OLYMON

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Please send your solution to

Mr. Rosu Mihai
54 Judith Crescent
Brampton, ON L6S 3J4

no later than April 30, 2008. Electronic files can be sent to *rosumihai@yahoo.ca*.

It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

Notes. $\lfloor x \rfloor$, the *floor of* x, is the largest integer n that does not exceed x, *i.e.*, that integer n for which $n \leq x < n+1$. $\{x\}$, the *fractional part of* x, is equal to $x - \lfloor x \rfloor$. The notation $[PQR]$ denotes the area of the triangle PQR. A *geometric progression* is a sequence for which the ratio of two successive terms is always the same; its nth term has the general form ar^{n-1}. A *truncated pyramid* is a pyramid with a similar pyramid on a base parallel to the base of the first pyramid removed. A polyhedron is inscribed in a sphere if each of its vertices lies on the surface of the sphere.

542. Solve the system of equations

$$\lfloor x \rfloor + 3\{y\} = 3.9 \ ,$$

$$\{x\} + 3\lfloor y \rfloor = 3.4 \ .$$

543. Let $a > 0$ and b be real parameters, and suppose that f is a function taking the set of reals to itself for which

$$f(a^3x^3 + 3a^2bx^2 + 3ab^2x) \leq x \leq a^3 f(x)^3 + 3a^2 b f(x)^2 + 3ab^2 f(x) \ ,$$

for all real x. Prove that f is a one-one function that takes the set of real numbers onto itself (*i.e.*, f is a *bijection*).

544. Define the real sequences $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ by $a_1 = 1$, $a_{n+1} = 5a_n + 4$ and $5b_n = a_n + 1$ for $n \geq 1$.

(a) Determine $\{a_n\}$ as a function of n.

(b) Prove that $\{b_n : n \geq 1\}$ is a geometric progression and evaluate the sum

$$S \equiv \frac{\sqrt{b_1}}{\sqrt{b_2} - \sqrt{b_1}} + \frac{\sqrt{b_2}}{\sqrt{b_3} - \sqrt{b_2}} + \cdots + \frac{\sqrt{b_n}}{\sqrt{b_{n+1}} - \sqrt{b_n}} \ .$$

545. Suppose that x and y are real numbers for which $x^3 + 3x^2 + 4x + 5 = 0$ and $y^3 - 3y^2 + 4y - 5 = 0$. Determine $(x+y)^{2008}$.

546. Let $a, a_1, a_2, \cdots, a_n$ be a set of positive real numbers for which

$$a_1 + a_2 + \cdots + a_n = a$$

and

$$\sum_{k=1}^{n} \frac{1}{a-a_k} = \frac{n+1}{a} \ .$$

Prove that

$$\sum_{k=1}^{n} \frac{a_k}{a-a_k} = 1 \ .$$

547. Let A, B, C, D be four points on a circle, and let E be the fourth point of the parallelogram with vertices A, B, C. Let AD and BC intersect at M, AB and DC intersect at N, and EC and MN intersect at F. Prove that the quadrilateral $DENF$ is concyclic.

548. In a sphere of radius R is inscribed a regular hexagonal truncated pyramid whose big base is inscribed in a great circle of the sphere (i.e., a whose centre is the centre of the sphere). The length of the side of the big base is three times the length of the side of a small base. Find the volume of the truncated pyramid as a function of R.