OLYMON

COMPLETE PROBLEM SET

No solutions. See yearly files.

February, 2009-xxx

PART 3

Problems 601-900

601. A convex figure lies inside a given circle. The figure is seen from every point of the circumference of the circle at right angles (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the centre of the circle is a centre of symmetry of the figure.

602. Prove that, for each pair (m, n) of integers with $1 \leq m \leq n$,

$$\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} \ .$$

(b) Suppose that $1 \leq r \leq n$; consider all subsets with r elements of the set $\{1, 2, 3, \cdots, n\}$. The elements of this subset are arranged in ascending order of magnitude. For $1 \leq i \leq r$, let t_i denote the ith smallest element in the subset, and let $T(n, r, i)$ denote the arithmetic mean of the elements t_i. Prove that

$$T(n, r, i) = i\left(\frac{n+1}{r+1}\right) \ .$$

603. For each of the following expressions severally, determine as many integer values of x as you can so that it is a perfect square. Indicate whether your list is complete or not.

(a) $1 + x$;

(b) $1 + x + x^2$;

(c) $1 + x + x^2 + x^3$;

(d) $1 + x + x^2 + x^3 + x^4$;

(e) $1 + x + x^2 + x^3 + x^4 + x^5$.

604. $ABCD$ is a square with incircle Γ. Let l be a tangent to Γ, and let A', B', C', D' be points on l such that AA', BB', CC', DD' are all prependicular to l. Prove that $AA' \cdot CC' = BB' \cdot DD'$.

605. Prove that the number $299\cdots998200\cdots029$ can be written as the sum of three perfect squares of three consecutive numbers, where there are $n-1$ nines between the first 2 and the 8, and $n-1$ zeros between the last pair of twos.

606. Let $x_1 = 1$ and let $x_{n+1} = \sqrt{x_n + n^2}$ for each positive integer n. Prove that the sequence $\{x_n : n > 1\}$ consists solely of irrational numbers and calculate $\sum_{k=1}^{n} \lfloor x_k^2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer that does not exceed x.

607. Solve the equation

$$\sin x \left(1 + \tan x \tan \frac{x}{2}\right) = 4 - \cot x \ .$$

608. Find all positive integers n for which n, $n^2 + 1$ and $n^3 + 3$ are simultaneously prime.

609. The first term of an arithmetic progression is 1 and the sum of the first nine terms is equal to 369. The first and ninth terms of the arithmetic progression coincide respectively with the first and ninth terms of a geometric progression. Find the sum of the first twenty terms of the geometric progression.

610. Solve the system of equations

$$\log_{10}(x^3 - x^2) = \log_5 y^2$$
$$\log_{10}(y^3 - y^2) = \log_5 z^2$$
$$\log_{10}(z^3 - z^2) = \log_5 x^2$$

where $x, y, z > 1$.

611. The triangle ABC is isosceles with $AB = AC$ and I and O are the respective centres of its inscribed and circumscribed circles. If D is a point on AC for which $ID \| AB$, prove that $CI \perp OD$.

612. $ABCD$ is a rectangle for which $AB > AD$. A rotation with centre A takes B to a point B' on CD; it takes C to C' and D to D'. Let P be the point of intersection of the lines CD and $C'D'$. Prove that $CB' = DP$.

613. Let ABC be a triangle and suppose that

$$\tan\frac{A}{2} = \frac{p}{u} \qquad \tan\frac{B}{2} = \frac{q}{v} \qquad \tan\frac{C}{2} = \frac{r}{w} \ ,$$

where p, q, r, u, v, w are positive integers and each fraction is written in lowest terms.

(a) Verify that $pqw + pvr + uqr = uvw$.

(b) Let f be the greatest common divisor of the pair $(vw - qr, qw + vr)$, g be the greatest common divisor of the pair $(uw - pr, pw + ur)$, and h be the greatest common divisor of the pair $(uv - pq, pv + qu)$. Prove that

$$fp = vw - qr \qquad fu = qw + vr$$
$$gq = uw - pr \qquad gv = pw + ur$$
$$hr = uv - pq \qquad hw = pv + qu \ .$$

(c) Prove that the sides of the triangle ABC are proportional to $fpu : gqv : hrw$.

614. Determine those values of the parameter a for which there exist at least one line that is tangent to the graph of the curve $y = x^3 - ax$ at one point and normal to the graph at another.

615. The function $f(x)$ is defined for real nonzero x, takes nonzero real values and satisfies the functional equation

$$f(x) + f(y) = f(xyf(x + y)) \ ,$$

whenever $xy(x + y) \neq 0$. Determine all possibilities for f.

616. Let T be a triangle in the plane whose vertices are lattice points (*i.e.*, both coordinates are integers), whose edges contain no lattice points in their interiors and whose interior contains exactly one lattice point. Must this lattice point in the interior be the centroid of the T?

617. Two circles are externally tangent at A and are internally tangent to a third circle Γ at points B and C. Suppose that D is the midpoint of the chord of Γ that passses through A and is tangent there to the two smaller given circles. Suppose, further, that the centres of the three circles are not collinear. Prove that A is the incentre of triangle BCD.

618. Let a, b, c, m be positive integers for which $abcm = 1 + a^2 + b^2 + c^2$. Show that $m = 4$, and that there are actually possibilities with this value of m.

619. Suppose that $n > 1$ and that S is the set of all polynomials of the form

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \ ,$$

whose coefficients are complex numbers. Determine the minimum value over all such polynomials of the maximum value of $|p(z)|$ when $|z| = 1$.

620. Let $a_1, a_2, \cdots, a_n$ be distinct integers. Prove that the polynomial

$$p(z) = (z - a_1)^2 (z - a_2)^2 \cdots (z - a_n)^2 + 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients.

621. Determine the locus of one focus of an ellipse reflected in a variable tangent to the ellipse.

622. Let I be the centre of the inscribed circle of a triangle ABC and let u, v, w be the respective lengths of IA, IB, IC. Let P be any point in the plane and p, q, r the respective lengths of PA, PB, PC. Prove that, with the sidelengths of the triangle given conventionally as a, b, c,

$$ap^2 + bq^2 + cr^2 = au^2 + bv^2 + cw^2 + (a + b + c)z^2 \ ,$$

where z is the length of IP.

623. Given the parameters a, b, c, solve the system

$$x + y + z = a + b + c;$$

$$x^2 + y^2 + x^2 = a^2 + b^2 + c^2;$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 \ .$$

624. Suppose that $x_i \geq 0$ and

$$\sum_{i=1}^{n} \frac{1}{1 + x_i} \leq 1 \ .$$

Prove that

$$\sum_{i=1}^{n} 2^{-x_i} \leq 1 \ .$$

625. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

626. Let ABC be an isosceles triangle with $AB = AC$, and suppose that D is a point on the side BC with $BC > BD > DC$. Let BE and CF be diameters of the respective circumcircles of triangles ABD and ADC, and let P be the foot of the altitude from A to BC. Prove that $PD : AP = EF : BC$.

627. Let

$$f(x, y, z) = 2x^2 + 2y^2 - 2z^2 + \frac{7}{xy} + \frac{1}{z} \ .$$

There are three pairwise distinct numbers a, b, c for which

$$f(a, b, c) = f(b, c, a) = f(c, a, b) \ .$$

Determine $f(a,b,c)$. Determine three such numbers a,b,c.

628. Suppose that AP, BQ and CR are the altitudes of the acute triangle ABC, and that

$$9\overrightarrow{AP} + 4\overrightarrow{BQ} + 7\overrightarrow{CR} = \overrightarrow{O} \ .$$

Prove that one of the angles of triangle ABC is equal to $60°$.

629. Let $a > b > c > d > 0$ and $a + d = b + c$. Show that $ad < bc$.

(b) Let a, b, p, q, r, s be positive integers for which

$$\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$$

and $qr - ps = 1$. Prove that $b \geq q + s$.

630. (a) Show that, if

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = -1 \ \ ,$$

then

$$\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} = 1 \ \ .$$

(b) Give an example of numbers α and β that satisfy the condition in (a) and check that both equations hold.

631. The sequence of functions $\{P_n\}$ satisfies the following relations:

$$P_1(x) = x \ , \qquad P_2(x) = x^3 \ ,$$

$$P_{n+1}(x) = \frac{P_n^3(x) - P_{n-1}(x)}{1 + P_n(x)P_{n-1}(x)} \ \ , \qquad n = 1, 2, 3, \cdots .$$

Prove that all functions P_n are polynomials.

632. Let a, b, c, x, y, z be positive real numbers for which $a \leq b \leq c$, $x \leq y \leq z$, $a + b + c = x + y + z$, $abc = xyz$, and $c \leq z$, Prove that $a \leq x$.

633. Let ABC be a triangle with $BC = 2 \cdot AC - 2 \cdot AB$ and D be a point on the side BC. Prove that $\angle ABD = 2\angle ADB$ if and only if $BD = 3CD$.

634. Solve the following system for real values of x and y:

$$2^{x^2+y} + 2^{x+y^2} = 8$$

$$\sqrt{x} + \sqrt{y} = 2 \ .$$

635. Two unequal spheres in contact have a common tangent cone. The three surfaces divide space into various parts, only one of which is bounded by all three surfaces; it is "ring-shaped". Being given the radii r and R of the spheres with $r < R$, find the volume of the "ring-shaped" region in terms of r and R.

636. Let ABC be a triangle. Select points D, E, F outside of ΔABC such that ΔDBC, ΔEAC, ΔFAB are all isosceles with the equal sides meeting at these outside points and with $\angle D = \angle E = \angle F$. Prove that the lines AD, BE and CF all intersect in a common point.