

OLYMON

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Please send your solution to

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no later than July 31, 2006. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

PROBLEMS FOR JUNE

444. (a) Suppose that a 6×6 square grid of unit squares (chessboard) is tiled by 1×2 rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.
- (b) Is the same thing true for an 8×8 array?
- (c) Is the same thing true for a 6×8 array?

445. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.

446. Suppose that you have a 3×3 grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

Two players A and B play a game. They take alternate turns, A putting a 0 in any unoccupied square of the grid and B putting a 1. The first player is A , and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

(For example, if there are three 0s down the diagonal, then B can place a 1 in any vacant square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

- (a) What is the maximum number of legitimate moves possible in a game?
- (b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?
- (c) Which player has a winning strategy? Explain.
447. A high school student asked to solve the surd equation

$$\sqrt{3x - 2} - \sqrt{2x - 3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples (a, b, c, d) for which this method leads to a correct solution of the surd equation

$$\sqrt{ax - b} - \sqrt{cx - d} = 1 .$$

448. A criminal, having escaped from prison, travelled for 10 hours before his escape was detected. He was then pursued and gained upon at 3 miles per hour. When his pursuers had been 8 hours on the way, they met an express (train) going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken? [from *The high school algebra* by Robertson and Birchard, approved for Ontario schools in 1886]

449. Let $S = \{x : x > -1\}$. Determine all functions from S to S which both

(a) satisfies the equation $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$, and

(b) $f(x)/x$ is strictly increasing or strictly decreasing on each of the two intervals $\{x : -1 < x < 0\}$ and $\{x : x > 0\}$.

450. The 4-sectors of an angle are the three lines through its vertex that partition the angle into four equal parts; adjacent 4-sectors of two angles that share a side consist of the 4-sector through each vertex that is closest to the other vertex.

Prove that adjacent 4-sectors of the angles of a parallelogram meet in the vertices of a square if and only if the parallelogram has four equal sides.

Solutions to February problems

430. Let triangle ABC be such that its excircle tangent to the segment AB is also tangent to the circle whose diameter is the segment BC . If the lengths of the sides BC , CA and AB of the triangle form, in this order, an arithmetic sequence, find the measure of the angle ACB .

Solution. Let M be the midpoint of BC , E the centre of the excircle tangent to AB and r its radius, T , S and K the respective points of tangency of this excircle to CB , CA and AB respectively, and P the point of tangency of the two circles of the problem. Following convention, let a , b , c be the sidelengths of the sides of triangle ABC and s its semiperimeter. We have that $|CS| = |CT| = s$, $|AK| = |AS| = s - b$ and $|BK| = |BT| = s - a$. Then

$$[ABC] = [ACE] + [BCE] - [ABE] = \frac{1}{2}rb + \frac{1}{2}ra - \frac{1}{2}rc = r(s - c) .$$

Consider the right triangle MET . We have that $|ME| = |MP| + |PE| = \frac{a}{2} + r$, $|MT| = |MB| + |BC| = \frac{a}{2} + s - a = s - \frac{a}{2}$ and $|ET| = r$, so that

$$\left(r + \frac{a}{2}\right)^2 = r^2 + \left(s - \frac{a}{2}\right)^2 \implies ra = s(s - a) .$$

Using Heron's formula, we find that

$$r(s - c) = [ABC] = \sqrt{s(s - a)(s - b)(s - c)} ,$$

so that

$$\frac{s(s - a)(s - c)}{a} = \sqrt{s(s - a)(s - b)(s - c)} \implies s(s - a)(s - c) = a^2(s - b) .$$

Since a, b, c are in arithmetic progression, for some number d , $a = b - d$, $c = b + d$ and $2s = 3b$, so that

$$\begin{aligned} \frac{3b}{2} \left(\frac{b}{2} - d \right) \left(\frac{b}{2} + d \right) &= (b - d)^2 \left(\frac{b}{2} \right) \\ \implies 3b(b^2 - 4d^2) &= 4b(b - d)^2 \\ \implies 3b^2 - 12d^2 &= 4b^2 - 8bd + 4d^2 \\ \implies 0 &= b^2 - 8bd + 16d^2 = (b - 4d)^2 . \end{aligned}$$

Hence $(a, b, c) = (3d, 4d, 5d)$ so that triangle ABC is right and $\angle C = 90^\circ$.

431. Prove the following trigonometric identity, for any natural number n :

$$\sin \frac{\pi}{4n+2} \cdot \sin \frac{3\pi}{4n+2} \cdot \sin \frac{5\pi}{4n+2} \cdots \sin \frac{(2n-1)\pi}{4n+2} = \frac{1}{2^n} .$$

Solution 1. [F. Barekat] Let $\theta = \pi/(4n+2)$. For each integer m , we have that

$$\sin m\theta = \frac{\sin 2m\theta}{2 \cos m\theta} = \frac{\sin 2m\theta}{2 \sin(\frac{\pi}{2} - m\theta)} .$$

When $m = 2k - 1$ ($1 \leq k \leq n$),

$$\frac{\pi}{2} - m\theta = \frac{\pi}{2} \left[1 - \frac{2k-1}{2n+1} \right] = \frac{\pi}{2} \left[\frac{2(n+1-k)}{2n+1} \right] = 2(n+1-k)\theta .$$

Hence

$$\begin{aligned} \prod_{k=1}^n \sin \frac{(2k-1)\pi}{4n+2} &= \prod_{k=1}^n \sin(2k-1)\theta = \prod_{k=1}^n \frac{\sin 2(2k-1)\theta}{2 \sin 2(n+1-k)\theta} \\ &= \frac{\prod_{k=1}^n \sin 2(2k-1)\theta}{\prod_{k=1}^n 2 \sin 2(n+1-k)\theta} \\ &= \frac{1}{2^n} \frac{\prod_{k=1}^n \sin 2(2k-1)\theta}{\prod_{k=1}^n \sin 2k\theta} = \frac{1}{2^n} \prod_{k=1}^n \frac{\sin 2(2k-1)\theta}{\sin 2k\theta} \\ &= \frac{1}{2^n} \prod \left\{ \frac{\sin 2i\theta}{\sin 2i\theta} : i \text{ odd}, 1 \leq i \leq n \right\} \cdot \prod \left\{ \frac{\sin 2(2n+i-j)\theta}{\sin 2j\theta} : j \text{ even}, 2 \leq j \leq n \right\} . \end{aligned}$$

Since $2(2n+1-j)\theta + 2j\theta = (4n+2)\theta = \pi$, all fractions in the products are equal to unity, and the required value is $1/2^n$.

Solution 2. We first illustrate the argument for $n = 3$. Consider a regular heptagon $ABCDEFGH$ with $|AB| = a$, $|AC| = b$ and $|AD| = c$. Let M be the midpoint of DE so that AM right bisects DE as well as the parallel diagonals CF and BG . Being one-quarter of the angle subtended by a side at the centre of the circumcircle, $\angle DAM = \pi/14$. Since $\frac{1}{2}|BC| = |AB| \sin \angle BAM$, $\frac{1}{2}b = a \sin \frac{5\pi}{14}$. Since $\frac{1}{2}|CF| = |AC| \sin \angle CAM$, $\frac{1}{2}c = b \sin \frac{3\pi}{14}$. Since $\frac{1}{2}|DE| = |AD| \sin \angle DAM$, $\frac{1}{2}a = c \sin \frac{\pi}{14}$. Multiplying these equations yields that

$$\frac{1}{2^3} = \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} .$$

In general, consider a regular $(2n+1)$ -gon $AB_1B_2 \cdots B_nC_n \cdots C_2C_1$, with M the midpoint of B_nC_n . Observe that the segments B_kC_k ($1 \leq k \leq n$) are all parallel. Suppose that $\theta = \pi/(4n+2)$. Then, for $1 \leq k \leq n$,

$$\angle B_kAM = \frac{[2n - (2k-1)]\pi}{4n+2} = [2n - (2k-1)]\theta = [2(n+1-k) - 1]\theta .$$

Therefore,

$$\frac{1}{2}|B_k C_k| = |AB_k| \sin[2(n+1-k)-1]\theta.$$

The chord AB_k has $k-1$ vertices of the polygon on one side and $2n-k$ vertices on the other side, while the chord $B_k C_k$ has $2k-1$ vertices of the polygon on one side and $2(n-k)$ vertices on the other. Hence $|AB_k| = |B_j C_j|$ where $j = \frac{1}{2}[2n+1-k]$ when k is odd and $j = \frac{1}{2}k$ when k is even. The sequence $\{|AB_k|\}$ is a permutation of the sequence $\{|B_k C_k|\}$. Hence

$$\begin{aligned} \frac{1}{2^n} \prod_{k=1}^n |AB_k| &= \frac{1}{2^n} \prod_{k=1}^n |B_k C_k| = \prod_{k=1}^n \frac{1}{2} |B_k C_k| \\ &= \prod_{k=1}^n |AB_k| \sin[2(n+1-k)-1]\theta = \prod_{k=1}^n |AB_k| \prod_{k=1}^n \sin(2k-1)\theta, \end{aligned}$$

from which the result follows.

432. Find the exact value of:

(a)

$$\sqrt{\frac{1}{6} + \frac{\sqrt{5}}{18}} - \sqrt{\frac{1}{6} - \frac{\sqrt{5}}{18}};$$

(b)

$$\sqrt{1 + \frac{2}{5}} \cdot \sqrt{1 + \frac{2}{6}} \cdot \sqrt{1 + \frac{2}{7}} \cdot \sqrt{1 + \frac{2}{8}} \cdots \sqrt{1 + \frac{2}{57}} \cdot \sqrt{1 + \frac{2}{58}}.$$

Solution. (a)

$$\sqrt{\frac{1}{6} + \frac{\sqrt{5}}{18}} = \frac{1}{6} \sqrt{6 + 2\sqrt{5}} = \frac{1}{6}(1 + \sqrt{5})$$

and

$$\sqrt{\frac{1}{6} - \frac{\sqrt{5}}{18}} = \frac{1}{6} \sqrt{6 - 2\sqrt{5}} = \frac{1}{6}(\sqrt{5} - 1),$$

so that the difference is equal to $1/3$.

(b) For each n , $\sqrt{1 + (2/n)} = \sqrt{(n+2)/n}$, so that the product is equal to

$$\sqrt{\frac{7}{5} \cdot \frac{8}{6} \cdot \frac{9}{7} \cdots \frac{59}{57} \cdot \frac{60}{58}} = \sqrt{\frac{59}{5} \cdot \frac{60}{6}} = \sqrt{118}.$$

433. Prove that the equation

$$x^2 + 2y^2 + 98z^2 = 77777 \dots 777$$

does not have a solution in integers, where the right side has 2006 digits, all equal to 7.

Solution. Since, modulo 7, squares have the values 0, 1, 2, 4, $x^2 + 2y^2 \equiv 0 \pmod{7}$ implies that $x \equiv y \equiv 0 \pmod{7}$, whence $x = 7u$ and $y = 7v$ for some integers u and v . Hence

$$0 \equiv 7u^2 + 14v^2 + 14z^2 = 11111 \dots 111$$

$\pmod{7}$. However $111111 = 7 \times 15873 \equiv 0 \pmod{7}$, and $2006 = 6 \times 334 + 2$. Thus,

$$11111 \dots 111 = 11 + 111111(10^2 + 10^8 + \cdots + 10^{2000}) \equiv 11 \equiv 4$$

(mod 7), and we arrive at a contradiction.

434. Find all natural numbers n for which $2^n + n^{2004}$ is equal to a prime number.

Solution. Let $N = 2^n + n^{2004}$. If n is even, then N is even and composite. Let n be odd and not a multiple of 3. Then $N \equiv 2^n + (n^2)^{1003} \equiv 2 + 1 \equiv 0 \pmod{3}$. When $n = 1$, $N = 3$ and is prime, while when $n > 1$, N exceeds 3 and is composite.

Finally, let n be a multiple of 3. Then $n = 3k$ for some integer k and $N = (2^k)^3 + (n^{668})^3$ is properly divisible by $2^k + n^{668}$. Hence N is prime exactly when $n = 1$.

435. A circle with centre I is the incircle of the convex quadrilateral $ABCD$. The diagonals AC and BD intersect at the point E . Prove that, if the midpoints of the segments AD , BC and IE are collinear, then $AB = CD$.

Solution. Let M be the midpoint of AD and N the midpoint of BC . Since the midpoints of AD , BC and IE are collinear, I and E are on opposite sides of MN . Wolog, let I lie inside $AMNB$ and E lie inside $DMNC$.

We have that

$$\begin{aligned} [MIN] &= [AMNB] - [ABI] - [AMI] - [BNI] \\ &= [AMNB] - [ABI] - \frac{1}{2}([ADI] + [BCI]) \\ &= [AMNB] - \frac{1}{2}[ABI] - \frac{1}{2}([ABI] + [ADI] + [BCI]) \\ &= [AMNB] - \frac{1}{2}[ABI] - \frac{1}{2}([ABCD] - [CDI]) . \end{aligned}$$

Similarly,

$$[MEN] = [DMNC] - \frac{1}{2}[DCE] - \frac{1}{2}([ABCD] - [ABE]) .$$

Since MN bisects IE , I and E are equidistant from MN and $[MIN] = [MEN]$. Now

$$[AMNB] = [AMN] + [ABN] = \frac{1}{2}[AND] + \frac{1}{2}[ABC]$$

and

$$[DMNC] = [MDN] + [DNC] = \frac{1}{2}[AND] + \frac{1}{2}[DBC] .$$

Hence

$$[AND] + [ABC] - [ABI] - [ABCD] + [CDI] = [AND] + [DBC] - [DEC] - [ABCD] + [ABE] ,$$

whence

$$[ABC] + [CDI] + [DEC] = [DBC] + [ABE] + [ABI] .$$

Since

$$[ABC] + [DEC] = [ABCDE] = [DBC] + [ABE] ,$$

$[CDI] = [ABI]$. But I is equidistant from CD and AB whence $AB = CD$,

436. In the Euro-African volleyball tournament, there were nine more teams participating from Europe than from Africa. In total, the European won nine times as many points as were won by all of the African teams. In this tournament, each team played exactly once against each other team; there were no ties; the winner of a game gets 1 point, the loser 0. What is the greatest possible score of the best African team?

Solution. Let a be the number of teams from Africa, so that $a + 9$ is the number of teams from Europe. Suppose that there were k African wins over Europeans. Then the total number of points taken by the African teams is $\binom{a}{2} + k = \frac{1}{2}a(a - 1) + k$, while the Europeans won

$$\binom{a+9}{2} + [a(a+9) - k] = \frac{(a+9)(3a+8)}{2} - k.$$

By the given conditions,

$$\frac{(a+9)(3a+8)}{2} - k = 9 \left[\frac{a(a-1)}{2} + k \right],$$

which simplifies to $3a^2 - 22a + (10k - 36) = 0$. This is a quadratic equation in a with discriminant equal to $916 - 120k = 4(229 - 30k)$. There are integer values of a satisfying the quadratic only if $229 - 30k$ is square with $k > 0$. Thus, $k = 2$ or $k = 6$.

When $k = 2$, $0 = 3a^2 - 22a - 16 = (a - 8)(3a + 2)$ so $a = 8$. In this case, there are 8 African teams, and any of these teams can get at most $7 + 2 = 9$ points, which can occur when one African team vanquishes all the other African teams as well as two European teams.

When $k = 6$, $0 = 3a^2 - 22a + 24 = (a - 6)(3a - 4)$ so $a = 6$. In this case, there are 6 African teams, and any of these teams can get at most $5 + 6 = 11$ points, which can occur when one African team vanquishes all the other African teams as well as six European teams.

Thus the greatest possible score for an African team is 11.