

## OLYMON

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no later than May 15, 2006 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

### PROBLEMS FOR APRIL

437. Let  $a, b, c$  be the side lengths and  $m_a, m_b, m_c$  the lengths of their respective medians, of an arbitrary triangle  $ABC$ . Show that

$$\frac{3}{4} < \frac{m_a + m_b + m_c}{a + b + c} < 1 .$$

Furthermore, show that one cannot find a smaller interval to bound the ratio.

438. Determine all sets  $(x, y, z)$  of real numbers for which

$$x + y = 2 \quad \text{and} \quad xy - z^2 = 1 .$$

439. A natural number  $n$ , less than or equal to 500, has the property that when one chooses a number  $m$  randomly among  $\{1, 2, 3, \dots, 500\}$ , the probability that  $m$  divides  $n$  (*i.e.*,  $n/m$  is an integer) is  $1/100$ . Find the largest such  $n$ .
440. You are to choose 10 distinct numbers from  $\{1, 2, 3, \dots, 2006\}$ . Show that you can choose such numbers with a sum greater than 10039 in more ways than you can choose such numbers with a sum less than 10030.
441. Prove that, no matter how 15 points are placed inside a circle of radius 2 (including the boundary), there exists a circle of radius 1 (including the boundary) containing at least 3 of the 15 points.
442. Prove that the regular tetrahedron has minimum diameter among all tetrahedra that circumscribe a given sphere. (The diameter of a tetrahedron is the length of its longest edge.)
443. For  $n \geq 3$ , show that  $n - 1$  straight lines are sufficient to go through the interior of every square of an  $n \times n$  chessboard. Are  $n - 1$  lines necessary?

### Solutions

423. Prove or disprove: if  $x$  and  $y$  are real numbers with  $y \geq 0$  and  $y(y + 1) \leq (x + 1)^2$ , then  $y(y - 1) \leq x^2$ .

*Solution 1.* The statement is true. The result holds when  $y \leq 1$  since  $y(y - 1) \leq 0 \leq x^2$ . Assume  $y \geq 1$ , so that  $\sqrt{y(y - 1)} - 1 > 0$ . We have that  $4y(y + 1) < (2y + 1)^2$ , so that  $2\sqrt{y(y + 1)} < 2y + 1$ . Hence

$$y(y - 1) < y^2 + y - 2\sqrt{y(y + 1)} + 1 = (\sqrt{y(y + 1)} - 1)^2 .$$

Since  $y(y+1) \leq (x+1)^2$  and  $|x+1| \leq |x|+1$ , it follows that

$$y(y+1) \leq (|x|+1)^2 \implies \sqrt{y(y+1)} - 1 \leq |x|$$

whence

$$y(y-1) < |x|^2 = x^2.$$

Thus the assertion holds.

*Solution 2.* [D. Dziabenko] The statement holds. If  $0 \leq y \leq 1$ , then  $y(y-1) \leq 0 \leq x^2$ . Assume henceforth that  $y > 1$ . If  $x + \frac{1}{2} \leq y$ , then

$$y(y-1) = y(y+1) - 2y \leq (x+1)^2 - 2\left(x + \frac{1}{2}\right) = x^2.$$

If  $x + \frac{1}{2} > y$ , then  $x > y - \frac{1}{2} > 0$ , whence

$$x^2 > \left(y - \frac{1}{2}\right)^2 = y(y-1) + \frac{1}{4} > y(y-1).$$

*Solution 3.* [G. Ghosn] The result holds. Let  $y > 0$ . The region in the cartesian plane defined by  $y(y+1) \leq (x+1)^2$  lies between the  $x$ -axis and the upper branch of the hyperbola with equation

$$\left(y + \frac{1}{2}\right)^2 - (x+1)^2 = \frac{1}{4}.$$

The region in the cartesian plane defined by  $y(y-1) \leq x^2$  lies between the  $x$ -axis and the upper branch of the hyperbola with equation

$$\left(y - \frac{1}{2}\right)^2 - x^2 = \frac{1}{4}.$$

The second hyperbola lies above the first when  $x = -1$ . Thus, if it can be shown that the two hyperbolas do not intersect, then the top branch of the second hyperbola lies above top branch of the first, and the result will follow.

But solving  $y(y+1) = (x+1)^2$  and  $y(y-1) = x^2$  leads to  $y = x + \frac{1}{2}$  and ultimately to  $(x + \frac{1}{2})^2 - (x + \frac{1}{2}) = x^2$  which has no solution (being equivalent to  $-1/4 = 0$ ).

424. Simplify

$$\frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2}$$

to a fraction whose numerator and denominator are of the form  $u\sqrt{v}$  with  $u$  and  $v$  each linear polynomials. For which values of  $x$  is the equation valid?

*Solution.* For a real expression, we require that  $x^2 \geq 2$ . Observe that  $x^3 - 3x + 2 = (x-1)^2(x+2)$  and that  $x^3 - 3x - 2 = (x+1)^2(x-2)$ . Thus, the denominator vanishes when  $x = -2$ , and we must exclude this value. Suppose, first, that  $x \geq 2$ . Then

$$\begin{aligned} \frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2} &= \frac{(x+1)^2(x-2) + (x^2 - 1)\sqrt{x^2 - 4}}{(x-1)^2(x+2) + (x^2 - 1)\sqrt{x^2 - 4}} \\ &= \frac{[(x+1)\sqrt{x-2}][(x+1)\sqrt{x-2} + (x-1)\sqrt{x+2}]}{[(x-1)\sqrt{x+2}][(x-1)\sqrt{x+2} + (x+1)\sqrt{x-2}]} \\ &= \frac{(x+1)\sqrt{x-2}}{(x-1)\sqrt{x+2}}. \end{aligned}$$

Now suppose that  $x \leq -2$ . Then

$$\begin{aligned} \frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2} &= \frac{-(x+1)^2(2-x) + (x^2-1)\sqrt{(2-x)(-2-x)}}{-(x-1)^2(-2-x) + (x^2-1)\sqrt{(2-x)(-2-x)}} \\ &= \frac{[(x+1)\sqrt{2-x}] [-(x+1)\sqrt{2-x} + (x-1)\sqrt{-2-x}]}{[(1-x)\sqrt{-2-x}] [(x-1)\sqrt{-2-x} - (x+1)\sqrt{2-x}]} \\ &= \frac{(x+1)\sqrt{2-x}}{(1-x)\sqrt{-2-x}}. \end{aligned}$$

*Comment.* Most solvers neglected to ensure that the quantities under the radical were nonnegative. This is the sort of "easy" question where many marks can be lost because of inattention to detail.

425. Let  $\{x_1, x_2, \dots, x_n, \dots\}$  be a sequence of nonzero real numbers. Show that the sequence is an arithmetic progression if and only if, for each integer  $n \geq 2$ ,

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \dots + \frac{1}{x_{n-1}x_n} = \frac{n-1}{x_1x_n}.$$

*Solution.* A constant sequence is an arithmetic progression and clearly satisfies the condition. Suppose that the sequence is a nonconstant arithmetic progression with common difference  $d$ . Then, for each positive index  $i$ , we have that

$$\frac{d}{x_i x_{i+1}} = \frac{x_{i+1} - x_i}{x_i x_{i+1}} = \frac{1}{x_i} - \frac{1}{x_{i+1}}.$$

Hence, for each  $n \geq 2$ ,

$$\begin{aligned} d \sum_{i=1}^{n-1} \frac{1}{x_i x_{i+1}} &= \sum_{i=1}^{n-1} \left( \frac{1}{x_i} - \frac{1}{x_{i+1}} \right) \\ &= \left( \frac{1}{x_1} - \frac{1}{x_n} \right) = \frac{x_n - x_1}{x_1 x_n} = \frac{(n-1)d}{x_1 x_n}, \end{aligned}$$

from which the condition follows.

On the other hand, suppose that the condition holds. Let  $d = x_2 - x_1$ , and suppose that we have established that  $x_n - x_1 = (n-1)d$  (this is true for  $n = 2$ ). Then, we have that

$$\frac{n}{x_1 x_{n+1}} - \frac{n-1}{x_1 x_n} = \frac{1}{x_n x_{n+1}}$$

so that

$$nx_n - (n-1)x_{n+1} = x_1 \implies (n-1)(x_{n+1} - x_1) = n(x_n - x_1) = n(n-1)d$$

from which  $x_{n+1} - x_1 = nd$ . It follows, by induction, that the  $\{x_n\}$  is an arithmetic progression.

426. (a) The following paper-folding method is proposed for trisecting an acute angle.

(1) transfer the angle to a rectangular sheet so that its vertex is at one corner  $P$  of the sheet with one ray along the edge  $PY$ ; let the angle be  $XPY$ ;

(2) fold up  $PY$  over  $QZ$  to fall on  $RW$ , so that  $PQ = QR$  and  $PY \parallel QZ \parallel RW$ , with  $QZ$  between  $PY$  and  $RW$ ;

(3) fold across a line  $AC$  with  $A$  on the sheet and  $C$  on the edge  $PY$  so that  $P$  falls on a point  $P'$  on  $QZ$  and  $R$  on a point  $R'$  on  $PX$ ;

(4) suppose that the fold  $AC$  intersects the fold  $QZ$  at  $B$  and carries  $Q$  to  $Q'$ ; make a fold along  $BQ'$ .

It is claimed that the fold  $BQ'$  passes through  $P$  and trisects angle  $XPY$ .

Explain why the fold described in (3) is possible. Does the method work? Why?

(b) What happens with a right angle?

(c) Can the method be adapted for an obtuse angle?

*Solution.* [F. Barekat] Let  $\angle XPY = 3\theta$ . Select  $B$  on  $QZ$  so that  $\angle PBR = 4\theta$  and draw the circle with centre  $B$  that passes through  $P$  and  $R$ . Suppose that this circle intersects  $QZ$  at  $P'$  and  $XP$  at  $R'$ . Since  $\angle PBR = 4\theta$ ,  $\angle P'PY = \angle PP'Q = \frac{1}{2}\angle PP'R = \theta$ . Also,  $\angle P'RR' = \angle XPP' = \angle XPY - \angle P'PY = 2\theta$ , so that  $\angle R'RW = \angle P'RR' - \angle WRP' = \theta$ .

Hence  $PP'$  and  $RR'$  are parallel chords in the circle, and their right bisectors pass through  $B$  and defines the required fold to interchange  $P$  and  $P'$ , and  $R$  and  $R'$ . Since  $BP = BP'$ ,  $\angle BPY = \angle QBP = \angle BPP' + \angle BP'P = 2\angle BP'P = 2\theta$ , so that  $\angle BPR' = \theta$  and  $BP$  trisects the angle.

The fold (reflection) fixes  $B$  and interchanges  $P$  and  $P'$ , and  $Q$  and  $Q'$ . Since  $P', B, Q$  are collinear, so are  $P, B, Q'$ . Hence the line through  $B$  and  $Q'$  also passes through  $P$  and so trisects the angle.

(b) When  $\angle XPY = 90^\circ$ , then  $X$  lies on  $PR$  and  $R$  and  $R'$  coincide. We have that  $PR = P'R = P'P$ , so that triangle  $PP'R$  is equilateral and  $RC$  is an altitude. Hence  $Q'$  is the midpoint of  $P'R$  and  $\angle Q'PP' = 30^\circ = \angle P'PX$ .

(c) One way to trisect an obtuse angle is to trisect its supplement, and subtract the result from  $60^\circ$ .

427. The radius of the inscribed circle and the radii of the three escribed circles of a triangle are consecutive terms of a geometric progression. Determine the largest angle of the triangle.

*Solution 1.* [F. Barekat] Let  $r$  be the inradius and  $r_a, r_b, r_c$  the respective exradii of the circles touching the sides  $a, b, c$ . Let  $s$  be the semiperimeter and  $\Delta$  the area of the triangle. Recall that

$$\Delta = rs = r_a(s - a) = r_b(s - b) = r_c(s - c)$$

[derive this], from which

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{s - a}{\Delta} + \frac{s - b}{\Delta} + \frac{s - c}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

so that  $r$  is the smallest term in the geometric progression. Suppose that  $A \leq B \leq C$ . Then

$$r_a = s \tan(A/2) \leq r_b = s \tan(B/2) \leq r_c = s \tan(C/2) .$$

Hence there is a number  $t > 1$  for which  $r_a = tr$ ,  $r_b = t^2r$ ,  $r_c = t^3r$  and  $s - b = t(s - c)$ ,  $s - a = t^2(s - c)$ ,  $s = t^3s$ . By Heron's formula,

$$\begin{aligned} t^3r(s - c) &= r_c(s - c) = rs = \Delta \\ &= \sqrt{s(s - a)(s - b)(s - c)} = \sqrt{t^6(s - c)^4} = t^3(s - c)^2 . \end{aligned}$$

Hence  $r = s - c$ , and so  $\tan(C/2) = r/(s - c) = 1$ . Thus  $C = 90^\circ$ .

*Solution 2.* [D. Dziabenko] Define the symbols as in the first solution. The equation  $r_a^{-1} + r_b^{-1} + r_c^{-1} = r^{-1}$  leads to  $t^3 = t^2 + t + 1$ . Now

$$\begin{aligned} \Delta &= \frac{1}{2}ab \sin C = \frac{1}{2}[(s - b + s - c)(s - a + s - c) \sin C \\ &= \frac{1}{2}[(t + 1)(t^2 + 1)(s - c)^2 \sin C] = t^3(s - c)^2 \sin C . \end{aligned}$$

On the other hand, by Heron's formula, we find that

$$\Delta = \sqrt{t^6(s - c)^4} = t^3(s - c)^2 .$$

Comparing the two expressions leads to  $\sin C = 1$ , so that  $C = 90^\circ$ .

*Solution 3.* [G. Ghosn; A. Remorov] With  $r \leq r_a \leq r_b \leq r_c$  as in the previous solutions and from the inverse proportionality of  $r : r_a : r_b : r_c$  and  $s : (s-a) : (s-b) : (s-c)$ , we have that  $(s-a)(s-b) = s(s-c)$ , whence

$$ab = (a+b-c)s \implies 2ab = (a+b)^2 - c^2 \implies c^2 = a^2 + b^2 .$$

Hence the triangle is right, and its largest angle,  $C$ , is  $90^\circ$ .

428.  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are three lines in space. Neither  $\mathbf{a}$  nor  $\mathbf{b}$  is perpendicular to  $\mathbf{c}$ . Points  $P$  and  $Q$  vary on  $\mathbf{a}$  and  $\mathbf{b}$ , respectively, so that  $PQ$  is perpendicular to  $\mathbf{c}$ . The plane through  $P$  perpendicular to  $\mathbf{b}$  meets  $\mathbf{c}$  at  $R$ , and the plane through  $Q$  perpendicular to  $\mathbf{a}$  meets  $\mathbf{c}$  at  $S$ . Prove that  $RS$  is of constant length.

*Solution.* Let the point  $P$  on line  $\mathbf{a}$  be given by  $\mathbf{p} + s\mathbf{u}$  and the point  $Q$  on line  $\mathbf{b}$  be given by  $\mathbf{q} + t\mathbf{v}$ , where  $\mathbf{p}$ ,  $\mathbf{q}$ ,  $\mathbf{u}$  and  $\mathbf{v}$  are fixed vectors and  $s, t$  parameters determining the points. Let  $\mathbf{r}$  be a point on  $\mathbf{c}$  and  $\mathbf{w}$  be the direction vector for  $\mathbf{c}$ . Wolog, we can normalize  $\mathbf{u}$  and  $\mathbf{v}$  so that  $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 1$ .

By hypothesis,  $[(\mathbf{p} + s\mathbf{u}) - (\mathbf{q} + t\mathbf{v})] \cdot \mathbf{w} = 0$ , whence  $s - t = (\mathbf{q} - \mathbf{p}) \cdot \mathbf{w}$ . Let  $R$  be given by  $\mathbf{r} + m\mathbf{w}$  and  $S$  by  $\mathbf{r} + n\mathbf{w}$ . Then

$$[(\mathbf{p} + s\mathbf{u}) - (\mathbf{r} + m\mathbf{w})] \cdot \mathbf{v} = 0$$

so that  $m = (\mathbf{p} - \mathbf{r}) \cdot \mathbf{v} + s(\mathbf{u} \cdot \mathbf{v})$ . Also

$$[(\mathbf{q} + t\mathbf{v}) - (\mathbf{r} + m\mathbf{w})] \cdot \mathbf{u} = 0$$

so that  $n = (\mathbf{q} - \mathbf{r}) \cdot \mathbf{u} + t(\mathbf{v} \cdot \mathbf{u})$ . Hence

$$m - n = (\mathbf{p} - \mathbf{r}) \cdot \mathbf{v} - (\mathbf{q} - \mathbf{r}) \cdot \mathbf{u} + (\mathbf{u} \cdot \mathbf{v})[(\mathbf{q} - \mathbf{p}) \cdot \mathbf{w}]$$

is independent of  $s$  and  $t$  and the result follows.

429. Prove that

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \binom{kn}{n} = (-1)^{n+1} n^n .$$

*Solution 1.* Let  $f(x) = (1 - (1+x)^n)^n$ . Since  $(1 - (1+x)^n) = -nx + x^2g(x)$ , for some polynomial  $g(x)$ , we have that

$$f(x) = (-1)^n n^n x^n + x^{n+1}h(x)$$

for some polynomial  $h(x)$ . On the other hand,

$$\begin{aligned} f(x) &= \sum_{k=0}^n (-1)^k \binom{n}{k} (1+x)^{nk} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{j=0}^{nk} \binom{nk}{j} x^j \\ &= \sum_{j=0}^{n^2} \left[ \sum_{k=\lceil j/n \rceil}^n (-1)^k \binom{n}{k} \binom{nk}{j} \right] x^j . \end{aligned}$$

Comparing the coefficients of  $x^n$  in the two expressions for  $f(x)$  yields that  $(-1)^n n^n = \sum_{k=1}^n (-1)^k \binom{n}{k} \binom{nk}{n}$ , from which the desired result follows.

*Solution 2.* Consider a set of  $n^2$  distinct objects arranged in a  $n \times n$  square array. There are  $n^n$  ways of choosing  $n$  of them so that one is chosen from each row. We count this in a different way, using the Principle

of Inclusion-Exclusion. Let  $f(r)$  be the number of ways of selecting the  $n$  objects so that they come from at most  $r$  distinct rows. There are  $\binom{n}{r}$  ways of selecting the  $r$  rows containing the objects, and  $rn$  objects to choose from. Hence  $f(r) = \binom{n}{r} \binom{rn}{n}$ . [Note that this doublecounts choices involving fewer than  $r$  rows.]

There are  $f(n) = \binom{n^2}{n}$  ways of choosing  $n$  objects from the array without restriction. But this includes the  $f(n-1)$  selections where they are drawn from at most  $n-1$  rows. But then  $f(n) - f(n-1)$  subtracts off those from  $n-2$  rows twice, so we need to add  $f(n-2)$  back. But then, in  $f(n) - f(n-1) + f(n-2)$ , we have added in each selection from  $n-3$  rows  $\binom{3}{3}$  times in  $f(n)$ , subtracted in  $\binom{3}{2}$  times in  $f(n-1)$ , added it back  $\binom{3}{1}$  times. So we need to add it back. Continuing in this way, we find that

$$n^n = \sum_{k=0}^n (-1)^{n-k} f(k) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{kn}{k}$$

from which the desired result follows.

*Solution 4.* [G. Ghosh] Let  $P(x) = (x^n - 1)^n = \sum_{k=0}^n \binom{n}{k} x^{kn} (-1)^{n-k}$ . We calculate the  $n$ th derivative of the two expressions.

Recall Leibniz' Rule that  $D^n(QR) = \sum_{k=0}^n \binom{n}{k} D^{n-k}(Q)D^k(R)$ . Taking  $Q(x) = (x-1)^n$  and  $R(x) = (x^{n-1} + x^{n-2} + \dots + 1)^n$ , we have that

$$\begin{aligned} D^n[(x^n - 1)^n] &= D^n[(x-1)^n(x^{n-1} + \dots + x + 1)^n] \\ &= n!(x^{n-1} + \dots + x + 1)^n + (x-1)S(x), \end{aligned}$$

for some polynomial  $S(x)$ . When  $x = 1$ , this takes the value  $n!n^n$ .

On the other hand,

$$\begin{aligned} D^n \left[ \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} x^{kn} \right] &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (kn)(kn-1) \dots (kn-n+1) x^{kn-n} \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{(kn)!}{(kn-n)!} x^{kn-n} \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{kn}{n} n! x^{kn-n}. \end{aligned}$$

When  $x = 1$ , this takes the value  $n! \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{kn}{n}$ . The desired result follows.