OLYMON

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Please send your solutions to

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individually as you solve the problems. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

New procedure. Instead of the monthly problem sets with a long deadline, I plan to send out one or two problems a week which should be solved as soon as you can. I will record the problems in order of receipt and acknowledge solvers. Solutions will be published when there is no more activity on a problem. This month, I will pose the challenge problems for which solutions have been received to give anyone else a chance to solve them before solutions appear.

For those of you who are looking for practice problems, you can access old Olymon problems and solutions on the website *www.math.utoronto.ca/barbeau/home.html* or *www.cms.math.ca*; on the CMS website, you can also access International Mathematical Talent Search Problems as well as problems posed on the Canadian Open Mathematics Challenge and the Canadian Mathematical Olympiad.

Problems for May, 2009

619. [Solved 2/3/09 by Jonathan Schneider] Suppose that $n > 1$ and that S is the set of all polynomials of the form

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \ ,$$

whose coefficients are complex numbers. Determine the minimum value over all such polynomials of the maximum value of $|p(z)|$ when $|z| = 1$.

620. [Solved 2/3/09 by Jonathan Schneider, 4/3/09 by Cameron Bruggeman, 18/4/09 by Hao Sun, and 27/4/09 by Ahmad Abdi] Let $a_1, a_2, \cdots, a_n$ be distinct integers. Prove that the polynomial

$$p(z) = (z - a_1)^2 (z - a_2)^2 \cdots (z - a_n)^2 + 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients.

621. [Solved 2/3/09 by Jonathan Schneider and 27/4 by Ahmad Abdi] Determine the locus of one focus of an ellipse reflected in a variable tangent to the ellipse.

622. [Solved 10/4/09 by Robin Cheng] Let I be the centre of the inscribed circle of a triangle ABC and let u, v, w be the respective lengths of IA, IB, IC. Let P be any point in the plane and p, q, r the respective lengths of PA, PB, PC. Prove that, with the sidelengths of the triangle given conventionally as a, b, c,

$$ap^2 + bq^2 + cr^2 = au^2 + bv^2 + cw^2 + (a + b + c)z^2 \ ,$$

where z is the length of IP.

623. [Solved 12/4 by Jonathan Schneider] Given the parameters a, b, c, solve the system

$$x + y + z = a + b + c;$$

$$x^2 + y^2 + x^2 = a^2 + b^2 + c^2;$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 \ .$$

624. [Solved 12/4/09 by Jonathan Schneider] Suppose that $x_i \geq 0$ and

$$\sum_{i=1}^{n} \frac{1}{1 + x_i} \leq 1 \ .$$

Prove that

$$\sum_{i=1}^{n} 2^{-x_i} \leq 1 \ .$$

Problem of the Week for May 3 - 9.

625. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

Challenge problem

C13. Determine all positive integers x and y for which $x^2 + 7 = 2^y$.

Solutions

605. Prove that the number $299 \cdots 998200 \cdots 029$ can be written as the sum of three perfect squares of three consecutive numbers, where there are $n - 1$ nines between the first 2 and the 8, and $n - 1$ zeros between the last pair of twos.

Solution. Let $a - 1$, a, $a + 1$ be the three consecutive numbers. The sum of their square is $3a^2 + 2$; setting this equal to the given number yields

$$\begin{aligned} a^2 &= 9 \cdot 10^{2n+1} + \cdots + 9 \cdot 10^{n+3} + 9 \cdot 10^{n+2} + 4 \cdot 10^{n+1} + 9 \\ &= (10^n - 1)10^{n+2} + 4 \cdot 10^{n+1} + 9 = 10^{2n+2} - 6 \cdot 10^{n+1} + 9 \\ &= (10^{n+1} - 3)^2 \ , \end{aligned}$$

so that $a = 10^{n+1} - 3$.

606. Let $x_1 = 1$ and let $x_{n+1} = \sqrt{x_n + n^2}$ for each positive integer n. Prove that the sequence $\{x_n : n > 1\}$ consists solely of irrational numbers and calculate $\sum_{k=1}^{n} \lfloor x_k^2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer that does not exceed x.

Solution. We prove that x_n is nonrational as well as positive for $n \geq 2$. Note that x_2 is nonrational. Suppose that $n \geq 2$ and that x_{n+1} were rational; then $x_n = x_{n+1}^2 - n^2$ would also be rational; repeating this would lead to x_2 being rational and a contradiction.

Observe that, for any positive integer $n \geq 2$,

$$x_n = \sqrt{x_{n-1} + (n-1)^2} > n - 1 \ .$$

We prove by induction that $x_n < n$. This is true for $n = 2$. If $x_{n-1} < n - 1$, then

$$x_n^2 = x_{n-1} + (n-1)^2 < (n-1)n < n^2 \ ,$$

and the desired result follows. Thus, for each $n \geq 2$, $\lfloor x_n \rfloor = n - 1$,

For $n \geq 3$,

$$\lfloor x_n^2 \rfloor = \lfloor x_{n-1} + (n-1)^2 \rfloor = (n-2) + (n-1)^2 = n^2 - n - 1 = n(n-1) - 1 \ .$$

Therefore

$$\begin{aligned}
\sum_{k=1}^{n} \lfloor x_k^2 \rfloor &= \lfloor x_1^2 \rfloor + \lfloor x_2^2 \rfloor + \sum_{k=3}^{n} \lfloor x_k^2 \rfloor \\
&= 3 + \left[(\sum_{k=3}^{n} k(k-1) \right] - (n-2) \\
&= 5 - n + \frac{1}{3} \sum_{k=3}^{n} [(k+1)k(k-1) - k(k-1)(k-2)] \\
&= 5 - n + \frac{1}{3}[(n+1)n(n-1) - 6] = 3 - n + \frac{1}{3}(n^3 - n) \\
&= \frac{1}{3}(n^3 - 4n + 9) \ ,
\end{aligned}$$

607. Solve the equation

$$\sin x \left(1 + \tan x \tan \frac{x}{2} \right) = 4 - \cot x \ .$$

Solution. For the equation to be defined, x cannot be a multiple of π, so that $\sin x \neq 0$. Rearranging the terms of the equation and manipulating yields that

$$\begin{aligned}
4 &= \cot x + \sin x \left(\frac{\cos x \cos \frac{x}{2} + \sin x \sin \frac{x}{2}}{\cos x \cos \frac{x}{2}} \right) \\
&= \cot x + \sin x \left(\frac{\cos(x - (x/2))}{\cos x \cos(x/2)} \right) \\
&= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \\
&= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{2}{\sin 2x} \ ,
\end{aligned}$$

whence $\sin 2x = \frac{1}{2}$. Therefore $x = (-1)^k \frac{\pi}{12} + \frac{k\pi}{2}$, where k is an integer.

608. Find all positive integers n for which n, $n^2 + 1$ and $n^3 + 3$ are simultaneously prime.

Solution. If $n = 2$, then the numbers are 2, 5 and 11 and all are prime. Otherwise, n must be odd. But in this case, the other two numbers are even exceeding 2 and so nonprime. Therefore $n = 2$ is the only possibility.

609. The first term of an arithmetic progression is 1 and the sum of the first nine terms is equal to 369. The first and ninth terms of the arithmetic progression coincide respectively with the first and ninth terms of a geometric progression. Find the sum of the first twenty terms of the geometric progression.

Solution. The sum of the first nine terms of an arithmetic progression is equal to 9/2 the sum of the first and ninth terms, from which it is seen that the ninth term is 81. Let r be the common ratio of the geometric progression whose first term is 1 and whose ninth term is 81. Then $r^8 = 81$, whence $r = \pm\sqrt{3}$. The sum of the first twenty terms of the geometric progression is $\frac{1}{2}(3^{10} - 1)(\pm\sqrt{3} + 1)$.

610. Solve the system of equations

$$\log_{10}(x^3 - x^2) = \log_5 y^2$$
$$\log_{10}(y^3 - y^2) = \log_5 z^2$$
$$\log_{10}(z^3 - z^2) = \log_5 x^2$$

where $x, y, z > 1$.

Solution. For $x > 1$, let

$$f(x) = 5^{\log_{10}(x^3 - x^2)} .$$

The three equations are $f(x) = y^2$, $f(y) = z^2$ and $f(z) = x^2$. Since $x^3 - x^2 = x^2(x - 1)$ is increasing, f is an increasing function. If, say, $x < y$, then $y < z$ and $z < x$, yielding a contradiction. Thus, we can only have that $x = y = z$ and so

$$\log_{10}(x^3 - x^2) = \log_5 x^2 .$$

Let $2t = \log_5 x^2$ so that $t > 0$, $x^2 = 5^{2t}$ and so $x = 5^t$. Therefore

$$5^{3t} - 5^{2t} = 10^{2t} \Longrightarrow 5^t - 1 = 4^t \Longrightarrow 5^t - 4^t = 1 .$$

Since $5^t - 4^t = 4^t[(5/4)^t - 1]$ is an increasing function of t, we see that the equation for t has a unique solution, namely $t = 1$. Therefore $x = 5$.

611. The triangle ABC is isosceles with $AB = AC$ and I and O are the respective centres of its inscribed and circumscribed circles. If D is a point on AC for which $ID \| AB$, prove that $CI \perp OD$.

Solution. Since ABC is isosceles, the points A, O, I lie on the right bisector of BC. Let AO meet BC at P, DI meet BC at E, DO meet BC at F and CI meet DF at Q.

Suppose that angle A is less than $60°$. Then O lies between I and A, and Q lies within triangle APB. Since $DE \| AB$ and O is the centre of the circumcircle of ABC, we have that

$$\angle CDI = \angle BAC = \angle COI ,$$

so that $CIOD$ is concyclic. Therefore

$$\begin{aligned}\angle CQD &= 180° - (\angle QOI + \angle QIO) = 180° - (\angle ICD + \angle PIC) \\ &= 180° - (\angle ICP + \angle PIC) = 90° .\end{aligned}$$

Suppose that angle A exceeds $60°$. Then I lies between O and A, and Q lies on the same side of AP as C. Since

$$\angle IDC + \angle IOC = \angle BAC + \angle AOC = 180° ,$$

the quadrilateral $IOCD$ is concyclic. Therefore

$$\begin{aligned}\angle CQD &= 180° - (\angle DCQ + \angle QDC) = 180° - (\angle QCP + \angle ODC) \\ &= 180° - (\angle QCP + \angle OIC) = 180° - (\angle ICP + \angle PIC) = 90° .\end{aligned}$$

Finally, if $\angle A = 60°$, then I and O coincide so that $DF = DE \| AB$ and the result is clear.