OLYMON

Produced by the Canadian Mathematical Society and the Department of Mathematics of the University of Toronto.

Issue 7:7

November, 2006

Please send your solution to

Dr. Lily Yen
7255 Hewitt Street
Burnaby, BC V5A 3M3

no later than December 21, 2006. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

PROBLEMS FOR NOVEMBER

465. For what positive numbers $a$ is $3\sqrt{2 + \sqrt{a}} + \sqrt{a} + 3\sqrt{2 - \sqrt{a}}$ an integer?

466. For a positive integer $m$, let $\overline{m}$ denote the sum of the digits of $m$. Find all pairs of positive integers $(m, n)$ with $m < n$ for which $(\overline{m})^2 = n$ and $(\overline{n})^2 = m$.

467. For which positive integers $n$ does there exist a set of $n$ distinct positive integers such that

(a) each member of the set divides the sum of all members of the set, and

(b) none of its proper subsets with two or more elements satisfies the condition in (a)?

468. Let $a$ and $b$ be positive real numbers satisfying $a + b \geq (a - b)^2$. Prove that

$$x^a(1 - x)^b + x^b(1 - x)^a \leq \frac{1}{2^{a+b-1}}$$

for $0 \leq x \leq 1$, with equality if and only if $x = \frac{1}{2}$.

469. Solve for $t$ in terms of $a, b$ in the equation

$$\sqrt{\frac{t^3 + a^3}{t + a}} + \sqrt{\frac{t^3 + b^3}{t + b}} = \sqrt{\frac{a^3 - b^3}{a - b}}$$

where $0 < a < b$.

470. Let $ABC$, $ACP$ and $BCQ$ be nonoverlapping triangles in the plane with angles $CAP$ and $CBQ$ right. Let $M$ be the foot of the perpendicular from $C$ to $AB$. Prove that lines $AQ$, $BP$ and $CM$ are concurrent if and only if $\angle BCQ = \angle ACP$.

471. Let $I$ and $O$ denote the incentre and the circumcentre, respectively, of triangle $ABC$. Assume that triangle $ABC$ is not equilateral. Prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + CA$, with equality holding only simultaneously.