $CL \parallel AD$ and $CL : AD = CE : AE$. Since $AD = DE$, $CL = CE$ and so $CN \perp LE$. Consider the trapezoid $CBLE$. The line $MN$ joins the midpoints of the nonparallel opposite sides and so $MN \parallel BL$. $MPNC$ is a quadrilateral with right angles at $M$ and $N$, and so is concyclic. Hence

$$\angle BPC = 2\angle MPC = 2\angle MNC = 2\angle NCE = \angle LCE = \angle BAC$$

Solution 6. [C. So] Let $F, N, G$ be the feet of the perpendiculars dropped from $B, M, C$ respectively to $DE$. Note that $FN = NG$, so that $MF = MG$. Let $\angle ADE = \angle AED = \theta$, $|AB| = c$, $|AC| = b$ and $h$ be the altitude of $\triangle ADE$. Then

$$|MN| = \frac{1}{2}[|BF| + |CG|] = \frac{1}{2}(b + c)\sin \theta = \frac{h}{2}$$

and

$$|DF| = b \cos \theta$$,

$$|GE| = c \cos \theta$$,

$$|DE| = 2(b + c) \cos \theta$$

Hence $|FG| = |DE| - |DF| - |GE| = \frac{1}{2}|DE|$. Since $\triangle ADE$ and $\triangle MFG$ are isosceles triangles with heights and bases in proportion, they are similar so that $\triangle MFG = \angle ADE = \theta$. Since $\angle BFP = \angle BMG = 90^\circ$, the quadrilateral $BFPN$ is concyclic and so $\angle CBP = \angle MFP = \theta$ (we are supposing that the configuration is labelled so $P$ lies between $F$ and $E$). Hence $\triangle ADE$ is similar to $\triangle PCB$ and so $\angle BPC = \angle BAC$.  10
Solution 7. [A. Chan] Let $\angle ADE = \angle AED = \theta$, so $\angle BAC = 180^\circ - 2\theta$. Suppose that $\angle ACB = \phi$, $\angle CPE = \sigma$ and $\angle BCP = \rho$. By the Law of Sines for triangles $ABC$ and $PCE$, we find that

$$\frac{2|PC| \cos \rho}{\sin 2\theta} = \frac{|AB|}{\sin \phi}$$

whence

$$\sin \sigma = \frac{|CE|}{|PC|} = \frac{|AB|}{|PC|} = \frac{2 \cos \rho \sin \phi}{\sin 2\theta}$$

and

$$\sin \sigma \cos \theta = \sin \phi \cos \rho .$$

Therefore

$$\sin(\theta + \sigma) + \sin(\sigma - \theta) = \sin(\phi + \rho) + \sin(\phi - \rho).$$

Since $\theta + \sigma = \phi + \rho$, $\sin(\sigma - \theta) = \sin(\phi - \rho)$. Either $(\sigma - \theta) + (\phi - \rho) = \pm 180^\circ$ or $\sigma - \theta = \phi - \rho$. In the first case, since $\theta + \sigma = \phi + \rho$, $|\sigma - \rho| = 90^\circ$, which is false.

Hence $\sigma - \theta = \phi - \rho$, so, with $\theta + \sigma = \phi + \rho$, we have that

$$2\theta = \theta + (\rho + \sigma - \phi) = \theta + (\rho + \rho - \sigma) = 2\rho$$

and the result follows.

Solution 8. [A. Murali] Let $F$ be the midpoint of $BC$. Observe that triangles $ADE$ and $PBC$ are isosceles with $AD = AE$ and $PB = PC$. Suppose that the line parallel to $AC$ through $D$ and the line parallel to $AD$ through $C$ meet at $N$, and let $CN$ intersect $DE$ at $M$. Since $ACND$ is a parallelogram, $DN = AC$. Since triangle $CME$ is similar to triangle $ADE$, it is isosceles with $CM = CE = AB$. Since $AD = CN$, $BMND$ is a parallelogram. In fact, $MN = BD = AC = DN = BM$, so that $BMND$ is a rhombus.

Since $P$ is a point on a diagonal of the rhombus $BMND$, $PB = PN$ and so triangles $PBM$ and $PNM$ are congruent, from which we see that $\angle PBM = \angle PNM$. Since $PC = PB = PN$, it follows that $\angle PBM = \angle PNC = \angle PCM$ and quadrilateral $BCMP$ is concyclic. Therefore, $\angle BPC = \angle BMC = \angle BAC (ABMC$ being a quadrilateral).

Solution 9. [C. Deng] If $BC$ were parallel to $DE$, then $BC$ would be a midline of triangle $ADE$ and $P$ would be the reflection of $A$ in the axis $BC$ yielding the desired result. Suppose that $BC$ and $DE$ are not parallel. Let $R$ be the circumradius of triangle $ADE$, $R_1$ the circumradius of triangle $BDP$ and $R_2$ the circumradius of triangle $CPE$. Observe that $AD = AE$ and $PB = PC$.

Let the circumcircles of triangles $BDP$ and $CEP$ intersect at $O$. The point $O$ lies inside triangle $ADE$. By the Extended Sine Law,

$$\frac{OP}{\sin \angle PBO} = 2R_1 = \frac{PB}{\sin \angle ADE} = \frac{PC}{\sin \angle AED} = 2R_2 = \frac{OP}{\sin \angle PCO} .$$

Since $\angle PCO = \angle PEO < \angle PEA < 90^\circ$, the angle $PCO$ is acute. Similarly, angle $PBO$ is acute. Therefore $\angle PBO = \angle PCO$, so that $\angle OBC = \angle OCB$ and $O$ is on the right bisector of $BC$. Since

$$DO = 2R_1 \sin \angle DPO = 2R_2 \sin \angle OPE = EO ,$$

the point $O$ is on the right bisector of $DE$, which is also the angle bisector of $\angle BAC$.

Since the quadrilaterals $OBDP$ and $OCEP$ are concyclic,

$$\angle BOC = 360^\circ - \angle BOP - \angle COP$$

$$= 36^\circ - (180^\circ - \angle BDP) - (180^\circ - \angle CEP)$$

$$= \angle ADE + \angle AED = 180^\circ - \angle BAC .$$
Hence quadrilateral $ABOC$ is concyclic. Also $\angle BCO = \angle CBO = \frac{1}{2} \angle BAC$.

From Ptolemy’s Theorem, we have that
\[ BC \cdot AO = AB \cdot CO + AC \cdot BO = (AB + AB \cdot BO = AD \cdot BO . \]
Therefore
\[ AO = AD \cdot \frac{BO}{BC} = AD \cdot \frac{\sin \angle BCO}{\sin BOC} = AD \cdot \frac{\sin \frac{1}{2} \angle BAC}{2 \sin \frac{1}{2} \angle BAC} = \frac{AD}{2 \cos \frac{1}{2} \angle BAC} = R . \]

Since $O$ is on the right bisector of $DE$ and $AO = R$, $O$ is the circumcentre of triangle $ADE$. Therefore
\[ \angle BPC = \angle BPO + \angle CPO = \angle BDO + \angle CEO = \angle OAB + \angle OAC = \angle A . \]