OLYMON

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PROBLEMS FOR APRIL

Please send your solution to
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no later than April 30, 2005 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

374. What is the maximum number of numbers that can be selected from $\{1, 2, 3, \cdots, 2005\}$ such that the difference between any pair of them is not equal to 5?

375. Prove or disprove: there is a set of concentric circles in the plane for which both of the following hold:

(i) each point with integer coordinates lies on one of the circles;

(ii) no two points with integer coefficients lie on the same circle.

376. A soldier has to find whether there are mines buried within or on the boundary of a region in the shape of an equilateral triangle. The effective range of his detector is one half of the height of the triangle. If he starts at a vertex, explain how he can select the shortest path for checking that the region is clear of mines.

377. Each side of an equilateral triangle is divided into 7 equal parts. Lines through the division points parallel to the sides divide the triangle into 49 smaller equilateral triangles whose vertices consist of a set of 36 points. These 36 points are assigned numbers satisfying both the following conditions:

(a) the number at the vertices of the original triangle are 9, 36 and 121;

(b) for each rhombus composed of two small adjacent triangles, the sum of the numbers placed on one pair of opposite vertices is equal to the sum of the numbers placed on the other pair of opposite vertices.

Determine the sum of all the numbers. Is such a choice of numbers in fact possible?

378. Let $f(x)$ be a nonconstant polynomial that takes only integer values when x is an integer, and let P be the set of all primes that divide $f(m)$ for at least one integer m. Prove that P is an infinite set.

379. Let n be a positive integer exceeding 1. Prove that, if a graph with $2n + 1$ vertices has at least $3n + 1$ edges, then the graph contains a circuit (*i.e.*, a closed non-self-intersecting chain of edges whose terminal point is its initial point) with an even number of edges. Prove that this statement does not hold if the number of edges is only $3n$.

380. Factor each of the following polynomials as a product of polynomials of lower degree with integer coefficients:

(a) $(x + y + z)^4 - (y + z)^4 - (z + x)^4 - (x + y)^4 + x^4 + y^4 + z^4$;

(b) $x^2(y^3 - z^3) + y^2(z^3 - x^3) + z^2(x^3 - y^3)$;

(c) $x^4 + y^4 - z^4 - 2x^2y^2 + 4xyz^2$;

(d) $(yz + zx + xy)^3 - y^3z^3 - z^3x^3 - x^3y^3$;

(e) $x^3y^3 + y^3z^3 + z^3x^3 - x^4yz - xy^4z - xyz^4$;

(f) $2(x^4 + y^4 + z^4 + w^4) - (x^2 + y^2 + z^2 + w^2)^2 + 8xyzw$;

(g) $6(x^5 + y^5 + z^5) - 5(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)$.

Solutions to the January problems.

353. The two shortest sides of a right-angled triangle, a and b, satisfy the inequality:

$$\sqrt{a^2 - 6a\sqrt{2} + 19} + \sqrt{b^2 - 4b\sqrt{3} + 16} \leq 3 \ .$$

Find the perimeter of this triangle.

Solution. The equation can be rewritten as

$$\sqrt{(a - 3\sqrt{2})^2 + 1} + \sqrt{(b - 2\sqrt{3})^2 + 4} \leq 3 \ .$$

Since the left side is at least equal to $1 + 2 = 3$, we must have equality and so $a = 3\sqrt{2}$ and $b = 2\sqrt{3}$. The hypotenuse of the triangle equal to $\sqrt{a^2 + b^2} = \sqrt{30}$, and so the perimeter is equal to $3\sqrt{2} + 2\sqrt{3} + \sqrt{30}$. ♠

354. Let ABC be an isosceles triangle with $AC = BC$ for which $|AB| = 4\sqrt{2}$ and the length of the median to one of the other two sides is 5. Calculate the area of this triangle.

Solution. Let M be the midpoint of BC, $\theta = \angle AMB$ and $x = |BM| = |MC|$. Then $|AC| = 2x$. By the Law of Cosines, $4x^2 = x^2 + 25 + 10x\cos\theta$ and $32 = x^2 + 25 - 10x\cos\theta$. Adding these two equations yields that $x^2 = 9$, so that $x = 3$. The height of the triangle from C is $\sqrt{4x^2 - 8} = \sqrt{28} = 2\sqrt{7}$. Hence the area of the triangle is $4\sqrt{14}$. ♠

355. (a) Find all natural numbers k for which $3^k - 1$ is a multiple of 13.

(b) Prove that for any natural number k, $3^k + 1$ is not a multiple of 13.

Solution 1. Let $k = 3q + r$. Since $3^3 \equiv 1 \pmod{13}$, $3^k - 1 \equiv 3^r - 1 \pmod{13}$ and $3^k + 1 \equiv 3^r + 1 \pmod{13}$. Since $3^0 = 1$, $3^2 = 9$, we see that only $3^k - 1$ is a multiple of 13 when k is a multiple of 3. ♠

Solution 2. Let p be a prime and $N = d_0 + d_1p + \cdots + d_rp^r = (d_rd_{r-1}\cdots d_1d_0)_p$ be an integer written to base p. Then $p^k = (100\cdots00)_p$, $p^k + 1 = (1000\cdots01)_p$ and $p^k - 1 = (\overline{p-1}, \overline{p-1}, \cdots \overline{p-1})_p$ where the first two have $k + 1$ digits and the last has k digits. Let $p = 3$, we see that $3^k - 1 = (222\cdots22)_3$ and $3^k + 1 = (100\cdots01)_3$. Since $13 = (111)_3$, we see that $3^k + 1$ is never a multiple of 13 and $3^k - 1$ is a multiple of 13 if and only if k is a multiple of 3. ♠

356. Let a and b be real parameters. One of the roots of the equation $x^{12} - abx + a^2 = 0$ is greater than 2. Prove that $|b| > 64$.

Solution 1. Clearly, $a \neq 0$. The equation can be rewritten $b = (x^{12} + a^2)/(ax)$. If $x > 2$, then

$$|b| = \frac{x^{12} + a^2}{|a|x} \geq \frac{2|a|x^6}{|a|x} = 2x^5 > 64 \ ,$$

by the arithmetic-geometric means inequality. ♠

Solution 2. [V. Krakovna] The equation can be rewritten

$$x^{12} + \left(\frac{bx}{2} - a\right)^2 = \frac{b^2x^2}{4} ,$$

whence $b^2x^2 = 4x^{12} + (bx - 2a)^2 \geq 4x^{12}$ and $b^2 \geq 4x^{10}$. If $|x| > 2$, then $b^2 \geq 2^{12}$ and so $|b| \geq 2^6$. ♠

357. Consider the circumference of a circle as a set of points. Let each of these points be coloured red or blue. Prove that, regardless of the choice of colouring, it is always possible to inscribe in this circle an isosceles triangle whose three vertices are of the same colour.

Solution 1. Consider any regular pentagon inscribed in the given circle. Since there are five vertices and only two options for their colours. the must be three vertices of the same colour. If they are adjacent, then two of the sides of the triangle they determine are sides of the pentagon, and so equal. If two are adjacent and the third opposite to the side formed by the first two, then once again they determine an isosceles triangle. As this covers all the possibilities, the result follows.

Solution 2. If at most finitely many points on the circumference are red, then it is possible to find an isosceles triangle with green vertices. (Why?) Suppose that there are infinitely many red points. Then there are two red points, P and Q, that are neither at the end of a diagonal nor two vertices of an inscribed equilateral triangle. Let U, V, W be three distinct points on the circumference of the circle unequal to P and Q for which $|UP| = |PQ| = |QV|$ and $|PW| = |QW|$. Then the triangles PQU, PQV, PQW and UVW are isoceles. Either one of the first three has red vertices, or the last one has green vertices. ♠

Rider. Can one always find both a red and a green isosceles triangle if there are infinitely many points of each colour?

358. Find all integers x which satisfy the equation

$$\cos\left(\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800})\right) = 1 .$$

Solution. We must have that

$$\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800}) = 2k\pi$$

for some integer k, whence

$$3x - \sqrt{9x^2 + 160x + 800} = 16k .$$

Multiplying by the surd conjugate of the left side yields

$$-160x - 800 = 16k(3x + \sqrt{9x^2 + 160x + 800})$$

so that

$$3x + \sqrt{9x^2 + 160x + 800} = \frac{1}{k}(-10x - 50) .$$

Therefore, $6x = 16k - (1/k)(10x + 50)$, whereupon $(3k + 5)x = 8k^2 - 25$. Multiplying by 9 yields that

$$9x(3k + 5) = 8(9k^2 - 25) - 25 = 8(3k - 5)(3k + 5) - 25 ,$$

whereupon $3k + 5$ is a divisor of 25, *i.e.*, one of the six numbers $\pm 1, \pm 5, \pm 25$. This leads to the three possibilities $(k, x) = (-2, -7), (0, -5), (-10, -31)$. The solution $x = -5$ is extraneous, so the given equation has only two integers solutions, $x = -31, -7$. ♠

359. Let ABC be an acute triangle with angle bisectors AA_1 and BB_1, with A_1 and B_1 on BC and AC, respectively. Let J be the intersection of AA_1 and BB_1 (the incentre), H be the orthocentre and O the circumcentre of the triangle ABC. The line OH intersects AC at P and BC at Q. Given that C, A_1, J and B_1 are vertices of a concyclic quadrilateral, prove that $PQ = AP + BQ$.

Solution. [Y. Zhao] Since CA_1JB_1 is concyclic, we have that

$$\angle C = 180^\circ - \angle AJB = \angle JAB + \angle JBA = \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90^\circ - \frac{1}{2}\angle C$$

so that $\angle C = 60^\circ$. (Here we used the fact that the sum of opposite angles of a concyclic quadrilateral is 180° and the sum-of-interior-angles theorem for triangle AJB.) Now, applying the same reasoning to $\angle AHB$ and using the fact that H is the orthocentre of triangle ABC, we find that

$$\begin{aligned}\angle AHB &= 180^\circ - \angle HAB - \angle HBA == 180^\circ - \angle HAB - \angle HCA \\ &= 180^\circ - (90^\circ - \angle ABC) - (90^\circ - \angle BAC) \\ &= \angle ABC + \angle BAC = 180^\circ - \angle ACB = 120^\circ \ .\end{aligned}$$

On the other hand, since O is the circumcentre of triangle ABC, $\angle AOB = 2\angle ACB = 120^\circ$. Therefore, $AOHB$ is concyclic. Now, $\angle PHA + \angle AHO = 180^\circ$ (supplementary angles) and $\angle OBA + \angle AHO = 180^\circ$ (opposite angles of concyclic quadrilateral), so that $\angle PHA = \angle OBA$. Next, in triangle AOB with $AO = OB$,

$$\angle OBA = \frac{1}{2}(180^\circ - \angle AOB) = 90^\circ - \frac{1}{2}\angle AOB = 90^\circ - \angle C = \angle PAH \ .$$

So, $\angle PHA = \angle PAH$; thus, triangle APH is isosceles and $AP = PH$. Similarly, $QB = QH$. Therefore $PQ = PH + QH = AP + BQ$ as desired. ♠

Solutions to the February problems.

348. (b) Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that both $f(x) - 3x$ and $f(x) - x^3$ are increasing functions. Must $f(x) - x - x^2$ also be increasing on all of the real numbers, or on at least the positive reals?

Solution 1. Let $u \geq v$. Suppose that $u+v \leq 2$. Then, since $f(x)-3x$ is increasing, $f(u)-3u \geq f(v)-3v$, whence

$$f(u) - f(v) \geq 3(u - v) \geq (u + v + 1)(u - v) = u^2 - v^2 + u - v \Longrightarrow f(u) - u - u^2 \geq f(v) - v - v^2 \ .$$

Suppose that $u + v \geq 2$. Then, since $f(x) - x^3$ is increasing,

$$f(u) - u^3 \geq f(v) - v^3 \Longrightarrow f(u) - f(v) \geq u^3 - v^3 = (u - v)(u^2 + uv + v^2) \ .$$

Now

$$2[(u^2 + uv + v^2) - (u + v + 1)] = (u + v)^2 + (u - 1)^2 + (v - 1)^2 - 4 \geq 0 \ ,$$

so that $u^2 + uv + v^2 \geq u + v + 1$ and

$$f(u) - f(v) \geq (u - v)(u + v + 1) = u^2 - v^2 + u - v \Longrightarrow f(u) - u - u^2 \geq f(v) - v - v^2 \ .$$

Hence $f(u) - u - u^2 \geq f(v) - v - v^2$ whenever $u \geq v$, so that $f(x) - x - x^2$ is increasing. ♠

Solution 2. [F. Barekat] Let $u \geq v$. Then, as in Solution 1, we find that $f(u) - f(v) \geq 3(u - v)$ and $f(u) - f(v) \geq u^3 - v^3 = (u - v)(u^2 + uv + v^2)$. If $1 \geq u \geq v$, then $3 \geq u + v + 1$, so that

$$f(u) - f(v) \geq 3(u - v) \geq (u + v + 1)(u - v) \ .$$

If $u \geq v \geq 1$, then $u^2 \geq u$, $v^2 \geq v$ and

$$f(u) - f(v) \geq (u-v)(u^2 + uv + v^2) \geq (u-v)(u+1+v) \ .$$

In either case, we have that $f(u) - u - u^2 \geq f(v) - v - v^2$. Finally, if $u \geq 1 \geq v$, then

$$f(u) - u - u^2 \geq f(1) - 2 \geq f(v) - v - v^2 \ .$$

The result follows. ♠

Comment. D. Dziabenko assumed that f was differentiable on $\mathbf{R}$, so that $f'(x) \geq 3$ and $f'(x) \geq 3x^2$ everywhere. Hence, for all x, $3f'(x) \geq 3x^2 + 6$, so that $f'(x) \geq x^2 + 2 \geq 2x + 1$. Hence, the derivative of $f(x) - x - x^2$ is always nonnegative, so that $f(x) - x - x^2$ is increasing. However, there is nothing in the hypothesis that forces f to be differentiable, so this is only a partial solution and its solver would have to settle for a grade of 2 out of 7. A little knowledge is a dangerous thing. If calculus is used, you need to make sure that everything is in place, all assumptions made identified and justified. Often, a more efficient and transparent solution exists without recourse to calculus.

360. Eliminate θ from the two equations

$$x = \cot\theta + \tan\theta$$
$$y = \sec\theta - \cos\theta \ ,$$

to get a polynomial equation satisfied by x and y.

Solution 1. We have that

$$x = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1}{\sin\theta\cos\theta} \Longrightarrow \sin\theta\cos\theta = \frac{1}{x} \ .$$

$$y = \frac{1}{\cos\theta} - \cos\theta = \frac{\sin^2\theta}{\cos\theta} \ .$$

Hence

$$\sin^3\theta = \frac{y}{x} \qquad \text{and} \qquad \cos^3\theta = \frac{1}{x^2 y}$$

so that

$$\left(\frac{y}{x}\right)^{\frac{2}{3}} + \left(\frac{1}{x^2 y}\right)^{\frac{2}{3}} = 1 \Longrightarrow (xy^2)^{\frac{2}{3}} + 1 = (x^2 y)^{\frac{2}{3}}$$
$$\Longrightarrow (x^2 y)^2 = 1 + (xy^2)^2 + 3(xy^2)^{\frac{2}{3}}(x^2 y)^{\frac{2}{3}} = 1 + x^2 y^4 + 3x^2 y^2 \ .$$

Hence

$$x^4 y^2 = 1 + x^2 y^4 + 3x^2 y^2 \ . ♠$$

Solution 2. [D. Dziabenko] Since $\sin^3\theta = y/x$ and $\cos^3\theta = 1/(x^2 y)$,

$$\begin{aligned} \frac{y^2}{x^2} + \frac{1}{x^4 y^2} &= \sin^6\theta + \cos^6\theta \\ &= (\sin^2\theta + \cos^2\theta)^3 - 3(\sin^2\theta + \cos^2\theta)\sin^2\theta\cos^2\theta \\ &= 1 - \frac{3}{x^2} \ . \end{aligned}$$

Hence $x^2 y^4 + 1 = x^4 y^2 - 3x^2 y^2$. ♠

Solution 3. [P. Shi] Using the fact that $\cos^3\theta = 1/(x^2 y)$ in the expression for y, we find that

$$y^3 = x^2 y - 3y - \frac{1}{x^2 y} \Longrightarrow x^2 y^4 = x^4 y^2 - 3x^2 y^2 - 1 \ . ♠$$

Solution 4. [Y. Zhao] Since $\sin^3\theta = y/x$ and $\cos^3\theta = 1/(x^2y)$, we have that

$$\sqrt[3]{\frac{y^2}{x^2}} + \sqrt[3]{\frac{1}{x^4y^2}} - 1 = 0 \ .$$

Using the identity $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ with the substitution $a = \sqrt[3]{y^2/x^2}$, $b = \sqrt[3]{1/(x^4y^2)}$, $c = -1$, we obtain that

$$\frac{y^2}{x^2} + \frac{1}{x^4y^2} - 1 + \frac{3}{x^2} = 0$$

or

$$x^2y^4 + 1 - x^4y^2 + 3x^2y^2 = 0 \ .\spadesuit$$

Comment. In a question like this, it is very easy to make a mechanical slip. Accordingly, it is prudent to make a convenient substitution of values to see if your identity works. For example, when $\theta = \pi/4$, $x = 2$ and $y = 1/\sqrt{2}$, and we find that the identity checks out.

361. Let $ABCD$ be a square, M a point on the side BC, and N a point on the side CD for which $BM = CN$. Suppose that AM and AN intersect BD and P and Q respectively. Prove that a triangle can be constructed with sides of length $|BP|$, $|PQ|$, $|QD|$, one of whose angles is equal to $60°$.

Solution 1. Let the sides of the square have length 1 and let $|BM| = u$. Then $|NC| = u$ and $|MC| = |ND| = 1-u$. Let $|BP| = a$, $|PQ| = b$ and $|QD| = c$. Since triangles APD and MPB are similar, $(a/u) = b+c$. Since triangle DQN and BQA are similar, $(c/(1-u))) = a+b$. Hence

$$(1-u+u^2)a = (2u-u^2)b \qquad \text{and} \qquad (1-u+u^2)c = (1-u^2)b$$

so that

$$a : b : c = (2u-u^2) : (1-u+u^2) : (1-u^2) \ .$$

Now

$$\begin{aligned}(2u-u^2)^2 &+ (1-u^2)^2 - 2(2u-u^2)(1-u^2)\cos 60° \\ &= (4u^2-4u^3+u^4) + (1-2u^2+u^4) - (2u-2u^3-u^2+u^4) \\ &= u^4 - 2u^3 + 3u^2 - 2u + 1 = (1-u+u^2)^2 \ .\end{aligned}$$

Thus $b^2 = a^2+c^2-ac$. Note that this implies that $(a-c)^2 < b^2 < (a+c)^2$, whence $a < b+c$, $b < a+c$ and $c < a+b$. Accordingly, a, b, c are the sides of a triangle and $b^2 = a^2+c^2-\frac{1}{2}ac\cos 60°$. From the law of cosines, the result follows. $\spadesuit$

Solution 2. [F. Barekat] *Lemma.* Let PQR be a right triangle with $\angle R = 90°$, $|QR| = p$ and $|PR| = q$. Then the length m of the bisector RT of angle R (with T on PQ) is equal to $(\sqrt{2}pq)/(p+q)$.

Proof. $[PQR] = [QRT] + [PRT] \Longrightarrow \frac{1}{2}pq = \frac{1}{2}pm\sin 45° + \frac{1}{2}qm\sin 45°$, from which the result follows. $\clubsuit$

Using the same notation as in Solution 1, the above lemma and the fact that BD bisects the right angles at B and D, we find that

$$a = \frac{\sqrt{2}u}{1+u} \ , \quad c = \frac{\sqrt{2}(1-u)}{2-u} \ , \quad b = \sqrt{2} - a - c \ .$$

Hence

$$\begin{aligned}(a^2+c^2-2ac\cos 60°) - b^2 &= (a^2+c^2-ac) - (2+a^2+c^2) + 2\sqrt{2}(a+c) - 2ac \\ &= 2\sqrt{2}(a+c) - 3ac - 2 \\ &= \frac{2}{(1+u)(2-u)}[2u(2-u) + 2(1+u)(1-u) - 3u(1-u) - (1+u)(2-u)] \\ &= \frac{2}{(1+u)(2-u)}[4u - 2u^2 + 2 - 2u^2 - 3u + 3u^2 - 2 - u + u^2] = 0 \ .\end{aligned}$$

Hence $b^2 = a^2 + c^2 - 2ac\cos 60^\circ$ and the result follows from the law of cosines. ♠

Solution 3. [V. Krakovna; R. Shapiro] Let a, b, c, u be as in Solution 1. Then $u = a/(b+c)$ and $1 - u = c/(a+b)$. Hence

$$1 = \frac{a}{b+c} + \frac{c}{a+b} = \frac{a^2 + c^2 + b(a+c)}{b^2 + b(a+c) + ac} ,$$

which is equivalent to $b^2 = a^2 + c^2 - ac$. As in Solution 1, we find that a, b, c are sides of a triangle and $b^2 = a^2 + c^2 - \frac{1}{2}ac\cos 60^\circ$. ♠

Comment. P. Shi used Menelaus' Theorem with triangle BOC and transversal APM and with triangle COD and transversal AQN to get

$$\frac{|BP|}{|PO|} \cdot \frac{|OA|}{|AC|} \cdot \frac{|CM|}{|MB|} = \frac{|DQ|}{|QO|} \cdot \frac{|OA|}{|AC|} \cdot \frac{|CN|}{|ND|} = 1 ,$$

where O is the centre of the square. Noting that $|PO| = (1/\sqrt{2}) - |BP|$ and $|QO| = (1/\sqrt{2}) - |DQ|$, we can determine the lengths of BP and DQ.

362. The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B, C meet the circle again at U, V, W, respectively. Prove that the area of triangle UVW is not less than the area of triangle ABC.

Solution 1. Let R be the common circumradius of the triangles ABC and UVW. Observe that $\angle WUA = \angle WCA = \frac{1}{2}\angle ACB$ and $\angle VUA = \angle VBA = \frac{1}{2}\angle ABC$, whence

$$U = \angle WUV = \frac{1}{2}(\angle ACB + \angle ABC) = \frac{1}{2}(B + C) ,$$

et cetera. Now

$$[ABC] = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C$$

and

$$[UVW] = \frac{uvw}{4R} = 2R^2 \sin U \sin V \sin W .$$

Since by the arithmetic-geometric means inequality,

$$\begin{aligned} \sqrt{\sin A \sin B} &= 2\sqrt{(\sin\frac{A}{2}\cos\frac{B}{2})(\cos\frac{A}{2}\sin\frac{B}{2})} \\ &\le \sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2} \\ &= \sin\frac{A+B}{2} = \sin W , \end{aligned}$$

et cetera, it follows that $[ABC] \le [UVW]$ with equality if and only if ABC is an equilateral triangle. ♠

Second solution. Since $\frac{1}{2}(A+B) = 90^\circ - \frac{C}{2}$, we find that

$$[ABC] = 2R^2 \sin A \sin B \sin C = 16R^2 \sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}$$

and

$$[UVW] = 2R^2 \sin U \sin V \sin W = 2R^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} ,$$

so that

$$\begin{aligned}\frac{[ABC]}{[UVW]} &= 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\ &= 8\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\\ &= \frac{8(s-a)(s-b)(s-c)}{abc} = \frac{8[ABC]^2}{sabc} = \frac{2r}{R} \le 1\end{aligned}$$

bu Euler's inequality for the inradius and the circumradius. The result follows. ♠

Third solution. Let I be the incentre and H the orthocentre of triangle ABC. Suppose the respective altitudes from A, B, C meet the circumcircle at P, Q, R. We have that

$$\angle BHP = \angle AHQ = 90^\circ - \angle HAC = 90^\circ - \angle PAC = 90^\circ - \angle PBC = \angle BPH$$

so that $BH = BP$. Similarly, $CH = CP$. Hence $\Delta HBC \equiv \Delta PBC$ (SSS). Similarly $\Delta HAC \equiv \Delta QAC$ and $\Delta HAB \equiv \Delta RAB$, whence $[ARBPCQ] = 2[ABC]$.

Let AU intersect VW at X. Then

$$\begin{aligned}\angle VXA &= \angle XWA + \angle XAW = \angle VWA + \angle UAB + \angle WAB\\ &= \angle VBA + \angle UAB + \angle WCB = \frac{1}{2}(\angle B + \angle A + \angle C) = 90^\circ\ ,\end{aligned}$$

so that UA is an altitude of triangle UVW, as similarly are VB and WC (so that I is the orthocentre of triangle UVW). Therefore, we have, as above, that $[UCVAWB] = 2[UVW]$.

But U is the midpoint of the arc BC, V the midpoint of the arc CA and W the modpoint of the orc AB. Thus, $[CPB] \le [CUB]$, $[AQC] \le [AVC]$ and $[BRA] \le [BWA]$. Therefore, $[ARBPCQ] \le [UCVAWB]$ and so $[ABC] \le [UVW]$. ♠

Comment. F. Barekat noted that for $0 < x < \pi$, the function $\log\sin x$ is concave so that $\sqrt{\sin u \sin v} \le \sin(\frac{1}{2}(u+v))$ for $0 \le u, v \le \pi$. (This can be seen by noting that the second derivative of $\log\sin x$ is $-\csc^2 x < 0$.) Then the solution can be completed as in Solution 1.

363. Suppose that x and y are positive real numbers. Find all real solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2}\ .$$

Preliminaries. It is clear that if one of x and y vanishes, then so must the other. Otherwise, there are two possibilities, according as x and y are both negative or both positive ($\sqrt{xy}$ needs to make sense). If x and y are both negative, then the only solution is $x = y$ as the equation asserts that the sum of the harmonic and geometric means of $-x$ and $-y$ is equal to the sum of the arithmetic mean and root-mean-square of these quantities. For unequal positive reals, each of the first two is less than each of the second two. Henceforth, we assume that x and y are positive. Let $h = 2xy/(x+y)$, $g = \sqrt{xy}$, $a = \frac{1}{2}(x+y)$ and $r = \sqrt{\frac{1}{2}(x^2+y^2)}$.

Solution 1. It is straightforward to check that $2a^2 = r^2 + g^2$ and that $g^2 = ah$. Suppose that $h + r = a + g$. Then

$$\begin{aligned}&r = a + g - h\\ &\implies 2a^2 - g^2 = r^2 = a^2 + g^2 + h^2 - 2ah - 2gh + 2ag\\ &\implies (a+h)(a-h) = a^2 - h^2 = 2(g^2 - ah) + 2g(a-h) = 0 + 2g(a-h)\\ &\implies a = h \quad \text{or} \quad a + h = 2g\ .\end{aligned}$$

In the latter case, $g = \frac{1}{2}(a+h) = \sqrt{ah}$, so that both possibilities entail $a = g = h$. But equality of these means occur if and only if $x = y$. ♠

Solution 2. [A. Cornuneanu] Observe that

$$r^2 - g^2 = \frac{1}{2}(x-y)^2 = 2a(a-h) \ .$$

Since, from the given equation,

$$r^2 - g^2 = (r-g)(r+g) = (a-h)(r+g) \ ,$$

it follows that $a = \frac{1}{2}(r+g)$. However, it can be checked that, in general,

$$a = \sqrt{\frac{r^2+g^2}{2}} \ ,$$

so that a is at once the arithmetic mean and the root-mean-square of r and g. But this can occur if and only if $r = g = a$ if and only if $x = y$. ♠

Solution 3. [R. O'Donnell] From the homogeneity of the given equation, we can assume without loss of generality that $g = 1$ so that $y = 1/x$. Then $h = 1/a$ and $r = \sqrt{2a^2-1}$. The equation becomes

$$r + \frac{1}{a} = a + 1 \quad \text{or} \quad \sqrt{2a^2-1} = 1 + a - \frac{1}{a} \ .$$

Squaring and manipulating leads to

$$0 = a^4 - 2a^3 + 2a - 1 = (a-1)^3(a+1)$$

whence $a = 1 = g$ and so $x = y = 1$. The main result follows from this. ♠

Solution 4. [D. Dziabenko] Let $2a = x+y$ and $2b = x-y$, so that $x = a+b$ and $y = a-b$, Then $a \neq 0$, $a \geq b$ and the equation becomes

$$\frac{a^2-b^2}{a} + \sqrt{a^2+b^2} = \sqrt{a^2-b^2} + a \Longleftrightarrow \sqrt{a^2+b^2} - \sqrt{a^2-b^2} = \frac{b^2}{a} \ .$$

Multiplying by $\sqrt{a^2+b^2} + \sqrt{a^2-b^2}$ yields that

$$2b^2 = (a^2+b^2) - (a^2-b^2) = \frac{b^2}{a}(\sqrt{a^2+b^2} + \sqrt{a^2-b^2}) \ ,$$

from which

$$\frac{\sqrt{a^2+b^2} + \sqrt{a^2-b^2}}{2} = a = \sqrt{\frac{(a^2+b^2)+(a^2-b^2)}{2}} \ .$$

The left side is the arithmetic mean and the right the root-mean-square of $\sqrt{a^2+b^2}$ and $\sqrt{a^2-b^2}$. These are equal if and only if $a^2+b^2 = a^2-b^2 \Leftrightarrow b = 0 \leftrightarrow x = y$. ♠

Solution 5. [M. Elqars] With $2a = x+y$ and $2b = x-y$, we have that $x^2+y^2 = 2(a^2+b^2)$ and $xy = a^2-b^2$. The equation becomes

$$\frac{a^2-b^2}{a} + \sqrt{a^2+b^2} = \sqrt{a^2-b^2} + a$$

$$\Longleftrightarrow \sqrt{a^2+b^2} - \sqrt{a^2-b^2} = a - \frac{a^2-b^2}{a} = \frac{b^2}{a}$$

$$\iff 2a^2 - 2\sqrt{a^4 - b^4} = \frac{b^4}{a^2}$$

$$\iff \sqrt{a^4 - b^4} = a^2 - \frac{b^4}{2a^2} = \frac{2a^4 - b^4}{2a^2} = \frac{a^4 - b^4}{2a^2} + \frac{a^2}{2}$$

$$\iff 0 = (a^4 - b^4) - 2a^2\sqrt{a^4 - b^4} + a^4 = [\sqrt{a^4 - b^4} - a^2]^2$$

$$\iff b = 0 \longleftrightarrow x = y \ .\spadesuit$$

364. Determine necessary and sufficient conditions on the positive integers a and b such that the vulgar fraction a/b has the following property: *Suppose that one successively tosses a coin and finds at one time, the fraction of heads is less than a/b and that at a later time, the fraction of heads is greater than a/b; then at some intermediate time, the fraction of heads must be exactly a/b.*

Solution. Consider the situation in which a tail is tossed first, and then a head is tossed thereafter. Then the fraction of heads after n tosses is $(n-1)/n$. Since any positive fraction a/b less than 1 exceeds this for $n = 1$ and is less than this for n sufficiently large, a/b can be realized as a fraction of head tosses only if it is of this form (*i.e.* $a = b - 1$).

On the other hand, suppose that $a/b = (n-1)/n$ for some positive integer n. There must exist numbers p and q for which the fraction p/q of heads at one toss is less than a/b and the fraction $(p+1)/(q+1)$ at the next toss is not less than a/b. Thus

$$\frac{p}{q} < \frac{n-1}{n} \leq \frac{p+1}{q+1} \ .$$

Hence $np < nq - q$ and $nq - q + n - 1 \leq np + n$, so that

$$np < nq - q \leq np + 1 \ .$$

Since the three members of this inequality are integers and the outer two are consecutive, we must have $nq - q = np + 1$, whence

$$\frac{n-1}{n} = \frac{p+1}{q+1} \ .$$

Hence the necessary and sufficient condition is that $a/b = (n-1)/n$ for some positive integer n. $\spadesuit$

Rider. What is the situation when the fraction of heads moves from a number greater than a/b to a number less than a/b?

365. Let $p(z)$ be a polynomial of degree greater than 4 with complex coefficients. Prove that $p(z)$ must have a pair u, v of roots, not necessarily distinct, for which the real parts of both u/v and v/u are positive. Show that this does not necessarily hold for polynomials of degree 4.

Solution. Since the degree of the polynomial exceeds 4, there must be two roots u, v in one of the four quadrants containing a ray from the origin along either the real or the imaginary axis along with all the points within the region bounded by this ray and the next such ray in the counterclockwise direction. The difference in the arguments between two such numbers must be strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\arg(u/v) = \arg u - \arg v$ and $\arg(v/u) = \arg v - \arg u$ both lie in this range, both u/v and v/u lie to the right of the imaginary axis, and so have positive real parts.

This result does not necessarily hold for a polynomial of degree 4, as witnessed by $z^4 - 1$ whose roots are $1, -1, i, -i$.

366. What is the largest real number r for which

$$\frac{x^2 + y^2 + z^2 + xy + yz + zx}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \geq r$$

holds for all positive real values of x, y, z for which $xyz = 1$.

Solution 1. Let u, v, w be positive reals for which $u^2 = yz$, $v^2 = zx$ and $w^2 = xy$. Then $\sqrt{x} = x\sqrt{yz} = xu$, $\sqrt{y} = yv$ and $\sqrt{z} = zw$, so that

$$\begin{aligned}(x^2+y^2+z^2)+(xy+yz+zx) &= (x^2+y^2+z^2)+(u^2+v^2+w^2)\\ &\geq 2\sqrt{x^2+y^2+z^2}\sqrt{u^2+v^2+w^2} \geq 2(xu+yv+zw)\ ,\end{aligned}$$

from the arithmetic-geometric means and the Cauchy-Schwarz inequalities. Hence, the inequality is always valid when $r \leq 2$. When $(x, y, z) = (1, 1, 1)$, the left side is equal to 2, so the inequality does not always hold when $r > 2$. Hence the largest value of r is 2. ♠

Solution 2. Applying the arithmetic-geometric means inequality to the left side yields

$$\begin{aligned}\frac{(x^2+yz)+(y^2+zx)+(z^2+xy)}{\sqrt{x}+\sqrt{y}+\sqrt{z}} &\geq \frac{2(\sqrt{x^2yz}+\sqrt{y^2zx}+\sqrt{z^2xy}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\\ &= \frac{2\sqrt{xyz}(\sqrt{x}+\sqrt{y}+\sqrt{z})}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = 2\ .\end{aligned}$$

Equality occurs when $(x, y, z) = (1, 1, 1)$. Hence the largest value of r for which the inequality always holds is 2. ♠

Solution 3. [V. Krakovna]

$$\begin{aligned}\frac{x^2+y^2+z^2+xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}} &= \frac{\frac{1}{2}(x^2+y^2)+\frac{1}{2}(y^2+z^2)+\frac{1}{2}(z^2+x^2)+xy+yz+zx}{x\sqrt{yz}+y\sqrt{xz}+z\sqrt{xy}}\\ &\geq \frac{(xy+yz+zx)+(xy+yz+zx)}{x(y+z)/2+y(x+z)/2+z(x+y)/2}\\ &= \frac{2(xy+yz+zx)}{xy+yz+zx} = 2\ .\end{aligned}$$

Equality occurs if and only if $x = y = z$.

Solution 4. [R. Shapiro] Applying the arithmetic-geometric means and Cauchy-Schwarz inequalities, we have that

$$\begin{aligned}x^2+y^2+z^2+xy+yz+zx &= (x^2+y^2+z^2)+(xy+yz+zx)\\ &\geq 2\sqrt{(x^2+y^2+z^2)(xy+yz+zx)}\\ &\geq 2(x\sqrt{yz}+y\sqrt{zx}+z\sqrt{xy}) = 2(\sqrt{x}+\sqrt{y}+\sqrt{z})\ ,\end{aligned}$$

with equality if and only if $x = \sqrt{yz}$, $y = \sqrt{zx}$, $z = \sqrt{xy}$, if and only if $x = y = z = 1$. ♠