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PROBLEMS FOR FEBRUARY

Please send your solution to
Prof. Edward J. Barbeau
Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3

no later than February 28, 2005 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

348. (b) Syppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that both $f(x)-3x$ and $f(x)-x^3$ are increasing functions. Must $f(x)-x-x^2$ also be increasing on all of the real numbers, or on at least the positive reals?

360. Eliminate θ from the two equations

$$x = \cot\theta + \tan\theta$$

$$y = \sec\theta - \cos\theta \ ,$$

to get a polynomial equation satisfied by x and y.

361. Let $ABCD$ be a square, M a point on the side BC, and N a point on the side CD for which $BM = CN$. Suppose that AM and AN intersect BD and P and Q respectively. Prove that a triangle can be constructed with sides of length $|BP|$, $|PQ|$, $|QD|$, one of whose angles is equal to $60°$.

362. The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B, C meet the circle again at U, V, W, respectively. Prove that the area of triangle UVW is not less than the area of triangle ABC.

363. Suppose that x and y are positive real numbers. Find all real solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2} \ .$$

364. Determine necessary and sufficient conditions on the positive integers a and b such that the vulgar fraction a/b has the following property: *Suppose that one successively tosses a coin and finds at one time, the fraction of heads is less than a/b and that at a later time, the fraction of heads is greater than a/b; then at some intermediate time, the fraction of heads must be exactly a/b.*

365. Let $p(z)$ be a polynomial of degree greater than 4 with complex coefficients. Prove that $p(z)$ must have a pair u, v of roots, not necessarily distinct, for which the real parts of both u/v and v/u are positive. Show that this does not necessarily hold for polynomials of degree 4.

366. What is the largest real number r for which

$$\frac{x^2+y^2+z^2+xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}} \geq r$$

holds for all positive real values of x, y, z for which $xyz = 1$.

Solutions for October problem.

Comment on problems 339 and 342. In both these problems, a condition was left out and made each of them trivial. Accordingly, problem 339 is marked out of 4 and problem 342 out of 3, for the basic solution. However, additional marks were provided for students who recognized that the problems might have been misstated and provided work that led to the intended solutions. While I try to make sure that the problems are correct, and certainly on contests, the problems are generally gone over very carefully, mistakes do occur. If on a contest, you feel that a mistake has been made in formulating a problem, then you should state clearly a *non-trivial* version of the problem and solve that. In the solutions below, the corrected version of the problem is given.

339. Let a, b, c be integers with $abc \neq 0$, and u, v, w be integers, not all zero, for which

$$au^2 + bv^2 + cw^2 = 0 \ .$$

Let r be any rational number. Prove that the equation

$$ax^2 + by^2 + cz^2 = r$$

is solvable for rational values of x, y, z.

Solution 1. Suppose, wolog, that $u \neq 0$. Try a solution of the form

$$(x, y, z) = (u(1 + t), vt, wt) \ .$$

Then $au^2 + 2au^2t + au^2t^2 + b^2v^2t^2 + cw^2t^2 = r$ implies that $2au^2t = r - au^2$, from which we find the value $t = (r - au^2)/((2au^2)$. Since a, b, c, u, v, w, r, t are all rational, so is the trial solution.

Solution 2. [R. Peng] Suppose that $r = p/q$. Then the equation with r on the right is satisfied by

$$(x, y, z) = \left(\frac{p}{2} + \frac{1}{2qa}, \left(\frac{p}{2} - \frac{1}{2qa} \right) \left(\frac{v}{u} \right), \left(\frac{p}{2} - \frac{1}{2qa} \right) \left(\frac{w}{u} \right) \right) \ .$$

Comment. If not all a, b, c are zero, then it is trivial to prove that the equation with r on the right has a solution; the only rôle that the equation with 0 on the right plays is to ensure that a, b, c do not all have the same sign. However, some made quite heavy weather of this. The hypothesis that integers are involved should alert you to the fact that some special character of the solution is needed. It is unreasonable to ask that the solution be in integers, but one could seek out rational solutions.

340. The lock on a safe consists of three wheels, each of which may be set in eight different positions. Because of a defect in the safe mechanism, the door will open if any two of the three wheels is in the correct position. What is the smallest number of combinations which must be tried by someone not knowing the correct combination to guarantee opening the safe?

Solution. The smallest number of combinations that will guarantee success is 32. Denote the eight positions of each whell by the digits 0, 1, 2, 3, 4, 5, 6, 7, so that each combination can be represented by an ordered triple (a, b, c) of three digits. We show that a suitably selected set of 32 combinations will do the job. Let $A = \{(a, b, c) : 0 \leq a, b, c \leq 3 \text{ and } a + b + c \equiv 0 (\text{mod } 4\}$ and $B = \{(u, v, w) : 4 \leq u, v, w \leq 7 \text{ and } u + v + w \equiv 0 (\text{mod })4\}$. Any entry in the triples of A and B is uniquely determined by the other two, and any ordered pair is a possibility for these two. Thus, each of A and B contains exactly 16 members. If (p, q, r) is any combination, then either two of p, q, r belong to the set $\{0, 1, 2, 3\}$ and agree with corresponding entries in a combination in A, or two belong to $\{4, 5, 6, 7\}$ and agree with corresponding entries in a combination in B.

We now show that at least 32 combinations are needed. Suppose, if possible, that a set S of combinations has three members whose first entry is 0: $(0,a,b),(0,c,d),(0,e,f)$. There will be twenty-five combinations of the form $(0,y,z)$ with $y \neq a,c,e$, $z \neq b,d,f$, that will not match in two entries any of these three. To cover such combinations, we will need at least 25 distinct combinations of the form (x,y,z) with $1 \leq x \leq 7$. None of the 28 combinations identified so far match the $7 \times 6 = 42$ combinations of the form (u,v,w), where $v \in \{a,c,e\}$, $w \in \{b,d,f\}$, $(v,w) \neq (a,b),(c,d),(e,f)$. Any combination of the form (u,v,t) or (u,t,w) can cover at most three of these and of the form (t,v,w) at most 7. Thus, S will need at least $37 = 3+25+(42/7)$ members to cover all the combinations. A similar argument obtains if there are only three members in S with any other given entry. If there are only one or two members in S with a given entry, say first entry 0, then at least 36 combinations would be needed to cover all the entries with first entry 0 and the other entries differing from the entries of these two elements of S.

Thus, a set of combinations will work only if there are at least four combinations with a specific digit in each entry, in particular at least four whose first entry is k for each of $k = 0,\cdots,7$. Thus, at least 32 entries are needed.

Comment. Some solvers formulated the problem in terms of the minimum number of rooks (castles) required to occupy or threaten every cell of a solid $8 \times 8 \times 8$ chessboard.

341. Let s,r,R respectively specify the semiperimeter, inradius and circumradius of a triangle ABC.

(a) Determine a necessary and sufficient condition on s,r,R that the sides a,b,c of the triangle are in arithmetic progression.

(b) Determine a necessary and sufficient condition on s,r,R that the sides a,b,c of the triangle are in geometric progression.

Comment. In the solutions, we will use the following facts, the establishment of which is left up to the reader:

$$a+b+c = 2s$$
$$ab+bc+ca = s^2+4Rr+r^2$$
$$abc = 4Rrs$$

An efficient way to get the second of these is to note that the square of the area is given by $r^2s^2 = s(s-a)(s-b)(s-c)$ from which

$$r^2s = s^3-(a+b+c)s^2+(ab+bc+ca)s-abc = s^3-2s^3+(ab+bc+ca)s-4Rrs\ .$$

Solution 1. (a) a,b,c are in arithmetic progression if and only if

$$\begin{aligned}0 &= (2a-b-c)(2b-c-a)(2c-a-b)\\ &= (2s-3a)(2s-3b)(2s-3c)\\ &= 8s^3-12s^2(a+b+c)+18s(ab+bc+ca)-27abc\\ &= 2s^3-36Rrs+18r^2s\ .\end{aligned}$$

Since $s \neq 0$, the necessary and sufficient condition that the three sides be in arithmetic progression is that $s^2+9r^2 = 18Rr$.

(b) First, note that

$$\begin{aligned}a^3+b^3+c^3 &= (a+b+c)^3-3(a+b+c)(ab+bc+ca)+3abc\\ &= 2s^3-12Rrs-6r^2s\ ,\end{aligned}$$

and

$$\begin{aligned}a^3b^3+b^3c^3+c^3a^3 &= (ab+bc+ca)^3-3abc(a+b+c)(ab+bc+ca)+3(abc)^2\\ &= (s^2+4Rr+r^2)^3-24Rrs^4-48R^2r^2s^2-24Rr^3s^2\ .\end{aligned}$$

a, b, c are in geometric progression if and only if

$$\begin{aligned} 0 &= (a^2 - bc)(b^2 - ca)(c^2 - ab) \\ &= abc(a^3 + b^3 + c^3) - (a^3b^3 + b^3c^3 + c^3a^3) \\ &= 32Rrs^4 - (s^2 + 4Rr + r^2)^3 \ , \end{aligned}$$

The necessary and sufficient condition is that

$$(s^2 + 4Rr + r^2)^3 = 32Rrs^4 \ .$$

Solution 2. The three sides of the triangle are the three real roots of the cubic equation

$$x^3 - 2sx^2 + (s^2 + r^2 + 4Rr)x - 4Rrs = 0 \ .$$

The three sides are in arithmetic progression if and only if one of them is equal to $2s/r$ and are in geometric progression if and only if one of them is equal to their geometric mean $\sqrt[3]{4Rrs}$.

(a) The condition is that $2s/3$ satisfies the cubic equation:

$$0 = 8s^3 - 6s(4s^2) + 9(s^2 + r^2 + 4Rr)(2s) - 108Rrs = 2s(s^2 + 9r^2 - 18Rr) \ .$$

(b) The condition is that $\sqrt[3]{4Rrs}$ satisfies the cubic equation: $2s(4Rrs)^{1/3} = s^2 + 4Rr + r^2$ or $32Rrs^3 = (s^2 + 4Rr + r^2)^3$.

Solution 3. [B.H. Deng] Assume that b lies between a and c, inclusive. (a) The three sides are in arithmetic progression if and only if $b = \frac{2}{3}s$ or $a + c = 2b$. Since $4Rrs = abc$, this is equivalent to $6Rr = ac$, which in turn is equivalent to

$$r^2 + s^2 + 4Rr = (a + c)b + ac = 2b^2 + ac = (8/9)s^2 + 6Rr$$

or $s^2 + 9r^2 = 18Rr$.

(b) The three sides are in geometric progression if and only if $b^3 = abc = 4Rrs$ and $ac = b^2$. This holds if and only if

$$r^2 + s^2 + 4Rr = (a + c)b + ac = (2s - b)b + ac = 2s\sqrt[3]{4Rrs} - b^2 + ac = 2s\sqrt[3]{4Rrs}$$

or $(r^2 + s^2 + 4Rr)^3 = 32Rrs^4$.

342. Prove that there are infinitely many solutions in positive integers, whose greatest common divisor is equal to 1, of the system

$$\begin{aligned} a + b + c &= x + y \\ a^3 + b^3 + c^3 &= x^3 + y^3 \ . \end{aligned}$$

Solution 1. Suppose that a, b, c are in arithmetic progression, so that $c = 2b - a$ and $x + y = 3b$. Then

$$x^2 - xy + y^2 = \frac{a^3 + b^3 + c^3}{a + b + c} = 3b^2 - 4ab + 2a^2$$

so that

$$3xy = (x + y)^2 - (x^2 - xy + y^2) = 6b^2 + 4ab - 2a^2$$

and

$$xy = 2b^2 + \frac{2a(2b - a)}{3} \ .$$

Therefore

$$(y-x)^2 = (x+y)^2 - 4xy = b^2 - \frac{8a(2b-a)}{3} = \frac{(3b-8a)^2 - 40a^2}{9} \ .$$

Let $p = 3b - 8a$, $q = 2a$. We can get solutions by solving $p^2 - 10q^2 = 9$. Three solutions are $(p,q) = (3,0), (7,2), (13,4)$. The fundamental solution of $u^2 - 10v^2 = 1$ is $(u,v) = (19,6)$. So from any solution $(p,q) = (r,s)$ of $p^2 - 10q^2 = 9$, we get another $(p,q) = (19r + 60s, 6r + 19s)$. For these to yield solutions $(a,b,c;x,y)$ of the original system, we require q to be even and $p + 4q$ to be divisible by 3. Since $19r + 60s \equiv r \pmod 3$ and $6r + 19s \equiv s \pmod 2$, if $(p,q) = (r,s)$ has these properties, then so also does $(p,q) = (19r+60s, 6r+19s)$. Starting with (r,s), we can define integers p and q, and then solve the equations $x + y = 3b$, $y - x = 1$. Since p and so b are odd, these equations have integer solutions. Here are some examples:

$$\begin{aligned}(p,q;a,b,c;x,y) =& (3,0;0,1,2;1,2), (57,18;9,43,77;64,65),\\ & (7,2;1,5,9;7,8), (253,80;40,191,342;286,287) \ .\end{aligned}$$

Solution 2. [D. Dziabenko] Let $a = 3d$. $c = 2b - 3d$, so that $x + y = 3b$ and a, b, c are in arithmetic progression. Then

$$\begin{aligned}a^3 + b^3 + c^3 &= 27d^3 + b^3 + 8b^3 - 36b^2d + 54bd^2 - 27d^3 \\ &= 9b^3 - 36b^2d + 54bd^2 = 9b(b^2 - 4b^2d + 6d^2) \ ,\end{aligned}$$

whence $x^2 - xy + y^2 = 3b^2 - 12bd + 18d^2$. Therefore

$$3xy = (x+y)^2 - (a^2 - xy + y^2) = 6b^2 + 12bd - 18d^2$$

so that $xy = 2b^2 + 4bd - 6d^2$ and

$$(x-y)^2 = (x+y)^2 - 4xy = b^2 - 16bd + 24d^2 = (b-8d)^2 - 40d^2 \ .$$

Let $b - 8d = p^2 + 10q^2$ and $d = pq$. Then

$$x - y = \sqrt{p^4 - 20p^2q^2 + 100q^4} = p^2 - 10q^2 \ .$$

Solving this, we find that

$$(a,b,c;x,y) = (3pq, p^2 + 8pq + 10q^2, 2p^2 + 13pq + 20q^2; 2p^2 + 12pq + 10q^2, p^2 + 12pq + 20q^2) \ .$$

Some numerical examples are

$$(a,b,c;p,q) = (3,19,35;24,33), (6,30,54;42,48), (6,57,108;66,105) \ .$$

Any common divisor of a and b must divide $3pq$ and $p^2 + 10q^2$, and so must divide both p and q. [Justify this; you need to be a little careful.] We can get the solutions we want by arranging that the p and q are coprime.

Solution 3. [F. Barekat] Let $m = a + b + c = x + y$ and $n = a^3 + b^3 + c^3 = x^3 + y^3$. Then

$$3xy = m^2 - \frac{n}{m} = \frac{m^3 - n}{m}$$

and

$$\begin{aligned}(x-y)^2 = \frac{x^3 + y^3}{x+y} - xy &= \frac{4n - m^3}{3n} \\ &= \frac{4(a^3 + b^3 + c^3) - (a+b+c)^3}{3(a+b+c)} \\ &= (c - a - b)^2 - \frac{4ab(a+b)}{a+b+c} \ .\end{aligned}$$

Select a, b, c so that

$$\frac{4ab(a+b)}{a+b+c} = 2(c-a-b) - 1 \ .$$

so that $x-y = c-a-b-1$. Then we can solve for rational values of x and y. If we can do this $x+y = a+b+c$ and $x-y = c-a-b-1$. Note that, these two numbers have different parity, so we will obtain fractional values of x and y, whose denominators are 2. However, the equations to be solved are homogeneous, so we can get integral solutions by doubling: $(2a, 2b, 2c; 2x, 2y)$.

Let $c = u(a+b)$. Then

$$4ab = 2(u-1)(u+1)(a+b) - (u+1) \ .$$

Let $u = 4v+3$, Then we get

$$ab = 4(v+1)(2v+1)(a+b) - 4(v+1) \ ,$$

from which

$$[a - 4(v+1)(2v+1)][b - 4(v+1)(2v+1)] = (v+1)[16(v+1)(2v+1)^2 - 1] \ .$$

We use various factorizations of the right side and this equation to determine integer values of a and b, from which the remaining variables c, x and y can be determined.

For example, $v = 0$ yields the equation $(a-4)(b-4) = 15$ from which we get the possibilities

$$(a, b, c) = (5, 19, 72), (7, 9, 48) \ .$$

Doubling to clear fractions, yields the solutions

$$(a, b, c; x, y) = (10, 38, 144; 49, 143), (14, 18, 96; 33, 95) \ .$$

Additional solutions come from $v = 1$:

$$(a, b, c; x, y) = (76, 130, 1442; 207, 1441), (50, 1196, 8722; 1247, 8721) \ .$$

Solution 4. [D. Rhee] An infinite set of solutions is given by the formula

$$\begin{aligned}(a, b, c; x, y) &= (2, n^2+3n, n^2+5n+4; n^2+4n+2, n^2+4n+4) \\ &= (2, n(n+3), (n+1)(n+4); (n+2)^2 - 2, (n+2)^2) \ .\end{aligned}$$

Examples are $(a, b, c; x, y) = (2, 4, 10; 7, 9), (2, 10, 18; 14, 16), (2, 18, 28; 23, 25)$.

Comment. M. Fatehi gave the solution

$$(a, b, c; x, y) = (5, 6, 22; 12, 21) \ .$$

343. A sequence $\{a_n\}$ of integers is defined by

$$a_0 = 0 \ , \quad a_1 = 1 \ , \quad a_n = 2a_{n-1} + a_{n-2}$$

for $n > 1$. Prove that, for each nonnegative integer k, 2^k divides a_n if and only if 2^k divides n.

Solution 1. Let m and n be two nonnegative integers. Then $a_{m+n} = a_m a_{n+1} + a_{m-1}a_n = a_{m+1}a_n + a_m a_{n-1}$. This can be checked for small values of m and n and established by induction. The induction step is

$$\begin{aligned}a_{m+n+1} &= 2a_{m+n} + a_{m+n-1} = 2(a_m a_{n+1} + a_{m-1}a_n) + (a_m a_n + a_{m-1}a_{n-1} \\ &= a_m(2a_{n+1} + a_n) + a_{m-1}(2a_n + a_{n-1}) = a_m a_{n+2} + a_{m-1}a_{n+1} \ .\end{aligned}$$

In particular, for each integer n,

$$a_{2n} = a_n(a_{n-1} + a_{n+1}) \ .$$

It is straightforward to show by induction from the recursion that a_n is odd whenever n is odd and even whenever n is even. Suppose now that n is even. Then $a_{n+1} = 2a_n + a_{n-1} \equiv a_{n-1} \equiv a_1 = 1 \pmod 4$, so that $a_{n-1} + a_{n+1} = 2b_n$ for some odd number b_n. Hence $a_{2n} = 2a_nb_n$. For $k = 0$, we have that $2^k|a_n$ if and only if $2^k|n$. Suppose that this has been established for $k = r$.

Suppose that $n = 2^{r+1}m$ for some integer m. Then $n/2$ is divisible by 2^r, and therefore so is $a_{n/2}$. Hence $a_n = 2a_{n/2}b_{n/2}$ is divisible by 2^{r+1}. On the other hand, suppose that n is not divisible by 2^{r+1}. If n is not divisible by 2^r, then a_n is not so divisible by the induction hypothesis, and so not divisible by 2^{r+1}. On the other hand, if $n = 2^rc$, with c odd, then a_n is divisible by 2^r. But $n/2 = 2^{r-1}c$, so $a_{n/2}$ is not divisible by 2^r. Hence $a_n = 2a_{n/2}b_{n/2}$ is not divisible by 2^{r+1}. The result follows.

Solution 2. For convenience, imagine that the sequence is continued backwards using the recursion $a_{n-2} = a_n - 2a_{n-1}$ for all integer values of the index n. We have for every integer n, $a_{n+1} = 2a_n + a_{n-1} \Rightarrow 2a_{n+1} = 4a_n + 2a_{n-1} \Rightarrow a_{n+2} - a_n = 4a_n + a_n - a_{n-2} \Rightarrow a_{n+2} = 6a_n - a_{n-2}$. Suppose, for some positive integer r, we have established that, for every integer n,

$$a_{n+2^r} = b_r a_n - a_{n-2^r}$$

where $b^r \equiv 2 \pmod 4$. This is true for $r = 1$ with $b_1 = 6$. Then

$$\begin{aligned} b_r a_{n+2^r} &= b_r^2 a_n - b_r a_{n-2^r} \\ \Longrightarrow a_{n+2^{r+1}} + a_n &= b_r^2 a_n - (a_n + a_{n-2^{r+1}}) \\ \Longrightarrow a_{n+2^r} &= b^{r+1} a_n - a_{n-2^{r+1}} \ , \end{aligned}$$

where $b_{r+1} = b_2^2 - 2 \equiv 2 \pmod 4$.

Observe that, since $a_{n+1} \equiv a_{n-1} \pmod 2$ and $a_0 = 0$, $a_1 = 1$, a_n is even if and only if n is even. When n is even, then $a_{n+2} \equiv a_{n-2} \pmod 4$, so that a_n is divisible by 4 if and only if n is.

Let $m \geq 2$ be a positive integer. Suppose that it has been established for $1 \leq s \leq m$, that 2^s divides a_n if and only if 2^s divides n. Then 2^{s+1} will divide a_n only if $n = 2^sp$ for some integer p. Now

$$a_{2^s} = b_{s-1}a_{2^{s-1}} - a_0 = b_{s-1}a_{2^{s-1}} \ ;$$

since $2\|b^{s-1}$ and $2^{s-1}\|a^{2^{s-1}}$, it follows that $2^s\|a_{2^s}$. (The notation $2^k\|q$ means that 2^k is the highest power of 2 that divides q.) Thus 2^{s+1} does not divide 2^s.

Suppose that it has been established for $1 \leq i \leq p$ that when $n = 2^si$, $2^{s+1}|n$ if and only if p is even. We have that

$$a_{2^s(p+1)} = b_s 2^s p - a_{2^s(p-1)} \ .$$

If p is even, then $b^s a_{2^sp} \equiv 0 \pmod{2^{s+1}}$, so that $a_{2^s(p+1)} \equiv a_2^s(p-1) \equiv 2^s \pmod{2^{s+1}}$, and $a_{2^s(p+1)}$ is not a multiple of 2^{s+1}. If p is odd, then each term on the right side of the foregoing equation is a multiple of 2^{s+1}, and therefore so is $a^{2^s(p+1)}$. The desired result follows by induction.

Solution 3. The characteristic equation for the recursion is $t^2 - 2t - 1 = 0$, with roots $t = 1 \pm \sqrt{2}$. Solving the recursion, we find that

$$\begin{aligned} a_n &= \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n - (1-\sqrt{2})^n] \\ &= \frac{1}{\sqrt{2}}\left[\sum_{k=0}^{\infty}\binom{n}{2k+1}2^k\sqrt{2}\right] \\ &= \sum_{k=0}^{\infty}\binom{n}{2k+1}2^k = n + \sum_{k=1}^{\infty}\binom{n}{2k+1}2^k \\ &= n + \sum_{k=1}^{\infty}\frac{n}{2k+1}\binom{n-1}{2k}2^k \ . \end{aligned}$$

(We use the convention that $\binom{i}{j} = 0$ when $i < j$. Suppose that $n = 2^r s$ where r is a nonnegative integer and s is odd. Since the odd number $2k+1$ divides $n\binom{n-1}{2k} = 2^r s\binom{n-1}{2k}$, $2k+1$ must divide $s\binom{n-1}{2k}$, so that 2^s must divide $n\binom{n-1}{2k} = \binom{n}{2k+1}$. Therefore, 2^{s+1} must divide each term $\binom{n}{2k+1}$ for $k \geq 1$. Therefore $a_n \equiv n \pmod{2^s}$ and the desired result follows.

Comment. Y. Zhao obtained by induction that

$$\begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$$

from which the matrix equation

$$\begin{pmatrix} a_{2n+1} & a_{2n} \\ a_{2n} & a_{2n-1} \end{pmatrix} = \begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{2n-1} \end{pmatrix}^2$$

yields the equation $a_{2n} = a_n(a_{n-1} + a_{n+1}$.

344. A function f defined on the positive integers is given by

$$f(1) = 1 \ , \quad f(3) = 3 \ , \quad f(2n) = f(n) \ ,$$

$$f(4n+1) = 2f(2n+1) - f(n)$$
$$f(4n+3) = 3f(2n+1) - 2f(n) \ ,$$

for each positive integer n. Determine, with proof, the number of positive integers no exceeding 2004 for which $f(n) = n$.

Solution. Let $g(n)$ be defined for positive integer n by writing n to base 2 and reversing the digits. Specifically, if $n = \sum_{k=0}^{r} a_k 2^r$ with each a_k equal to 0 or 1 and $a_r = 1$, then $g(n) = \sum_{k=0}^{n} a_{r-k} 2^k$. We prove that $g(n)$ has the properties ascribed to $f(n)$. It is checked that $g(1) = g(2) = g(3) = 1$. Let $n = a_r 2^r + a_{r-1}2^{r-1} + \cdots + a_1 2 + a_0$. Then $2n = a_r 2^{r+1} + \cdots + a_0 2 + 0$ and $g(2n) = 0 \cdot 2^{r+1} + a_0 2^r + \cdots + a_{r-1}2 + a_r = g(n)$.

Since $4n+1 = a_r 2^{r+2} + a_{r-1}2^{r+1} + \cdots + a_1 2^3 + a_0 2 + 0 \cdot 2 + 1$,

$$\begin{aligned} g(n) + g(4n+1) &= (a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a_r) + (2^{r+2} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a_r) \\ &= 2^{r+2} + a_0 2^{r+1} + a_1 r^r + \cdots + a_{r-1}2^2 + a_r 2 = 2(2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a_r \\ &= 2g(a_2 2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a_r) = 2g(2n+1) \ . \end{aligned}$$

(This uses the fact that $a2^i + a2^i = a2^{i+1}$.)

Since $4n+3 = a_r 2^{r+2} + a_{r-1}2^{r+1} + \cdots + a_1 2^3 + a_0 2 + 1 \cdot 2 + 1$,

$$\begin{aligned} &2g(n) + g(4n+3) \\ &\quad = (a_0 2^{r+1} + a_1 2^r + a_2 2^{r-1} + \cdots + a_{r-1}2^2 + a_r 2) \\ &\qquad + (2^{r+2} + 2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1}2 + a^r) \\ &\quad = (2^{r+2} + a_0 2^{r+1} + a_1 2^r + \cdots + a_r 2) + (2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_r) \\ &\quad = 2g(2n+1) + g(2n+1) = 3g(2n+1) \ . \end{aligned}$$

We show by induction that $f(n) = g(n)$ for every positive integer n. This is true for $1 \leq n \leq 4$. Suppose it holds for $1 \leq n \leq 4m$. Then

$$f(4m+1) = 2f(2m+1) - f(m) = 2g(2m+1) - g(m) = g(4m+1) \ ;$$

$$f(4m+2) = f(2m+1) = g(2m+1) = g(4m+2) \ ;$$
$$f(4m+3) = 3f(2m+1) - 2f(m) = 3g(2m+1) - 2g(m) = g(4m+3) \ ;$$
$$f(4m+4) = f(2m+2) = g(2m+2) = f(4m+4) \ .$$
Thus we have a description of $f(n)$.

For $f(n) = n$, it is necessary and sufficient that n is a palindrome when written to base 2. We need to find the number of palindromes between 1 and $2004 = (11111010100)_2$ inclusive. The number of $(2r-1)-$ and $2r-$digit palindromes is each 2^{r-1} as the first and last digits must be 1 and there are $r-1$ other matching pairs of digits or central digits that can be set to either 0 or 1. The number of palindromes up to $2^{11} - 1 = 2047$ is $2(1+2+4+8+16) + 32 = 94$. The only palindromes between 2004 and 2048 are $(11111011111)_2$ and $(11111111111)_2$, and these should not be counted. Therefore, there are exactly 92 palindromes, and therefor 92 solutions of $f(n) = n$ between 1 and 2004, inclusive.

345. Let $\mathfrak{C}$ be a cube with edges of length 2. Construct a solid figure with fourteen faces by cutting off all eight corners of $\mathfrak{C}$, keeping the new faces perpendicular to the diagonals of the cuhe and keeping the newly formed faces identical. If the faces so formed all have the same area, determine the common area of the faces.

Solution 1. In the situation where the cuts pass through the midpoints of the edges, yielding a cube-octahedron with six square and eight equilateral-triangular sides, we find that the square faces have area 2 and the triangular faces have area $(\sqrt{3}/4)(\sqrt{2}) = \sqrt{6}/4 < 2$. Moving the cuts closer to the vertices yields triangular faces of area less than 2 and octahedral faces of area greater than 2. Thus, for equal areas of the corner and face figures, the cuts must be made a a distance exceeding 1 from each vertex.

The corner faces of the final solid are hexagons formed by large equilateral triangles with smaller equilateral triangles clipped off each vertex; the other faces are squares (diamonds) in the middle of the faces of the cube. Let the square faces have side length x. The vertices of this face are distant $1 - (x/\sqrt{2})$ from the edge of the cube, so that smaller equilateral triangles of side $\sqrt{2}(1 - (x/\sqrt{2})) = \sqrt{2} - x$ are clipped off from a larger equilateral triangle of side $2(\sqrt{2} - x) + x = 2\sqrt{2} - x$. The areas of the hexagonal faces of the solid figure are each
$$\frac{\sqrt{3}}{4}[(2\sqrt{2} - x)^2 - 3(\sqrt{2} - x)^2] = \frac{\sqrt{3}}{2} + \frac{x\sqrt{6}}{2} - \frac{x^2\sqrt{3}}{2} \ .$$
For equality, we need
$$x^2 = \frac{\sqrt{3}}{2}[1 + x\sqrt{2} - x^2] \ ,$$
or
$$(2+\sqrt{3})x^2 - x\sqrt{6} - \sqrt{3} = 0 \ .$$
Hence
$$x = \frac{\sqrt{6} + \sqrt{8\sqrt{3} + 18}}{2(2+\sqrt{3})}$$
and the common area is
$$x^2 = \frac{6 + 2\sqrt{3} + \sqrt{27 + 12\sqrt{3}}}{7 + 4\sqrt{3}} = (6 + 2\sqrt{3} + \sqrt{27 + 12\sqrt{3}})(7 - 4\sqrt{3}) \ .$$

Solution 2. Let the cut be made distant u from a vartex. As in Solution 1, we argue that $1 < u < 2$. Then the edge of the square face of the final solid is distant $u/\sqrt{2}$ from the vertex of the cube and $\sqrt{2}(1 - (u/2))$ from the centre of the face. Thus, the square face has side length $\sqrt{2}(2 - u)$ and area $8 - 8u + 2u^2$.

The hexagonal face of the solid consists of an equilateral triangle of side $\sqrt{2}u$ with three equilateral triangles of side $\sqrt{2}(u-1)$ clipped off. Its area is $(\sqrt{3}/2)[-2u^2 + 6u - 3$. For equality of area of all the faces, we require that
$$2(8 - 8u + 2u^2) = \sqrt{3}(-2u^2 + 6u - 3)$$

or

$$2(2+\sqrt{3})u^2 - 2(8+3\sqrt{3})u + (16+3\sqrt{3}) = 0 \; .$$

Solving this equation and taking the root less than 2 yields that

$$u = \frac{(8+3\sqrt{3}) - \sqrt{9+4\sqrt{3}}}{2(2+\sqrt{3})} \; ,$$

whence

$$2 - u = \frac{\sqrt{3} + \sqrt{9+4\sqrt{3}}}{2(2+\sqrt{3})} \; .$$

Thus, the common area is

$$2(2-u)^2 = \frac{6+2\sqrt{3}+\sqrt{27+12\sqrt{3}}}{7+4\sqrt{3}} = (6+2\sqrt{3}+\sqrt{27+12\sqrt{3}})(7-4\sqrt{3}) \; .$$

Solutions for Novmeber problems.

Notes. A real-valued function $f(x)$ of a real variable is *increasing* if and only if $u < v$ implies that $f(u) \leq f(v)$. The *circumcircle* of a triangle is that circle that passes through its three vertices; its centre is the *circumcentre* of the triangle. The *incircle* of a triangle is that circle that is tangent internally to its three sides; its centre is the *incentre* of the triangle.

346. Let n be a positive integer. Determine the set of all integers that can be written in the form

$$\sum_{k=1}^{n} \frac{k}{a_k}$$

where $a_1, a_2, \cdots, a_n$ are all positive integers.

Solution 1. The sum cannot exceed $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. We prove by induction that the set of integers that can be written in the required form consists of the integers from 1 to $\frac{1}{2}n(n+1)$ inclusive. Observe that 1 is representable for any n (for example, by making each a_k equal to kn). Also, $\frac{1}{2}n(n+1)$ is representable, by taking $a_1 = a_2 = \cdots = a_n$. Thus, the result holds for $n = 1$ and $n = 2$.

Suppose that it holds for $n = m \geq 2$. Then, by taking $a_{m+1} = m+1$ and appending $(m+1)/(m+1)$ to each integer representable by an $m-$term sum, we find that each integer between 2 and $\frac{1}{2}m(m+1) + 1 = (m^2+m+2)/2$ inclusive can be represented with a $(m+1)-$ term sum. By taking $a_{m+1} = 1$ and appending $m+1$ to each integer representable by an $m-$term sum, we find that each integer between $m+2$ and $\frac{1}{2}m(m+1) + (m+1) = \frac{1}{2}(m+1)(m+2)$ can be represented with an $(m+1)-$term sum. Since $[(m^2+m+2)/2] - (m+2) = \frac{1}{2}(m^2-m-2) = \frac{1}{2}(m-2)(m+1) \geq 0$ for $m \geq 2$, $\frac{1}{2}(m^2+m+2) \geq m+2$. Thus, we can represent all numbers between 1 and $\frac{1}{2}(n+1)(n+2)$ inclusive. The result follows.

Solution 2. [Y. Zhao] `Lemma.` For any integer k with $1 \leq k \leq \frac{1}{2}n(n+1)$, there is a subset T_k of $\{1, 2, \cdots, n\}$ for which the sum of the numbers in T_k is k.

`Proof.` T_k is the entire set when $k = \frac{1}{2}n(n+1)$. For the other values of k, we give a proof by induction on k. A singleton suffices for $1 \leq k \leq n$. Suppose that the result holds for $k = m-1 < \frac{1}{2}n(n+1)$. Then T_{m-1} must lack at least one number. If $1 \notin T_{m-1}$, let $T_m = T_{m-1} \cup \{1\}$. If $1 \in T_{m-1}$, let $i > 1$ be the least number not in T_{m-1}. Then let $T_m = T_{m-1} \cup \{i+1\} \backslash \{i\}$. ♠

Now to the result. Let $2 \leq k \leq \frac{1}{2}n(n+1)$ be given and determine T_{k-1}. Define $a_i = 1$ when $i \in T_{k-1}$ and $a_i = \frac{1}{2}n(n+1) - (k-1)$ when $i \notin T_{k-1}$. Then

$$\sum \left\{ \frac{i}{a_i} : i \notin T_{k-1} \right\} = \left[\binom{n+1}{2} - (k-1) \right]^{-1} \sum \{ i : i \notin T_{k-1} \}$$
$$= \left[\binom{n+1}{2} - (k-1) \right]^{-1} \left[\binom{n+1}{2} - \sum \{ i : i \in T_{k-1} \right] = 1 \ .$$

Since one can give a representation for 1 and since no number exceeding $\binom{n+1}{2}$ can be represented, it follows that the set of representable integers consists of those between 1 and $\binom{n+1}{2}$ inclusive.

347. Let n be a positive integer and $\{a_1, a_2, \cdots, a_n\}$ a finite sequence of real numbers which contains at least one positive term. Let S be the set of indices k for which at least one of the numbers

$$a_k, a_k + a_{k+1}, a_k + a_{k+1} + a_{k+2}, \cdots, a_k + a_{k+1} + \cdots + a_n$$

is positive. Prove that

$$\sum \{ a_k : k \in S \} > 0 \ .$$

Solution. We prove the result by induction on n. When $n = 1$, the result is obvious. Let $m \geq 2$ and suppose that the result holds for all $n \leq m-1$. Suppose a suitable sequence $\{a_1, a_2, \cdots, a_m\}$ is given. If $a_1 \notin S$, then $\sum \{a_k : k \in S\} > 0$, by the induction hypothesis applied to the $(m-1)-$element set $\{a_2, \cdots, a_m\}$. Suppose that $a_1 \in S$ and that r is the smallest index for which $a_1 + a_2 + \cdots + a_r > 0$. Then, for $1 \leq i \leq r-1$, $a_1 + \cdots + a_i \leq 0$ and so

$$(a_{i+1} + \cdots + a_r) = (a_1 + \cdots + a_r) - (a_1 + \cdots + a_i) > 0 \ ,$$

i.e., $a_2, a_3, \cdots, a_r \in S$. Hence $\sum \{a_k : 2 \leq k \leq r\} > 0$. If there are no elements of S that exceed r. then the desired conclusion follows. Otherwise, by the induction hypothesis applied to the $(m-r)-$ element set $\{a_{r+1}, \cdots, a_m\}$, we have that $\sum \{a_k : k \in S, r+1 \leq k \leq m\} > 0$. The desired conclusion follows.

Comment. Most solvers had much more elaborate solutions which essentially used this idea. No one recognized that the proliferation of cases could be sidestepped by the technical use of an induction argument.

348. Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that $f(x) - x^3$ is an increasing function. Must $f(x) - x - x^2$ also be increasing?

Solution. The answer is *no*. Consider $f(x) = x^3 + x$. Then $x = f(x) - x^3$ is increasing, but $g(x) = f(x) - x - x^2 = x^3 - x^2$ is not. Indeed, $g(0) = 0$ while $g(\frac{1}{2}) = -\frac{1}{8} < 0$.

Comment. See Problem 348.(b) included with the February set.

349. Let s be the semiperimeter of triangle ABC. Suppose that L and N are points on AB and CB produced (*i.e.*, B lies on segments AL and CN) with $|AL| = |CN| = s$. Let K be the point symmetric to B with respect to the centre of the circumcircle of triangle ABC. Prove that the perpendicular from K to the line NL passes through the incentre of triangle ABC.

Let the incentre of the triangle be I.

Solution 1. Let P be the foot of the perpendicular from I to AK, and Q the foot of the perpendicular from I to CK. Since BK is a diameter of the circumcircle of triangle ABC, $\angle BAK = \angle BCK = 90°$ and $IP \| BA$, $IQ \| BC$. Now $|IP| = s - a = |BN|$, $|IQ| = s - c = |BL|$ and $\angle PIQ = \angle ABC = \angle NBL$, so that $\Delta IPQ \equiv \Delta BNL$ (SAS). Select R on IP and S on IQ (possibly produced) so that $IR = IQ$, $IS = IP$.

Thus, $\Delta ISR \equiv \Delta BNL$ and $RS \| NL$ (why?). Since $IPKQ$ is concyclic, $\angle KIP + \angle IRS = \angle KIP + \angle IQP = \angle KIP + \angle IKP = 90°$. Therefore IK is perpendicular to RS, and so to NL.

Solution 2. `Lemma.` Let W, X, Y, Z be four points in the plane. Then $WX \perp YZ$ if and only ig $|WY|^2 - |WZ|^2 = |XY|^2 - |XZ|^2$.

`Proof.` Note that

$$2(\overrightarrow{W} - \overrightarrow{X}) \cdot (\overrightarrow{Z} - \overrightarrow{Y}) = (\overrightarrow{Y} - \overrightarrow{W})^2 - (\overrightarrow{Z} - \overrightarrow{W})^2 - (\overrightarrow{Y} - \overrightarrow{X})^2 + (\overrightarrow{Z} - \overrightarrow{X})^2 \ .\spadesuit\ .$$

Since BK is a diameter of the circumcircle, $\angle LAK = \angle NCK = 90°$. We have that

$$|KL|^2 - |KN|^2 = (|KA|^2 + |AL|^2) - (|KC|^2 + |CN|^2) = |KA^2| - |KC|^2$$
$$= (|BK|^2 - |AB|^2) - (|BK|^2 - |BC|^2) = |BC|^2 - |AB|^2 \ .$$

Let U and V be the respective feet of the perpendiculars from I to BA and BC. Observe that $|AU| = |BN| = s - a$, $|CV| = |BL| = s - c$ and $|BU| = |BV| = s - b$, so that $|UL| = |BC| = a$, $|VN| = |AB| = c$. Then

$$|IL|^2 - |IN|^2 = (|IU|^2 + |UL|^2) - (|IV|^2 + |VN|^2) = |UL|^2 - |VN|^2 = |BC|^2 - |AB|^2 \ ,$$

which, along with the lemma, implies the result.

350. Let $ABCDE$ be a pentagon inscribed in a circle with centre O. Suppose that its angles are given by $\angle B = \angle C = 120°$, $\angle D = 130°$, $\angle E = 100°$. Prove that BD, CE and AO are concurrent.

Solution 1. [P. Shi; Y. Zhao] The vertices $ABCDE$ are the vertices A_1, A_5, A_7, A_{11}, A_{12} of a regular $18-$gon. (Since A, B, C, D, E lie on a circle, the position of the remaining vertices are determined by that of A are the angle sizes. Alternatively, look at the angles subtended at the centre by the sides of the pentagon.) Since the sum of the angles of a pentagon is $540°$, $\angle A = 70°$. Since $\angle AED > 90°$, D and E lie on the same side of the diameter through A, and B and C lie on the other side. Thus, the line AO produced intersects the segment CD. Consider the triangle ACD for which AO produced, BD and CE are cevians. We apply the trigonometric version of Ceva's theorem.

We have that $\angle CAO = 30°$, $\angle ADB = 40°$, $\angle DCE = 10°$, $\angle OAD = 10°$, $\angle BDC = 20°$ and $\angle ECA = 70°$. Hence

$$\frac{\sin \angle CAO \cdot \sin \angle ADB \cdot \sin \angle DCE}{\sin \angle OAD \cdot \sin \angle BDC \cdot \sin \angle ECA} = \frac{(\frac{1}{2}) \sin 40° \sin 10°}{\sin 10° \sin 20° \sin 70°} = \frac{\cos 20°}{\sin 70°} = 1 \ .$$

Hence the three lines AO, BD and CE are concurrent as desired.

Solution 2. [C. Sun] Let BD and CE intersect at P. We can compute the following angles: $\angle EAB = \angle EBA = 70°$, $\angle AEB = 40°$, $\angle BEP = \angle PDC = 20°$, $\angle EBP = \angle PCD = 10°$, $\angle PBC = \angle PED = 40°$, $\angle BCP = \angle EDP = 110°$ and $\angle BPC = \angle EPD = 30°$. Since triangle ABE is acute, the circumcentre O of it (and the pentagon) lie in its interior, and $\angle AOB = \frac{1}{2}\angle AEB = 80°$. Since triangle ABE is isosceles, $\angle BAO = \angle ABO = 50°$.

Let Q be the foot of the perpendicular from E to AB and R the foot of the perpendicular from B to EP produced. Since $\Delta EQB \equiv \Delta ERB$ (ASA), $QB = RB$. Since $\angle BPR = 30°$, $\angle PBR = 60°$ and $PB = 2RB = 2QB = AB$. Hence ABP is isosceles with apex $\angle ABP = 80°$. Thus, $\angle BAP = \angle BPA = 50°$. Hence $\angle BAO = \angle BAP = 50°$, so that AO must pass through P and the result follows.

351. Let $\{a_n\}$ be a sequence of real numbers for which $a_1 = 1/2$ and, for $n \geq 1$,

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1} \ .$$

Prove that, for all n, $a_1 + a_2 + \cdots + a_n < 1$.

Solution. Let $b_n = 1/a_n$ for $n \geq 1$, whence $b_1 = 2$ and $b_{n+1} = b_n^2 - b^n + 1 = b_n(b_n - 1) + 1$ for $n \geq 1$. Then $\{b_n\}$ is an increasing sequence of integers and

$$\frac{1}{b_n} = \frac{1}{b_n - 1} - \frac{1}{b_{n+1} - 1}$$

for $n \geq 1$. Hence

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \\ &= \left[\frac{1}{b_1 - 1} - \frac{1}{b_2 - 1}\right] + \left[\frac{1}{b_2 - 1} - \frac{1}{b_3 - 1}\right] + \cdots + \left[\frac{1}{b_n - 1} - \frac{1}{b_{n+1} - 1}\right] \\ &= \frac{1}{b_1 - 1} - \frac{1}{b_{n+1} - 1} = 1 - \frac{1}{b_{n+1} - 1} < 1 \ . \end{aligned}$$

352. Let $ABCD$ be a unit square with points M and N in its interior. Suppose, further, that MN produced does not pass through any vertex of the square. Find the smallest value of k for which, given any position of M and N, at least one of the twenty triangles with vertices chosen from the set $\{A, B, C, D, M, N\}$ has area not exceeding k.

Solution 1. Wolog, suppose that M lies in the interior of triangle ABN. Then

$$[ABM] + [AMN] + [BMN] + [CND] = [ABN] + [CND] = \frac{1}{2} \ ,$$

so that at least one of the four triangles on the left has area not exceeding $1/8$. Hence $k \leq 1/8$. We give a configuration for which each of the twenty triangles has area not less than $1/8$, so that $k = 1/8$.

Suppose that M and N are both located on the line joining the midpoints of AD and BC with M distant $1/4$ from the side AD and N distant $1/4$ from the side BC. Then

$$\frac{1}{4} = [ABM] = [CDM] = [ABN] = [CDN]$$

$$\frac{3}{8} = [BCM] = [DAN]$$

$$\frac{1}{2} = [ABC] = [BCD] = [CDA] = [DAB]$$

$$\frac{1}{8} = [DAM] = [BCN] = [AMN] = [BMN] = [CMN] = [DMN] = [AMC] = [ANC] = [BMD] = [BND] \ .$$

Solution 2. [L. Fei] Suppose that all triangle have area exceed k. Then M and N must be in the interior of the square distant more than $2k$ from each edge to ensure that areas of triangles like ABM exceed k. Similarly, M and N must be distant more than $k\sqrt{2}$ from the diagonals of the square. For points M and N to be available that satisfy both conditions, we need to find a point that is distant at least $2k$ from the edges and $k\sqrt{2}$ from the diagonal; such a point would lie on a midline of the square. The condition is that $2k + \sqrt{2}(k\sqrt{2}) < \frac{1}{2}$ or $k < \frac{1}{8}$. On the other hand, we can give a configuration in which each area is at least equal to k and some areas are exactly $\frac{1}{8}$. This would have M and N on the same midline, each equidistant from an edge and the centre of the square.