

## OLYMON

Produced by the **Canadian Mathematical Society** and the **Department of Mathematics of the University of Toronto**.

**Issue 10:2**

**February, 2009**

Please send your solutions to

E.J. Barbeau  
Department of Mathematics  
University of Toronto  
40 St. George Street  
Toronto, ON M5S 2E4

no later than March 5, 2009. Electronic files can be sent to [barbeau@math.utoronto.ca](mailto:barbeau@math.utoronto.ca). However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

### Problems for February, 2009

598. Let  $a_1, a_2, \dots, a_n$  be a finite sequence of positive integers. If possible, select two indices  $j, k$  with  $1 \leq j < k \leq n$  for which  $a_j$  does not divide  $a_k$ ; replace  $a_j$  by the greatest common divisor of  $a_j$  and  $a_k$ , and replace  $a_k$  by the least common multiple of  $a_j$  and  $a_k$ . Prove that, if the process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.
599. Determine the number of distinct solutions  $x$  with  $0 \leq x \leq \pi$  for each of the following equations. Where feasible, give an explicit representation of the solution.
- (a)  $8 \cos x \cos 2x \cos 4x = 1$ ;  
(b)  $8 \cos x \cos 4x \cos 5x = 1$ .
600. Let  $0 < a < b$ . Prove that, for any positive integer  $n$ ,

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}} \leq \sqrt[n]{\frac{a^n + b^n}{2}}.$$

601. A convex figure lies inside a given circle. The figure is seen from every point of the circumference of the circle at right angles (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the centre of the circle is a centre of symmetry of the figure.
602. Prove that, for each pair  $(m, n)$  of integers with  $1 \leq m \leq n$ ,

$$\sum_{i=1}^n i(i-1)(i-2)\cdots(i-m+1) = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1}.$$

(b) Suppose that  $1 \leq r \leq n$ ; consider all subsets with  $r$  elements of the set  $\{1, 2, 3, \dots, n\}$ . The elements of this subset are arranged in ascending order of magnitude. For  $1 \leq i \leq r$ , let  $t_i$  denote the  $i$ th smallest element in the subset, and let  $T(n, r, i)$  denote the arithmetic mean of the elements  $t_i$ . Prove that

$$T(n, r, i) = i \binom{n+1}{r+1}.$$

603. For each of the following expressions severally, determine as many integer values of  $x$  as you can so that it is a perfect square. Indicate whether your list is complete or not.

- (a)  $1 + x$ ;
- (b)  $1 + x + x^2$ ;
- (c)  $1 + x + x^2 + x^3$ ;
- (d)  $1 + x + x^2 + x^3 + x^4$ ;
- (e)  $1 + x + x^2 + x^3 + x^4 + x^5$ .

604.  $ABCD$  is a square with incircle  $\Gamma$ . Let  $l$  be a tangent to  $\Gamma$ , and let  $A', B', C', D'$  be points on  $l$  such that  $AA', BB', CC', DD'$  are all perpendicular to  $l$ . Prove that  $AA' \cdot CC' = BB' \cdot DD'$ .

Note: there was an error in the statement of one of the December problems. I will state it as intended and repose it with this set. Solutions may be sent to E. Barbeau:

585. Calculate the number

$$b = \lfloor (\sqrt{n-1} + \sqrt{n} + \sqrt{n+1})^2 \rfloor .$$

### Challenge Problems

I am starting new feature of Olymon, a list of problems that look interesting and appropriate for which I do not have a solution. They may turn out to be trivial, easy, difficult or impossible. There is no fixed deadline for their solution, but I will acknowledge successful solutions in the order in which I receive them. When I have a solution in hand, then I will add them to the regular Olymon stock for everyone to have a go at them.

**C1.** The function  $f(x)$  is defined for real nonzero  $x$ , takes nonzero real values and satisfies the functional equation

$$f(x) + f(y) = f(xyf(x+y)) ,$$

whenever  $xy(x+y) \neq 0$ . Determine all possibilities for  $f$ .

**C2.** Let  $T$  be a triangle in the plane whose vertices are lattice points (*i.e.*, both coordinates are integers), whose edges contain no lattice points in their interiors and whose interior contains exactly one lattice point. Must this lattice point in the interior be the centroid of the  $T$ ?

**C3.** Two circles are externally tangent at  $A$  and are internally tangent to a third circle  $\Gamma$  at points  $B$  and  $C$ . Suppose that  $D$  is the midpoint of the chord of  $\Gamma$  that passes through  $A$  and is tangent there to the two smaller given circles. Suppose, further, that the centres of the three circles are not collinear. Prove that  $A$  is the incentre of triangle  $BCD$ .

**C4.** Let  $a, b, c, m$  be positive integers for which  $abcm = 1 + a^2 + b^2 + c^2$ . Show that  $m = 4$ , and that there are actually possibilities with this value of  $m$ .

**C5.** Solve the equation

$$x^{12} - x^9 + x^4 - x = 1 .$$

### Solutions.

584. Let  $n$  be an integer exceeding 2 and suppose that  $x_1, x_2, \dots, x_n$  are real numbers for which  $\sum_{i=1}^n x_i = 0$  and  $\sum_{i=1}^n x_i^2 = n$ . Prove that there are two numbers among the  $x_i$  whose product does not exceed  $-1$ .

*Solution.* We can suppose that the  $x_i$  are ordered in increasing sequence and that there is a positive integer  $k$  with  $x_1 \leq x_2 \leq \cdots \leq x_k \leq 0 \leq x_{k+1} \leq \cdots \leq x_n$ . Then, noting that  $-x_1 \geq 0$ , we have that

$$\sum_{i=1}^k x_i^2 \leq \sum_{i=1}^k x_1 x_i = -x_1(x_{k+1} + x_{k+2} + \cdots + x_n) \leq -(n-k)x_1 x_n$$

and

$$\sum_{i=k+1}^n x_i^2 \leq \sum_{i=k+1}^n x_n x_i = -x_n(x_1 + x_2 + \cdots + x_k) \leq -kx_1 x_n .$$

Finally,  $n = \sum_{i=1}^n x_i^2 \leq -nx_1 x_n$ ; thus  $x_1 x_n \leq -1$ .

585. Calculate the number

$$a = \lfloor \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} \rfloor^2 ,$$

where  $\lfloor x \rfloor$  denotes the largest integer than does not exceed  $x$  and  $n$  is a positive integer exceeding 1.

*Solution.* It does not appear that there is a neat expression for this. One can obtain without too much trouble the inequality

$$3\sqrt{n-1} < \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} < 3\sqrt{n} ,$$

from which we can find that when  $k^2 + 1 \leq n \leq k^2 + (2k/3)$ , then  $\sqrt{a} = 3k$ , when  $k^2 + (2k/3) + (10/9) < n \leq k^2 + (4k/3) + (1/3)$ , then  $\sqrt{a} = 3k + 1$ , and when  $k^2 + 4k + (13/9) < n \leq (k+1)^2$ , then  $\sqrt{a} = 3k + 2$ . However, this leaves the difficulty of getting the right expression for the gaps between the various ranges of  $n$ .

586. The function defined on the set  $\mathbf{C}^*$  of all nonzero complex numbers satisfies the equation

$$f(z)f(iz) = z^2 ,$$

for all  $z \in \mathbf{C}^*$ . Prove that the function  $f(z)$  is odd, i.e.,  $f(-z) = -f(z)$  for all  $z \in \mathbf{C}^*$ . Give an example of a function that satisfies this condition.

*Solution.* Note that  $f(z) \neq 0$  for all  $x \in \mathbf{C}^*$ . Replacing  $z$  by  $iz$  leads to  $f(iz)f(-z) = -z^2$ , from which we have that

$$f(z)f(iz) + f(iz)f(-z) = 0 \implies f(z) + f(-z) = 0 .$$

Therefore the function is odd.

An example is given by  $f(z) = (-1 + i)z/\sqrt{2}$ .

587. Solve the equation

$$\tan 2x \tan \left( 2x + \frac{\pi}{3} \right) \tan \left( 2x + \frac{2\pi}{3} \right) = \sqrt{3} .$$

*Solution.* Using the standard trigonometric identities for  $\sin A \sin B$ ,  $\cos A \cos B$ ,  $\cos 2A$  and  $\sin 2A$ , we have that

$$\begin{aligned} \sqrt{3} &= \tan 2x \left( \frac{\sin(2x + (\pi/3)) \sin(2x + (2\pi/3))}{\cos(2x + (\pi/3)) \cos(2x + (2\pi/3))} \right) \\ &= \tan 2x \left( \frac{\cos(\pi/3) - \cos(4x + \pi)}{\cos(\pi/3) + \cos(4x + \pi)} \right) \\ &= \tan 2x \left( \frac{1 + 2 \cos 4x}{1 - 2 \cos 4x} \right) = \tan 2x \left( \frac{1 + 2(2 \cos^2 2x - 1)}{1 - 2(1 - 2 \sin^2 2x)} \right) \\ &= \left( \frac{\sin 2x}{\cos 2x} \right) \left( \frac{4 \cos^2 2x - 1}{4 \sin^2 2x - 1} \right) = \frac{2 \sin 4x \cos 2x - \sin 2x}{2 \sin 4x \sin 2x - \cos 2x} \\ &= \frac{\sin 6x + \sin 2x - \sin 2x}{\cos 2x - \cos 6x - \cos 2x} = \frac{\sin 6x}{-\cos 6x} = -\tan 6x . \end{aligned}$$

Therefore  $x = -10^\circ + k \cdot 30^\circ$  for some integer  $k$ .

588. Let the function  $f(x)$  be defined for  $0 \leq x \leq \pi/3$  by

$$f(x) = \sec\left(\frac{\pi}{6} - x\right) + \sec\left(\frac{\pi}{6} + x\right).$$

Determine the set of values (its image or range) assumed by the function.

*Solution.* Making use of the inequality  $(1/a) + (1/b) \geq 2/\sqrt{ab}$  for  $a, b > 0$ , we find that

$$f(x) \geq \frac{2}{\sqrt{\cos((\pi/6) - x) \cos((\pi/6) + x)}} \geq \frac{2}{\sqrt{(1/4) + ((\cos 2x)/2)}}.$$

Since  $0 \leq x \leq \pi/3$  implies that  $-\frac{1}{2} \leq \cos 2x \leq 1$ , it follows that

$$0 \leq \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \leq \frac{\sqrt{3}}{2},$$

and

$$f(x) \geq \frac{4}{\sqrt{3}}.$$

Since  $f(x)$  is continuous,  $f(0) = 4/\sqrt{3}$  and  $f(x)$  grows without bound when  $x$  approaches  $\pi/3$ , the image of  $f$  on  $[0, \pi/3)$  is  $[4/\sqrt{3}, \infty)$ .

589. In a circle,  $A$  is a variable point and  $B$  and  $C$  are fixed points. The internal bisector of the angle  $BAC$  intersects the circle at  $D$  and the line  $BC$  at  $G$ ; the external bisector of the angle  $BAC$  intersects the circle at  $E$  and the line  $BC$  at  $F$ . Find the locus of the intersection of the lines  $DF$  and  $EG$ .

*Solution.* Suppose without loss of generality that  $AB > AC$ . If  $M$  is the midpoint of  $BC$ , since  $BG : GC = AB : AC$ ,  $BG > GC$  so that  $G$  lies between  $M$  and  $C$  and  $A$  lies between  $E$  and  $F$ . Let  $P$  be the intersection of  $DF$  and  $EG$ .

Observe that  $D$  is the midpoint of the arc  $BC$  and that  $AD \perp EF$ . Therefore  $DA$  is an altitude of triangle  $DEF$  and  $DE$  is a diameter of the circle. Therefore  $DE$  must pass through  $M$ , and so  $FM \perp DE$ , *i.e.*,  $FM$  is an altitude of triangle  $DEF$ . The intersection of these two altitudes,  $G$ , is the orthocentre of triangle  $ABC$  and so  $EG \perp DF$ . Thus,  $\angle EPD = 90^\circ$ , so that  $P$  must lie on the given circle.

Conversely, let  $P$  be a point on the given circle. Wolog, we may assume that  $P$  lies between  $D$ , the midpoint of arc  $BC$  and  $C$ . Let  $DE$  be the diameter of the circle that right bisects  $BC$ . Suppose that  $DP$  produced intersects  $BC$  produced at  $F$  and that  $EF$  intersects the circle at  $A$ . This is the point  $A$  that produced the point  $P$  as described in the problem. Thus, the locus is indeed the given circle with the exception of the points  $B$  and  $C$ .

590. Let  $SABC$  be a regular tetrahedron. The points  $M, N, P$  belong to the edges  $SA, SB$  and  $SC$  respectively such that  $MN = NP = PM$ . Prove that the planes  $MNP$  and  $ABC$  are parallel.

*Solution.* Let  $|SM| = a$ ,  $|SN| = b$  and  $|SP| = c$ . From the Law of Cosines, we have that  $|MN|^2 = a^2 + b^2 - ab$ , *etc.*, whence  $a^2 + b^2 - ab = b^2 + c^2 - bc = c^2 + a^2 - ac = 0$ . This implies that  $a = b = c$  [prove it], so that  $SM : SA = SN : SB = SP : SC$  and the result follows.