

## OLYMON

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Please mail your solutions to Ms. Valeria Pandelieva, 641 Kirkwood Avenue, Ottawa, ON K1Z 5X5 no later than **February 15, 2005**. Please make sure that your name, full mailing address and email address appears on the front page of your solutions. If you do not write your family name last, please indicate it with an asterisk (\*).

353. The two shortest sides of a right-angled triangle,  $a$  and  $b$ , satisfy the inequality:

$$\sqrt{a^2 - 6a\sqrt{2} + 19} + \sqrt{b^2 - 4b\sqrt{3} + 16} \leq 3 .$$

Find the perimeter of this triangle.

354. Let  $ABC$  be an isosceles triangle with  $AC = BC$  for which  $|AB| = 4\sqrt{2}$  and the length of the median to one of the other two sides is 5. Calculate the area of this triangle.
355. (a) Find all natural numbers  $k$  for which  $3^k - 1$  is a multiple of 13.  
(b) Prove that for any natural number  $k$ ,  $3^k + 1$  is not a multiple of 13.
356. Let  $a$  and  $b$  be real parameters. One of the roots of the equation  $x^{12} - abx + a^2 = 0$  is greater than 2. Prove that  $|b| > 64$ .
357. Consider the circumference of a circle as a set of points. Let each of these points be coloured red or blue. Prove that, regardless of the choice of colouring, it is always possible to inscribe in this circle an isosceles triangle whose three vertices are of the same colour.
358. Find all integers  $x$  which satisfy the equation

$$\cos \left( \frac{\pi}{8} (3x - \sqrt{9x^2 + 160x + 800}) \right) = 1 .$$

359. Let  $ABC$  be an acute triangle with angle bisectors  $AA_1$  and  $BB_1$ , with  $A_1$  and  $B_1$  on  $BC$  and  $AC$ , respectively. Let  $J$  be the intersection of  $AA_1$  and  $BB_1$  (the incentre),  $H$  be the orthocentre and  $O$  the circumcentre of the triangle  $ABC$ . The line  $OH$  intersects  $AC$  at  $P$  and  $BC$  at  $Q$ . Given that  $C$ ,  $A_1$ ,  $J$  and  $B_1$  are vertices of a concyclic quadrilateral, prove that  $PQ = AP + BQ$ .

## Solutions

*Notes.* A sequence  $x_1, x_2, \dots, x_k$  is in *arithmetic progression* iff  $x_{i+1} - x_i$  is constant for  $1 \leq i \leq k - 1$ . A *triangular number* is a positive integer of the form

$$T(x) \equiv \frac{1}{2}x(x+1) = 1 + 2 + \dots + x ,$$

where  $x$  is a positive integer.

$[\dots]$  refers to the area of the figure between brackets.

332. What is the minimum number of points that can be found (a) in the plane, (b) in space, such that each point in, respectively, (a) the plane, (b) space, must be at an irrational distance from at least one of them?

*Solution 1.* We solve the problem in space, as the planar problem is subsumed in the spatial problem. Two points will never do, as we can select on the right bisector of the segment joining them a point that is the same rational distance from both of them (why?).

However, we can find three points that will serve. Select three collinear points  $A, B, C$  such that  $|AB| = |BC| = u$  where  $u^2$  is not rational. Let  $P$  be any point in space. If  $P, A, B, C$  are collinear and  $|PA| = a, |PC| = c$ , then  $|PB|$  is equal to either  $a - u$  or  $c - u$ . If  $a$  and  $c$  are rational, then both  $a - u$  and  $c - u$  are nonrational. Hence at least one of the three distances is rational.

If  $P, A, C$  are not collinear, then  $PB$  is a median of triangle  $PAC$ . Let  $b = |PB|$ . Then  $a^2 + c^2 = 2b^2 + 2u^2$  (why?). Since  $u^2$  is non rational, at least one of  $a, b, c$  is nonrational.

*Comment.* You can check that  $a^2 + c^2 = 2(b^2 + u^2)$  holds for the collinear case as well.

*Solution 2.* [F. Barekat] As in the foregoing, the number has to be at least three. Consider the points  $(0, 0, 0), (u, 0, 0)$  and  $(v, 0, 0)$ , where  $v$  is irrational and  $u$  and  $v^2$  are rational. Let  $P \sim (x, y, z)$ . Then the distances from  $P$  to the three points are the respective square roots of  $x^2 + y^2 + z^2, x^2 - 2ux + u^2 + y^2 + z^2$  and  $x^2 - 2vx + v^2 + y^2 + z^2$ . If the first of these is irrational, then we have one irrational distance. Suppose that  $x^2 + y^2 + z^2$  is rational. If  $x$  is irrational, then

$$x^2 - 2ux + u^2 + y^2 + z^2 = (x^2 + y^2 + z^2 + u^2) - 2ux$$

is irrational. If  $x$  is rational, then

$$x^2 - 2vx + v^2 + y^2 + z^2 = (x^2 + y^2 + z^2 + v^2) - 2vx$$

is irrational. Hence not all the three distances can be rational.

333. Suppose that  $a, b, c$  are the sides of triangle  $ABC$  and that  $a^2, b^2, c^2$  are in arithmetic progression.

- (a) Prove that  $\cot A, \cot B, \cot C$  are also in arithmetic progression.  
 (b) Find an example of such a triangle where  $a, b, c$  are integers.

*Solution 1.* [F. Barekat] (a) Suppose, without loss of generality that  $a \leq b \leq c$ . Let  $AH$  be an altitude of the triangle. Then

$$\cot B = \frac{|BH|}{|AH|} \quad \text{and} \quad \cot C = \frac{|CH|}{|AH|}.$$

Therefore,

$$2[ABC](\cot B + \cot C) = a|AH| \left( \frac{|BH| + |CH|}{|AH|} \right) = a^2.$$

Similar equalities hold for  $b^2$  and  $c^2$ . Therefore

$$2b^2 = a^2 + c^2 \Leftrightarrow 2(\cot A + \cot C) = (\cot B + \cot C) + (\cot B + \cot A) \Leftrightarrow \cot A + \cot C = 2 \cot B.$$

The result follows from this.

(b) Observe that  $a^2 + c^2 = 2b^2$  if and only if  $(c - a)^2 + (c + a)^2 = (2b)^2$ . So, if  $(x, y, z)$  is a Pythagorean triple with  $z$  even and  $x$  and  $y$  of the same parity, then

$$(a, b, c) = \left( \frac{y - x}{2}, \frac{z}{2}, \frac{y + x}{2} \right).$$

Let  $(x, y, z) = (m^2 - n^2, 2mn, m^2 + n^2)$  where  $m$  and  $n$  have the same parity and  $m > n$ . Then

$$(a, b, c) = \left( \frac{n^2 + 2mn - m^2}{2}, \frac{m^2 + n^2}{2}, \frac{m^2 + 2mn - n^2}{2} \right).$$

To ensure that these are sides of a triangle, we need to impose the additional conditions that

$$n^2 + 2mn - m^2 > 0 \Leftrightarrow 2n^2 > (m - n)^2 \Leftrightarrow m < (\sqrt{2} + 1)n$$

and

$$m^2 + 2mn - n^2 < (m^2 + n^2) + (n^2 + 2mn - m^2) \Leftrightarrow m < n\sqrt{3}.$$

Thus, we can achieve our goal as long as  $n^2 < m^2 < 3n^2$  and  $m \equiv n \pmod{2}$ .

For example,  $(m, n) = (5, 3)$  yields  $(a, b, c) = (7, 17, 23)$ .

*Comment.* Take  $(x, y, z) = (2mn, m^2 - n^2, m^2 + n^2)$  to give the solution

$$(a, b, c) = \left( \frac{m^2 - 2mn - n^2}{2}, \frac{m^2 + n^2}{2}, \frac{m^2 + 2mn - n^2}{2} \right).$$

For example,  $(m, n) = (5, 1)$  yields  $(a, b, c) = (7, 13, 17)$ . Here are some further numerical examples; note how they come in chains with the first of each triple equal to the last of the preceding one:

$$(a, b, c) = [(1, 5, 7), (7, 13, 17), (17, 25, 31), (31, 41, 49), \dots]$$

$$(a, b, c) = (7, 17, 23), (23, 37, 47), (47, 65, 79), \dots$$

*Solution 2.* Since  $c^2 = a^2 + b^2 - 2ab \cos C$ , we have that

$$\cot C = \frac{a^2 + b^2 - c^2}{2ab \sin C} = \frac{a^2 + b^2 - c^2}{4[ABC]}$$

with similar equations for  $\cot A$  and  $\cot B$ . Hence

$$\cot A + \cot C - 2 \cot B = (2[ABC])^{-1}(2b^2 - a^2 - c^2).$$

Thus,  $a^2, b^2, c^2$  are in arithmetic progression if and only if  $\cot A, \cot B, \cot C$  are in arithmetic progression.

(b) We need to solve the Diophantine equation  $a^2 + c^2 = 2b^2$  subject to the condition that  $c < a + b$ . The inequality is equivalent to  $2b^2 - a^2 < (a + b)^2$  which reduces to  $(b - a)^2 < 3a^2$  or  $b < (1 + \sqrt{3})a$ . To ensure the inequality, let us try  $b = 2a + k$ , so that  $b^2 = 4a^2 + 4ka + k^2$  and we have to solve  $7a^2 + 8ka + 2k^2 = c^2$ . Upon multiplication by 7 and shifting terms, the equation becomes

$$(7a + 4k)^2 - 7c^2 = 2k^2.$$

(Note that  $b/a = 2 + (k/a) < 1 + \sqrt{3}$  as long as  $a > \frac{1}{2}(\sqrt{3} + 1)k$ .)

We solve a *Pell's Equation*  $x^2 - 7y^2 = 2k^2$  with the condition that  $x \equiv 4k \pmod{7}$ . There is a standard technique for solving such equations. We find the fundamental solution of  $x^2 - 7y^2 = 1$ ; this is the solution with the smallest positive values of  $x$  and  $y$ , and in this case is  $(x, y) = (8, 3)$ . We need a particular solution of  $x^2 - 7y^2 = 2k^2$ ; the solution  $(x, y) = (3k, k)$  will do. Then we get an infinite set of solutions  $(x_n, y_n)$  for  $x^2 - 7y^2 = 2k^2$  by defining  $(x_0, y_0) = (3k, k)$  and, for  $n \geq 1$ ,

$$x_n + y_n\sqrt{7} = (8 + 3\sqrt{7})(x_{n-1} + y_{n-1}\sqrt{7}).$$

(Note that the same equation holds when we replace the plus signs in the three terms by minus signs, so we can see that this works by multiplying this equation by its surd conjugate.) Separating out the terms, we get the recursion

$$x_n = 8x_{n-1} + 21y_{n-1}$$

$$y_n = 3x_{n-1} + 8y_{n-1}$$

for  $n \geq 1$ . Observe that  $x_n \equiv x_{n-1} \pmod{7}$ . Thus, to get the solution we want, we need to select  $k$  such that  $k \equiv 0 \pmod{7}$ .

An infinite family of solutions of  $x^2 - 7y^2 = 2k^2$  starts with

$$(x, y) = (3k, k), (45k, 17k), (717k, 271k), \dots$$

All but the first of these will yield a triangle, since we will have  $7a = x - 4k \geq 41k$  whence  $a > 5k > \frac{1}{2}(\sqrt{3} + 1)k$ . Let  $k = 7$ . Then, we get the triangles  $(a, b, c) = (41, 89, 119), (713, 1433, 1897), \dots$ . We get only similar triangles to these from other multiples of 7 for  $k$ .

334. The vertices of a tetrahedron lie on the surface of a sphere of radius 2. The length of five of the edges of the tetrahedron is 3. Determine the length of the sixth edge.

*Solution 1.* Let  $ABCD$  be the tetrahedron with the lengths of  $AB, AC, AD, BC$  and  $BD$  all equal to 3. The plane that contains the edge  $AB$  and passes through the centre of the sphere is a plane of symmetry for the tetrahedron and is thus orthogonal to  $CD$ . This plane meets  $CD$  in  $P$ , the midpoint of  $CD$ . Likewise, the plane orthogonal to  $AB$  passing through the midpoint  $M$  of  $AB$  is a plane of symmetry of the tetrahedron that passes through  $C, D$  and  $P$ , as well as the centre  $O$  of the sphere.

Consider the triangle  $OCM$  with altitude  $CP$ . Since  $OM$  is the altitude of the triangle  $OAB$  with  $|OA| = |OB| = 2$  and  $|AB| = 3$ ,  $|OM| = (\sqrt{7})/2$ . Since  $MC$  is an altitude of the equilateral triangle  $ABC$ ,  $|MC| = (3\sqrt{3})/2$ . Since  $OC$  is a radius of the sphere,  $|OC| = 2$ .

Let  $\theta = \angle OCM$ . Then, by the Cosine Law,  $\frac{7}{4} = 4 + \frac{27}{4} - 2\sqrt{27} \cos \theta$ , so that  $\cos \theta = (\sqrt{3})/2$  and  $\sin \theta = 1/2$ . Hence, the area  $[OCM]$  of triangle  $OCM$  is equal to  $\frac{1}{2}|OC||MC| \sin \theta = (3\sqrt{3})/4$ . But this area is also equal to  $\frac{1}{2}|CP||OM| = ((\sqrt{7})/4)|CP|$ . Therefore,

$$|CD| = 2|CP| = 2(3\sqrt{3})/(\sqrt{7}) = (6\sqrt{3})/(\sqrt{7}) = (6\sqrt{21})/7.$$

*Comment.* An alternative way is to note that  $CMP$  is a right triangle with hypotenuse  $CM$  with  $O$  a point on  $PM$ . Let  $u = |CP|$ . We have that  $|CM| = (3\sqrt{3})/2$ ,  $|OM| = (\sqrt{7})/2$ ,  $|CO| = 2$ . so that  $|OP|^2 = 4 - u^2$  and  $|MP| = [(\sqrt{7})/2] + \sqrt{4 - u^2}$ . Hence, by Pythagoras' Theorem,

$$\begin{aligned} \frac{27}{4} &= \left[ \frac{\sqrt{7}}{2} + \sqrt{4 - u^2} \right]^2 + u^2 \\ \implies \frac{27}{4} &= \frac{7}{4} + \sqrt{7}\sqrt{4 - u^2} + 4 \\ \implies u^2 &= 27/7 \implies u = (3\sqrt{3})/\sqrt{7}. \end{aligned}$$

Hence  $|CD| = 2u = (6\sqrt{3})/\sqrt{7}$ .

*Solution 2.* [F. Barekat] As in Solution 1, let  $CD$  be the odd side, and let  $O$  be the centre of the sphere. Let  $G$  be the centroid of the triangle  $ABC$ . Since the right bisecting plane of the three sides of triangle  $ABC$  each pass through the centroid  $G$  and the centre  $O$ ,  $OG \perp ABC$ . Observe that  $|CG| = \sqrt{3}$  and  $|GM| = (\sqrt{3})/2$ , where  $M$  is the midpoint of  $AB$ .

As  $CGO$  is a right triangle,  $|GO|^2 = |CO|^2 - |CG|^2 = 4 - 3 = 1$ , so that  $|GO| = 1$ . Hence,  $|OM|^2 = |OG|^2 + |GM|^2 = 1 + \frac{3}{4} = \frac{7}{4}$ , so that  $|OM| = (\sqrt{7})/2$ . Therefore,

$$\sin \angle OMC = \sin \angle OMG = |OG|/|OM| = 2/(\sqrt{7}) .$$

With  $P$  the midpoint of  $CD$ , the line  $MP$  (being the intersection of the planes right bisecting  $AB$  and  $CD$ ) passes through  $O$ . Since  $MC$  and  $MD$  are altitudes of equilateral triangles,  $|MC| = |MD| = 3(\sqrt{3})/2$ , so that  $MCD$  is isosceles with  $MP$  bisecting the apex angle. Hence

$$|CD| = 2|MC| \sin\left(\frac{1}{2}\angle CMD\right) = 2|MC| \sin \angle OMC = (6\sqrt{3})/(\sqrt{7}) .$$

*Solution 3.* Let  $A$  be at  $(-3/2, 0, 0)$  and  $B$  be at  $(3/2, 0, 0)$  in space. The locus of points equidistant from  $A$  and  $B$  is the plane  $x = 0$ . Let the centre of the sphere be at the point  $(0, u, 0)$ , so that  $u^2 + 9/4 = 4$  and  $u = (\sqrt{7})/2$ . Suppose that  $C$  is at the point  $(0, y, z)$  where  $\frac{9}{4} + y^2 + z^2 = 9$ . We have that  $(y - (\sqrt{7})/2)^2 + z^2 = 4$  whence

$$(\sqrt{7})y + (9/4) = y^2 + z^2 = 9 - (9/4)$$

and  $y = 9/(2\sqrt{7})$ . Therefore  $z = (3\sqrt{3})/(\sqrt{7})$ .

We find that  $C \sim (0, 9/(2\sqrt{7}), (3\sqrt{3})/(\sqrt{7}))$  and  $D \sim (0, 9/(2\sqrt{7}), (-3\sqrt{3})/(\sqrt{7}))$ , whence  $|CD| = (6\sqrt{3})/(\sqrt{7})$ .

*Solution 4.* [Y. Zhao] Use the notation of Solution 1, and note that the right bisecting plane of  $AB$  and  $CD$  intersect in a diameter of the sphere. Let us first determine the volume of the tetrahedron  $ABDO$ . Consider the triangle  $ABD$  with centroid  $X$ . We have that  $|AB| = |BD| = |AD| = 3$ ,  $|AX| = \sqrt{3}$  and  $[ABD] = (9\sqrt{3})/4$ . Since  $O$  is equidistant from the vertices of triangle  $ABD$ ,  $OX \perp ABD$ . By Pythagoras' Theorem applied to triangle  $AXO$ ,  $|OX| = 1$ , so that the volume of tetrahedron  $ABDO$  is  $(1/3)|OX|[ABD] = (3\sqrt{3})/4$ .

Triangle  $ABO$  has sides of lengths 2, 2, 3, and (by Heron's formula) area  $(3\sqrt{7})/4$ . Since  $CD \perp ABO$ ,  $CD$  is the production of an altitude of tetrahedron  $ABDO$ ; the altitude has length  $\frac{1}{2}|CO|$ . Hence

$$|CD| = \frac{2[3\text{Volume}(ABDO)]}{[ABO]} = \frac{2(9\sqrt{3})/4}{(3\sqrt{7})/4} = \frac{6\sqrt{3}}{\sqrt{7}} .$$

*Solution 5.* [A. Wice] Let the tetrahedron  $ABCD$  have its vertices on the surface of the sphere of equation  $x^2 + y^2 + z^2 = 4$  with  $A$  at  $(0, 0, 2)$ . The remaining vertices have coordinates  $(u, v, w)$  and satisfy the brace of equations:  $u^2 + v^2 + (w - 2)^2 = 9$  and  $u^2 + v^2 + w^2 = 4$ . Hence,  $w = -1/4$ . Thus, the points  $B, C, D$  lie on the circle of equations  $z = -1/4$ ,  $x^2 + y^2 = 63/16$ . The radius  $R$  of this circle and the circumradius of triangle  $BCD$  is  $3\sqrt{7}/4$ .

Triangle  $BCD$  has sides of length  $(b, c, d) = (b, 3, 3)$  and area  $bcd/4R = \frac{1}{2}b\sqrt{9 - (b^2/4)}$ . Hence

$$\frac{3}{\sqrt{7}} = \frac{1}{2}\sqrt{9 - (b^2/4)} \Leftrightarrow 36 = 7(9 - (b^2/4)) \Leftrightarrow 144 = 7(36 - b^2) \Leftrightarrow b = \frac{6\sqrt{3}}{\sqrt{7}} .$$

Thus, the length of the remaining side is  $(6\sqrt{3})/\sqrt{7}$ .

335. Does the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{12}{a + b + c}$$

have infinitely many solutions in positive integers  $a, b, c$ ?

*Comment.* The equation is equivalent to

$$(a + b + c)(bc + ca + ab + 1) = 12abc .$$

This is quadratic in each variable, and for any integer solution, has integer coefficients. The general idea of the solution is to start with a particular solution ( $(a, b, c) = (1, 1, 1)$  is an obvious one), fix two of the variables at these values and regard the equation as a quadratic in the third. Since the sum and the product of the roots are integers and one root is known, one can find another root and so bootstrap one's way to other solutions. Thus, we have the quadratic for  $a$ :

$$(b + c)a^2 + [(b + c)^2 + (bc + 1) - 12bc]a + (b + c)(bc + 1) = 0 ,$$

so that if  $(a, b, c)$  satisfies the equation, then so also does  $(a', b, c)$  where

$$a + a' = \frac{9bc - b^2 - c^2 - 1}{b + c} \quad \text{and} \quad aa' = bc + 1 .$$

We can start constructing solutions using these relations:

$$(1, 1, 1), (1, 1, 2), (1, 2, 3), (2, 3, 7), (2, 5, 7), (3, 7, 11), (5, 7, 18), (5, 13, 18), \dots .$$

However, some triples do not lead to a second solution in integers. For example,  $(b, c) = (2, 5)$  leads to  $(7, 2, 5)$  and  $(11/7, 2, 5)$ , and  $(b, c) = (3, 11)$  leads to  $(7, 3, 11)$  and  $(34/7, 3, 11)$ . So we have no guarantee that this process will not peter out.

*Solution 1.* [Y. Zhao] Yes, there are infinitely many solutions. Specialize to the case that  $c = a + b$ . Then the equation is equivalent to

$$2(a + b)[(a + b)^2 + ab + 1] = 12ab(a + b) \iff a^2 - 3ab + b^2 + 1 = 0 .$$

This has at least one solution  $(a, b) = (1, 1)$ . Suppose that  $(a, b) = (p, q)$  is a solution with  $p \leq q$ . Then  $(a, b) = (q, 3q - p)$  is a solution. (To see this, note that  $x^2 - 3qx + (q^2 + 1) = 0$  is a quadratic equation with one root  $x = p$  and root sum  $3q$ .) Observe that  $q < 3q - p$ .

Define a sequence  $\{x_n\}$  for  $n \geq 0$  by  $x_0 = x_1 = 1$  and  $x_n = 3x_{n-1} - x_{n-2}$  for  $n \geq 2$ . Then  $(a, b) = (x_0, x_1)$  satisfies the equation, and by induction, so does  $(a, b) = (x_n, x_{n+1})$  for  $n \geq 1$ . Since  $x_{n+1} - x_n = 2x_n - x_{n-1}$ , one sees by induction that  $\{x_n\}$  is strictly increasing for  $n \geq 1$ . Hence, an infinite set of solutions for the given equation is given by

$$(a, b, c) = (x_n, x_{n+1}, x_n + x_{n+1})$$

for  $n \geq 0$ . Some examples are

$$(a, b, c) = (1, 1, 2), (1, 2, 3), (2, 5, 7), (5, 13, 18), \dots$$

*Comment.* If  $\{f_n\}$  is the Fibonacci sequence defined by  $f_0 = 0, f_1 = 1$  and  $f_n = f_{n-1} + f_{n-2}$  for  $n \geq 2$ , then the solutions are  $(a, b, c) = (f_{2k-1}, f_{2k+1}, f_{2k-1} + f_{2k+1})$ .

*Solution 2.* Yes. Let  $u_0 = u_1 = 1, v_0 = 1, v_1 = 2$  and

$$u_n = 4u_{n-1} - u_{n-2}$$

$$v_n = 4v_{n-1} - v_{n-2}$$

for  $n \geq 2$ , so that  $\{u_n\} = \{1, 1, 3, 11, 41, \dots\}$  and  $\{v_n\} = \{1, 2, 7, 26, 97, \dots\}$ . It can be proven by induction that both sequences are strictly increasing for  $n \geq 1$ . We prove that the equation of the problem is satisfied by

$$(a, b, c) = (u_n, v_n, u_{n+1}), (v_n, u_{n+1}, v_{n+1})$$

for  $n \geq 0$ . In other words, the equation holds if  $(a, b, c)$  consists of three consecutive terms of the sequence  $\{1, 1, 1, 2, 3, 7, 11, 26, 41, 97, \dots\}$ .

Observe that, for  $n \geq 2$ ,

$$\begin{aligned} u_n v_n - u_{n+1} v_{n-1} &= u_n(4v_{n-1} - v_{n-2}) - (4u_n - u_{n-1})v_{n-1} \\ &= u_{n-1}v_{n-1} - u_n v_{n-2} \end{aligned}$$

so that, by induction, it can be established that  $u_n v_n - u_{n+1} v_{n-1} = -1$ . Similarly,

$$\begin{aligned} u_n v_{n+1} - u_{n+1} v_n &= u_n(4v_n - v_{n-1}) - (4u_n - u_{n-1})v_n \\ &= u_{n-1}v_n - u_n v_{n-1} = 1 . \end{aligned}$$

It can be checked that  $(a, b, c) = (1, 1, 1)$  satisfies the equation. Suppose, as an induction hypothesis, that  $(a, b, c) = (u_n, v_n, u_{n+1})$  satisfies the equation. Then  $u_n$  is a root of the quadratic

$$\begin{aligned} 0 &= [x + v_n + u_{n+1}][x(v_n + u_{n+1}) + v_n u_{n+1} + 1] - 12v_n u_{n+1} x \\ &= (v_n + u_{n+1})x^2 + (v_n^2 + u_{n+1}^2 + 1 - 9v_n u_{n+1})x + (v_n + u_{n+1})(v_n u_{n+1} + 1) . \end{aligned}$$

The second root is

$$v_{n+1} = \frac{v_n u_{n+1} + 1}{u_n}$$

so  $(a, b, c) = (v_n, u_{n+1}, v_{n+1})$  satisfies the equation. Similarly,  $v_n$  is a root of a quadratic, the product of whose roots is  $u_{n+1}v_{n+1} + 1$ . The other root of this quadratic is

$$u_{n+2} = \frac{u_{n+1}v_{n+1} + 1}{v_n}$$

and so,  $(a, b, c) = (u_{n+1}, v_{n+1}, u_{n+2})$  satisfies the equation.

*Solution 3.* [P. Shi] Yes. Let  $u_0 = v_0 = 1$ ,  $v_0 = 1$ ,  $v_1 = 2$ , and

$$u_n = \frac{u_{n-1}v_{n-1} + 1}{v_{n-2}} \quad \text{and} \quad v_n = \frac{u_n v_{n-1} + 1}{u_{n-1}}$$

for  $n \geq 2$ . We establish that, for  $n \geq 1$ ,

$$u_n = 2v_{n-1} - u_{n-1} ; \tag{1}$$

$$v_n = 3u_n - v_{n-1} ; \tag{2}$$

$$3u_n^2 + 2v_{n-1}^2 + 1 = 6u_n v_{n-1} ; \tag{3}$$

$$3u_n^2 + 2v_n^2 + 1 = 6u_n v_n . \tag{4}$$

Note that  $(3u_n^2 + 2v_n^2 + 1 - 6u_n v_n) - (3u_n^2 + 2v_{n-1}^2 + 1 - 6u_n v_{n-1}) = 2(v_n - v_{n-1})(v_n + v_{n-1} - 3u_n)$ , so that the truth of any two of (2), (3), (4) implies the truth of the third.

The proof is by induction. The result holds for  $n = 1$ . Suppose it holds for  $1 \leq n \leq k - 1$ . From (3),

$$3u_{k-1}^2 + 2v_{k-2}^2 + 1 = 6u_{k-1}v_{k-2} \implies$$

$$(3u_{k-1} - v_{k-2})u_{k-1} + 1 = 2(3u_{k-1} - v_{k-2})v_{k-2} - u_{k-1}v_{k-2} .$$

Substituting in (2) gives

$$u_k = \frac{u_{k-1}v_{k-1} + 1}{v_{k-2}} = \frac{u_{k-1}(3u_{k-1} - v_{k-2}) + 1}{v_{k-2}} = 2v_{k-1} - u_{k-1}$$

which establishes (1) for  $n = k$ . From (4),

$$(2v_{k-1} - u_{k-1})v_{k-1} + 1 = [3(2v_{k-1} - u_{k-1}) - v_{k-1}]u_{k-1}$$

whence

$$v_k = \frac{u_k v_{k-1} + 1}{u_{k-1}} = \frac{(2v_{k-1} - u_{k-1})v_{k-1} + 1}{u_{k-1}} = 3u_k - v_{k-1}$$

which establishes (2) for  $n = k$ . From (4),

$$\begin{aligned} 3(2v_{k-1} - u_{k-1})^2 + 2v_{k-1}^2 + 1 &= 6(2v_{k-1} - u_{k-1})v_{k-1} \\ \implies 3u_k^2 + 2v_{k-1}^2 + 1 &= 6u_k v_k. \heartsuit \end{aligned}$$

Let  $w_{2n} = u_n$  and  $w_{2n+1} = v_n$  for  $n \geq 0$ . Then

$$w_n = \frac{w_{n-1}w_{n-2} + 1}{w_{n-3}}$$

for  $n \geq 3$ . It can be checked that  $(a, b, c) = (1, 1, 1) = (w_0, w_1, w_2)$  satisfies the equation. Suppose that the equation is satisfied by  $(a, b, c) = (w_{n-1}, w_n, w_{n+1})$ . Thus

$$(x + w_n + w_{n+1})(x(w_n + w_{n+1}) + w_n w_{n+1} + 1) = 12xw_n w_{n+1} = 0$$

is a quadratic equation in  $x$  one of whose roots is  $x = w_{n-1}$ . The quadratic can be rewritten

$$x^2 + [(w_n + w_{n+1}) - (11w_n w_{n+1} - 1)/(w_n + w_{n+1})]x + (w_n w_{n+1} + 1) = 0.$$

Since the product of the roots is equal to  $w_n w_{n+1} + 1$ , the second root is equal to

$$\frac{w_n w_{n+1} + 1}{w_{n-1}} = w_{n+2}.$$

The result follows.

*Solution 4.* Let  $c = a + b$ . Then the equation becomes  $2c(ab + c^2 + 1) = 12abc$ , whence  $ab = (c^2 + 1)/5$ . Hence,  $a, b$  are roots of the quadratic equation

$$0 = t^2 - ct + \left(\frac{c^2 + 1}{5}\right) = \frac{1}{4} \left[ (2t - c)^2 - \left(\frac{5c^2 - 20}{25}\right) \right].$$

For this to have an integer solution, it is necessary that  $5c^2 - 20 = s^2$ , the square of an integer  $s$ . Now  $s^2 - 5c^2 = -20$  is a Pell's equation, three of whose solutions are  $(s, c) = (0, 2), (5, 3), (15, 7)$ . Since  $x^2 - 5y^2 = 1$  is satisfied by  $(x, y) = (9, 4)$ , solutions  $(s_n, c_n)$  of  $s^2 - 5c^2 = -20$  are given by the recursion

$$s_{n+1} = 9s_n + 20c_n$$

$$c_{n+1} = 4s_n + 9c_n$$

for  $n \geq 0$ . where  $(s_0, c_0)$  is a starter solution. Taking  $(s_0, c_0) = (0, 2)$ , we get the solutions

$$(s; a, b, c) = (0; 1, 1, 2), (40; 5, 13, 18), (720; 89, 233, 322), \dots$$

Taking  $(s_0, c_0) = (5, 3)$ , we get the solutions

$$(s; a, b, c) = (5; 1, 2, 3), (105; 13, 34, 47), (1885; 233, 610, 843), \dots$$

Taking  $(s_0, c_0) = (15, 7)$ , we get the solutions

$$(s; a, b, c) = (15; 2, 5, 7), (275; 34, 89, 123), \dots .$$

*Comment.* Note that the values of  $c$  seem to differ from a perfect square by 2; is this a general phenomenon?

*Solution 5.* [Z. Guo] Let  $r_1 = r_2 = 1$ , and, for  $n \geq 1$ ,

$$r_{2n+1} = 2r_{2n} - r_{2n-1}$$

$$r_{2n+2} = 3r_{2n+1} - r_{2n} .$$

Thus,  $\{r_n\} = \{1, 1, 1, 2, 3, 7, 11, 26, 41, 87, 133, \dots\}$ . Observe that  $(a, b, c) = (r_m, r_{m+1}, r_{m+2})$  satisfies the equation for  $m = 1, 2$ .

For each positive integer  $n$ , let  $k_n = r_{2n} - r_{2n-1}$ , so that

$$r_{2n-1} = r_{2n} - k_n ,$$

$$r_{2n+1} = r_{2n} + k_n ,$$

$$r_{2n+2} = 2r_{2n} + 3k_n .$$

Suppose that  $(a, b, c) = (r_{2m-1}, r_{2m}, r_{2m+1})$  and  $(a, b, c) = (r_{2m}, r_{2m+1}, r_{2m+2})$  satisfy the equation. Substituting  $(a, b, c) = (r_{2m} - k_m, r_{2m}, r_{2m} + k_m)$  into the equation and simplifying yields that  $r_{2m} = \sqrt{3k_m^2 + 1}$ . (This can be verified by substituting  $(a, b, c) = (r_{2m}, r_{2m} + k_m, 2r_{2m} + 3k_m)$  into the equation.) In fact, this condition is equivalent to these values of  $(a, b, c)$  satisfying the equation.

We have that

$$\begin{aligned} 3k_{m+1}^2 + 1 &= 3(r_{2m+2} - r_{2m+1})^2 + 1 \\ &= 3(r_{2m} + 2k_m)^2 + 1 \\ &= 3(r_{2m}^2 + 4k_m^2 + 4r_{2m}k_m) + 1 \\ &= 3(7k_m^2 + 1) + 12r_{2m}k_m + 1 = 21k_m^2 + 12r_{2m}k_m + 4 . \end{aligned}$$

Thus

$$\begin{aligned} r_{2m+2}^2 &= (2r_{2m} + 3k_m)^2 = 4r_{2m}^2 + 9k_m^2 + 12r_{2m}k_m \\ &= 12k_m^2 + 4 + 9k_m^2 + 12r_{2m}k_m = 3k_{m+1}^2 + 1 . \end{aligned}$$

This is the condition that

$$(a, b, c) = (r_{2m+2} - k_{m+1}, r_{2m+2}, r_{2m+2} + k_{m+1}) = (r_{2m+1}, r_{2m+2}, r_{2m+3})$$

and

$$(a, b, c) = (r_{2m+2}, r_{2m+2} + k_{m+1}, 2r_{2m+2} + 3k_{m+1}) = (r_{2m+2}, r_{2m+3}, r_{2m+4})$$

satisfy the equation. The result follows by induction.

*Solution 6.* [D. Rhee] Try for solutions of the form

$$(a, b, c) = \left( x, \frac{1}{2}(x + y), y \right)$$

where  $x$  and  $y$  are positive integers of the same parity. Plugging this into the equation and simplifying yields the equivalent equation

$$x^2 - 4xy + y^2 + 2 = 0 \iff x^2 - 4yx + (y^2 + 2) = 0 .$$

Suppose that  $z_1 = 1$ ,  $z_2 = 3$  and  $z_{n+1} = 4z_n - z_{n-1}$  for  $n \geq 1$ . Then, it can be shown by induction that  $z_{n+1} > z_n$  and  $z_n$  is odd for  $n \geq 1$ . We prove by induction that  $(x, y) = (z_n, z_{n+1})$  is a solution of the quadratic equation in  $x$  and  $y$ .

This is true for  $n = 1$ . Suppose that it holds for  $n \geq 1$ . Then  $z_n$  is a root of the quadratic equation  $x^2 - 4z_{n+1}x + (z_{n+1}^2 + 2) = 0$ . Since the sum of the roots is the integer  $4z_{n+1}$ , the second root is  $4z_{n+1} - z_n = z_{n+2}$  and the desired result holds because of the symmetry of the equation in  $x$  and  $y$ .

Thus, we obtain solutions  $(a, b, c) = (1, 2, 3), (3, 7, 11), (11, 26, 41), (41, 97, 153), \dots$  of the given equation.

*Solution 7.* [J. Park; A. Wice] As before, we note that if  $(a, b, c) = (u, v, w)$  satisfies the equation, then so also does  $(a, b, c) = (v, w, (vw + 1)/u)$ . Define a sequence  $\{x_n\}$  by  $x_1 = x_2 = x_3 = 1$  and

$$x_{n+3} = \frac{x_{n+2}x_{n+1} + 1}{x_n}$$

for  $n \geq 1$ . We prove by induction, that for each  $n$ , the following properties hold:

- (a)  $x_1, \dots, x_{n+3}$  are integers; in particular,  $x_n$  divides  $x_{n+2}x_{n+1} + 1$ ;
- (b)  $x_{n+1}$  divides  $x_n + x_{n+2}$ ;
- (c)  $x_{n+2}$  divides  $x_n x_{n+1} + 1$ ;
- (d)  $x_1 = x_2 = x_3 < x_4 < x_5 < \dots < x_n < x_{n+1} < x_{n+2}$ ;
- (e)  $(a, b, c) = (x_n, x_{n+1}, x_{n+2})$  satisfies the equation.

These hold for  $n = 1$ . Suppose they hold for  $n = k$ . Since

$$x_{k+2}x_{k+3} + 1 = \frac{x_{k+2}(x_{k+1}x_{k+2} + 1)}{x_k} + 1 = \frac{x_{k+2}^2 x_{k+1} + (x_k + x_{k+2})}{x_k},$$

from (b) we find that  $x_{k+1}$  divides the numerator. Since, by (a),  $x_k$  and  $x_{k+1}$  are coprime,  $x_{k+1}$  must divide  $x_{k+2}x_{k+3} + 1$ .

Now

$$x_{k+1} + x_{k+3} = \frac{(x_k x_{k+1} + 1) + x_{k+1} x_{k+2}}{x_k}.$$

By (a),  $x_k$  and  $x_{k+2}$  are coprime, and, by (c),  $x_{k+2}$  divides the numerator. Hence  $x_{k+2}$  divides  $x_{k+1} + x_{k+3}$ .

Since  $x_k = (x_{k+1}x_{k+2} + 1)/x_{k+3}$  is an integer,  $x_{k+3}$  divides  $x_{k+1}x_{k+2} + 1$ . Next,

$$x_{k+3} = \frac{x_{k+1}x_{k+2} + 1}{x_k} > \left(\frac{x_{k+1}}{x_k}\right)x_{k+2} > x_{k+2}.$$

Finally, the theory of the quadratic delivers (e) for  $n = k + 1$ . The result follows.

336. Let  $ABCD$  be a parallelogram with centre  $O$ . Points  $M$  and  $N$  are the respective midpoints of  $BO$  and  $CD$ . Prove that the triangles  $ABC$  and  $AMN$  are similar if and only if  $ABCD$  is a square.

*Comment.* Implicit in the problem, but what should have been stated, is that the similarity intended is in the order of the vertices as given, *i.e.*,  $AB : BC : CA = AM : MN : NA$ . In the first solution, other possible orderings of the vertices in the similarity are considered (in which case, the result becomes false); in the remaining solutions, the restricted sense of the similarity is discussed.

*Solution 1.* Let  $P$  be the midpoint of  $CN$ . Observe that

$$[AMD] = \frac{3}{4}[ABD] = \frac{3}{4}[ABC]$$

and

$$[AMC] = \frac{1}{2}[ABC].$$

As  $N$  is the midpoint of  $CD$ , the area of triangle  $AMN$  is the average of these two (why?), so that

$$[AMN] = \frac{5}{8}[ABC] .$$

Thus, in any case, we have the ratio of the areas of the two triangles, so, if they are similar, we know exactly what the factor of similarity must be.

Let  $|AB| = |CD| = 2a$ ,  $|AD| = |BC| = 2b$ ,  $|AC| = 2c$  and  $|BD| = 2d$ . Recall, that if  $UVW$  is a triangle with sides  $2u$ ,  $2v$ ,  $2w$  opposite the respective vertices  $U, V, W$  and  $m$  is the length of the median from  $U$ , then

$$m^2 = 2v^2 + 2w^2 - u^2 .$$

Since  $AN$  is a median of triangle  $ACD$ , with sides  $2a$ ,  $2b$ ,  $2c$ ,

$$|AN|^2 = 2c^2 + 2b^2 - a^2 .$$

Since  $AM$  is a median of triangle  $ABO$ , with sides  $2a$ ,  $c$ ,  $d$ ,

$$|AM|^2 = 2a^2 + (c^2/2) - (d^2/4) .$$

Since  $CM$  is a median of triangle  $CBO$ , with sides  $2b$ ,  $c$ ,  $d$ ,

$$|CM|^2 = 2b^2 + (c^2/2) - (d^2/4) .$$

Since  $MN$  is a median of triangle  $MCD$  with sides  $|CM|$ ,  $3d/2$ ,  $2a$ ,

$$\begin{aligned} |MN|^2 &= \frac{|CM|^2}{2} + \frac{1}{2} \left( \frac{3d}{2} \right)^2 - a^2 \\ &= b^2 + \frac{c^2}{4} - \frac{d^2}{8} + \frac{9d^2}{8} - a^2 = b^2 + \frac{c^2}{4} + d^2 - a^2 . \end{aligned}$$

Suppose that triangle  $ABC$  and  $AMN$  are similar (with any ordering of the vertices). Then

$$\begin{aligned} 8(|AM|^2 + |AN|^2 + |MN|^2) &= 5(|AB|^2 + |AC|^2 + |BC|^2) \\ \iff 24b^2 + 22c^2 + 6d^2 &= 20a^2 + 20b^2 + 20c^2 \\ \iff 2b^2 + c^2 + 3d^2 &= 10a^2 . \end{aligned} \tag{1}$$

Also, for the parallelogram with sides  $2a$ ,  $2b$  and diagonals  $2c$ ,  $2d$ , we have that

$$c^2 + d^2 = 2(a^2 + b^2) . \tag{2}$$

Equations (1) and (2) together yield  $d^2 = 4a^2 - 2b^2$  and  $c^2 = 4b^2 - 2a^2$ . We now have that

$$\frac{|AN|^2}{|AC|^2} = \frac{2c^2 + 2b^2 - a^2}{4c^2} = \frac{4c^2 + c^2}{8c^2} = \frac{5}{8}$$

which is the desired ratio. Thus, when triangles  $ABC$  and  $AMN$  are similar, the sides  $AN$  and  $AC$  are in the correct ratio, and so must correspond in the similarity. (There is a little more to this than meets the eye. This is obvious when triangle  $AMN$  is scalene; if triangle  $AMN$  were isosceles or equilateral, then it is self-congruent and the similarity can be set up to make  $AN$  and  $BC$  correspond.)

We also have that

$$\frac{|MN|^2}{|AB|^2} = \frac{4b^2 + c^2 + 4d^2 - 4a^2}{16a^2} = \frac{4b^2 + 4b^2 - 2a^2 + 16a^2 - 8b^2 - 4a^2}{16a^2} = \frac{10a^2}{16a^2} = \frac{5}{8}$$

and

$$\frac{|AM|^2}{|BC|^2} = \frac{8a^2 + 2c^2 - d^2}{16b^2} = \frac{8a^2 + 8b^2 - 4a^2 - 4a^2 + 2b^2}{16b^2} = \frac{10b^2}{16b^2} = \frac{5}{8}.$$

Thus, if triangle  $ABC$  and  $AMN$  are similar, then

$$AB : BC : AC = MN : AM : AN$$

and  $\cos \angle ABC = \cos \angle ADC = 3(a^2 - b^2)/(2ab)$ . The condition that the cosine has absolute value not exceeding 1 yields that  $(\sqrt{10} - 1)b \leq 3a \leq (\sqrt{10} + 1)b$ .

Now we look at the converse. Suppose that  $ABCD$  is a parallelogram with sides  $2a$  and  $2b$  as indicated above and that  $\cos \angle ABC = \cos \angle ADC = 3(a^2 - b^2)/(2ab)$ . Then, the lengths  $2c$  and  $2d$  of the diagonals are given by

$$4c^2 = |AC|^2 = 4a^2 + 4b^2 - 12a^2 + 12b^2 = 16b^2 - 8a^2 = 8(2b^2 - a^2)$$

and

$$4a^2 = |BD|^2 = 4a^2 + 4b^2 + 12a^2 - 12b^2 = 16a^2 - 8b^2 = 8(2a^2 - b^2)$$

so that  $|AC| = 2\sqrt{4b^2 - 2a^2}$  and  $|BD| = 2\sqrt{4a^2 - 2b^2}$ . Using the formula for the lengths of the medians, we have that

$$\begin{aligned} |AN|^2 &= 2(4b^2 - 2a^2) + 2b^2 - a^2 = 5(2b^2 - a^2) \\ |AM|^2 &= 2a^2 + \frac{4b^2 - 2a^2}{2} - \frac{4a^2 - 2b^2}{4} = \frac{5b^2}{2} \\ |MN|^2 &= b^2 + \left(\frac{4b^2 - 2a^2}{4}\right) + (4a^2 - 2b^2) - a^2 = \frac{5a^2}{2}. \end{aligned}$$

Thus

$$\frac{|AN|^2}{|AC|^2} = \frac{|AM|^2}{|BC|^2} = \frac{|MN|^2}{|AB|^2} = \frac{5}{8}$$

and triangles  $ABC$  and  $AMN$  are similar with  $AB : BC : AC = MN : AM : AN$ .

If triangles  $ABC$  and  $AMN$  are similar with  $AB : BC : AC = AM : MN : AN$ , then we must have that  $AB = BC$ , *i.e.*  $a = b$  and so  $\angle ABC = \angle ADC = 90^\circ$ , *i.e.*,  $ABCD$  is a square.

*Comment.* A direct geometric argument that triangles  $ABC$  and  $AMN$  are similar when  $ABCD$  is a square can be executed as follows. By reflection about an axis through  $M$  parallel to  $BC$ , we see that  $MN = MC$ . By reflection about axis  $BD$ , we see that  $AM = CM$  and  $\angle BAM = \angle BCM$ . Hence  $AM = MN$  and  $\angle BAM = \angle BCM = \angle CMP = \angle NMP$  where  $P$  is the midpoint of  $CN$ .

Consider a rotation with centre  $A$  through an angle  $BAM$  followed by a dilation of factor  $|AM|/|AB|$ . This transformation takes  $A \rightarrow A$ ,  $B \rightarrow M$  and the line  $BC$  to a line through  $M$  making an angle  $\angle BAM$  with  $BC$ ; the image line must be  $MN$ . Since  $AB = BC$  and  $AM = MN$ ,  $C \rightarrow N$ . Thus, the image of triangle  $ABC$  is triangle  $AMN$  and the two triangles are similar.

An alternative argument uses the fact that triangles  $AOM$  and  $ADN$  are similar, so that  $\angle AMO = \angle AND$  and  $AMND$  is concyclic. From this follows  $\angle AMN = 180^\circ - \angle ADN = 90^\circ$ .

*Solution 2.* [S. Eastwood; Y. Zhao] If  $ABCD$  is a square, we can take  $B$  at 1 and  $D$  at  $i$ , whereupon  $M$  is at  $(3+i)/4$  and  $N$  at  $(1+2i)/2$ . The vector  $\overrightarrow{MN}$  is represented by

$$\frac{1+2i}{2} - \frac{3+i}{4} = \frac{i(3+i)}{4}.$$

Since  $\overrightarrow{AM}$  is represented by  $(3+i)/4$ ,  $MN$  is obtained from  $AM$  by a  $90^\circ$  rotation about  $M$  so that  $MN = AM$  and  $\angle AMN = 90^\circ$ . Hence the triangles  $ABC$  and  $AMN$  are similar.

For the converse, let the parallelogram  $ABCD$  be represented in the complex plane with  $A$  at 0,  $B$  at  $z$  and  $D$  at  $w$ , Then  $C$  is at  $z+w$ ,  $M$  is at  $(3z+w)/4$  and  $N$  is at  $\frac{1}{2}(z+w) + \frac{1}{2}w = (z+2w)/2$ . Suppose

that triangles  $ABC$  and  $AMN$  are similar. Then, since  $\angle AMN = \angle ABC$  and  $AM : AN = AB : AC$ , we must have that

$$\begin{aligned} \frac{\frac{1}{4}(3z+w)}{\frac{1}{2}(z+2w)} &= \frac{z}{z+w} \iff (3z+w)(z+w) = 2z(z+2w) \\ \iff 3z^2 + 4zw + w^2 &= 2z^2 + 4zw \iff z^2 + w^2 = 0 \iff z = \pm iw . \end{aligned}$$

Thus  $AD$  is obtained from  $AB$  by a  $90^\circ$  rotation and  $ABCD$  is a square.

*Comment.* Strictly speaking, the reasoning in the last paragraph is reversible, so we could use it for the proof in both directions. However, the particularization may aid in understanding what is going on.

*Solution 3.* [F. Barekat] Let  $ABCD$  be a square. Then  $\triangle ADC \sim \triangle AOB$  so that  $AB : AC = OB : DC = MB : NC$ . Since also  $\angle ABM = \angle ACN = 45^\circ$ ,  $\triangle AMB \sim \triangle ANC$ . Therefore  $AM : AB = AN : AC$ . Also

$$\angle MAB = \angle NAC \implies \angle CAB = \angle CAM + \angle MAB = \angle CAM + \angle NAC = \angle NAM .$$

Therefore  $\triangle AMN \sim \triangle ABC$ .

On the other hand, suppose that  $\triangle AMN \sim \triangle ABC$ . Then  $AM : AN = AB : AC$  and

$$\angle NAC = \angle NAM - \angle CAM = \angle CAB - \angle CAM = \angle MAB ,$$

whence  $\triangle AMB = \triangle ANC$ .

Therefore,  $AB : AC = MB : NC = BO : DC$ . Since also  $\angle ABO = \angle ACD$ ,  $\triangle ABO \sim \triangle ACD$ , so that  $\angle ABO = \angle ACD$  and  $\angle AOB = \angle ADC$ . Because  $\angle ABO = \angle ACD$ ,  $ABCD$  is a concyclic quadrilateral and  $\angle DAB + \angle DCB = 180^\circ$ . Since  $ABCD$  is a parallelogram,  $\angle DAB = \angle DCB = 90^\circ$  and  $ABCD$  is a rectangle. Thus  $\angle AOB = \angle ADC = 90^\circ$ , from which it can be deduced that  $ABCD$  is a square.

*Solution 4.* [B. Deng] Let  $ABCD$  be a square. Since  $MN = MC = MA$ , triangles  $MNC$  and  $AMN$  are isosceles. We have that

$$AM^2 + MN^2 = 2AM^2 = 2(AO^2 + OM^2) = (5/4)AB^2$$

and

$$AN^2 = AD^2 + DN^2 = (5/4)AB^2 = AM^2 + MN^2 ,$$

whence  $\angle AMN = 90^\circ$  and  $\triangle AMN \sim \triangle ABC$ .

Suppose on the other hand that  $\triangle AMN \sim \triangle ABC$ . Then  $AN : AM = AC : AB$  and  $\angle NAM = \angle CAB$  together imply that

$$\angle NAC = \angle MAB \implies \triangle NAC \sim \triangle MAB \implies \angle NCA = \angle ABM .$$

But  $\angle NCA = \angle OAB \implies \angle OAB = \angle OBA \implies ABCD$  is a rectangle.

Let  $H$  be the foot of the perpendicular from  $M$  to  $CN$ . Then  $ON$ ,  $MH$  and  $BC$  are all parallel and  $M$  is the midpoint of  $OB$ . Hence  $H$  is the midpoint of  $CN$  and  $\triangle HMN \cong \triangle HMC$ . Therefore  $MC = MN$ . Now, from the median length formula,

$$AM^2 = \frac{1}{4}(2BA^2 + 2AO^2 - BO^2) = \frac{1}{4}(2BA^2 + AO^2)$$

and

$$CM^2 = \frac{1}{4}(2BC^2 + 2CO^2 - BO^2) = \frac{1}{4}(2BC^2 + CO^2)$$

whence

$$(2BA^2 + AO^2) : (2BC^2 + CO^2) = AM^2 : CM^2 = AM^2 : MN^2 = BA^2 : BC^2$$

so that

$$2BA^2 \cdot BC^2 + AO^2 \cdot BC^2 = 2BA^2 \cdot BC^2 + BA^2 \cdot CO^2 \Rightarrow BC^2 = BA^2 \Rightarrow BC = BA$$

and  $ABCD$  is a square.

337. Let  $a, b, c$  be three real numbers for which  $0 \leq c \leq b \leq a \leq 1$  and let  $w$  be a complex root of the polynomial  $z^3 + az^2 + bz + c$ . Must  $|w| \leq 1$ ?

*Solution 1.* [L. Fei] Let  $w = u + iv$ ,  $\bar{w} = u - iv$  and  $r$  be the three roots. Then  $a = -2u - r$ ,  $b = |w|^2 + 2ur$  and  $c = -|w|^2 r$ . Substituting for  $b$ ,  $ac$  and  $c$ , we find that

$$|w|^6 - b|w|^4 + ac|w|^2 - c^2 = 0$$

so that  $|w|^2$  is a nonnegative real root of the cubic polynomial  $q(t) = t^3 - bt^2 + act - c^2 = (t - b)t^2 + c(at - c)$ . Suppose that  $t > 1$ , then  $t - b$  and  $at - c$  are both positive, so that  $q(t) > 0$ . Hence  $|w| \leq 1$ .

*Solution 2.* [P. Shi; Y. Zhao]

$$\begin{aligned} 0 &= (1 - w)(w^3 + aw^2 + bw + c) \\ &= -w^4 + (1 - a)w^3 + (a - b)w^2 + (b - c)w + c \\ \implies w^4 &= (1 - a)w^3 + (a - b)w^2 + (b - c)w + c \\ \implies |w|^4 &\leq (1 - a)|w|^3 + (a - b)|w|^2 + (b - c)|w| + c. \end{aligned}$$

Suppose, if possible, that  $|w| > 1$ . Then

$$|w|^4 \leq |w|^3[(1 - a) + (a - b) + (b - c) + c] = |w|^3$$

which implies that  $|w| \leq 1$  and yields a contradiction. Hence  $|w| \leq 1$ .

*Solution 3.* There must be one real solution  $v$ . If  $v = 0$ , then the remaining roots  $w$  and  $\bar{w}$ , the complex conjugate of  $w$ , must satisfy the quadratic equation  $z^2 + az + b = 0$ . Therefore  $|w|^2 = w\bar{w} = b \leq 1$  and the result follows. Henceforth, let  $v \neq 0$ .

Observe that

$$f(-1) = -1 + a - b + c = -(1 - a) - (b - c) \leq 0$$

and that

$$f(-c) = -c^3 + ac^2 - bc + c \geq -c^3 + c^3 - bc + c = c(1 - b) \geq 0,$$

so that  $-1 \leq v \leq -c$ . The polynomial can be factored as

$$(z - v)(z^2 + pz + q)$$

where  $c = -qv$  so that  $q = c/(-v) \leq 1$ . But  $q = w\bar{w}$ , and the result again follows.

338. A *triangular triple*  $(a, b, c)$  is a set of three positive integers for which  $T(a) + T(b) = T(c)$ . Determine the smallest triangular number of the form  $a + b + c$  where  $(a, b, c)$  is a triangular triple. (*Optional investigations:* Are there infinitely many such triangular numbers  $a + b + c$ ? Is it possible for the three numbers of a triangular triple to each be triangular?)

*Solution 1.* [F. Barekat] For each nonnegative integer  $k$ , the triple

$$(a, b, c) = (3k + 2, 4k + 2, 5k + 3)$$

satisfies  $T(a) + T(b) = T(c)$ . Indeed,

$$(3k + 2)(3k + 3) + (4k + 2)(4k + 3) = (9k^2 + 15k + 6) + (16k^2 + 20k + 6) = 25k^2 + 35k + 12 = (5k + 3)(5k + 4).$$

We have that  $a+b+c = 12k+7$ , so we need to determine whether there are triangular numbers congruent to 7 modulo 12. Suppose that  $T(x)$  is such. Then  $x(x+1)$  must be congruent to 14 modulo 24. Now, modulo 24,

$$x^2 + x - 14 \equiv x^2 + x - 110 = (x-10)(x+11) \equiv (x-10)(x-13)$$

so  $T(x)$  leaves a remainder 7 upon division by 12 if and only if  $x = 10 + 24m$  or  $x = 13 + 24n$  for some nonnegative integers  $m$  and  $n$ . This yields

$$k = 4 + 21m + 24m^2$$

and

$$k = 7 + 27n + 24n^2$$

for nonnegative integers  $m$  and  $n$ . The smallest triples according to these formula are  $(a, b, c) = (14, 18, 23)$  and  $(a, b, c) = (23, 30, 38)$ , with the respective values of  $a+b+c$  equal to  $55 = T(10)$  and  $91 = T(13)$ . However, it may be that there are others that do not come under this set of formulae.

The equation  $a(a+1) + b(b+1) = c(c+1)$  is equivalent to  $(2a+1)^2 + (2b+1)^2 = (2c+1)^2 + 1$ . It is straightforward to check whether numbers of the form  $n^2 + 1$  with  $n$  odd is the sum of two odd squares. We get the following triples with sums not exceeding  $T(10) = 55$ :

$$(2, 2, 3), (3, 5, 6), (5, 6, 8), (4, 9, 10), (6, 9, 11), (8, 10, 13), (5, 14, 15),$$

$$(9, 13, 16), (11, 14, 18), (6, 20, 21), (12, 17, 21), (9, 21, 23), (11, 20, 23), (14, 18, 23) .$$

The entries of none except the last of these sum to a triangular number.

*Solution 2.* [Y. Zhao] Let  $(a, b, c)$  be a triangular triple and let  $n = a + b + c$ . Now

$$\begin{aligned} 2T(a) + 2T(b) - 2T(n-a-b) &= a^2 + a + b^2 + b - (n-a-b)^2 - (n-a-b) \\ &= (n+1)(n+2) - 2(n+1-a)(n+1-b) . \end{aligned}$$

Thus,  $(a, b, n-a-b)$  is a triangular triple if and only if  $(n+1)(n+2) = 2(n+1-a)(n+1-b)$ . In this case, neither  $n+1$  nor  $n+2$  can be prime, as each factor on the right side is strictly less than either of them. When 1 and 2 are added to each of the first nine triangular numbers, 1, 3, 6, 10, 15, 21, 28, 36, 45, we get at least one prime. Hence  $n \geq 55$ . It can be checked that  $(a, b, c) = (14, 18, 23)$  is a triangular triple.

We claim that, for every nonnegative integer  $k$ ,  $T(24k+10) = a+b+c$  for some triangular triple  $(a, b, c)$ . Observe that, with  $n = T(24k+10) = 288k^2 + 252k + 55$ ,

$$\begin{aligned} (n+1)(n+2) &= (T(24k+10) + 1)(T(24k+10) + 2) \\ &= (288k^2 + 252k + 56)(288k^2 + 252k + 57) \\ &= 12(72k^2 + 63k + 14)(96k^2 + 84k + 19) . \end{aligned}$$

Select  $a$  and  $b$  so that

$$n+1-a = 3(72k^2 + 63k + 14) = 216k^2 + 189k + 42$$

and

$$n+1-b = 2(96k^2 + 84k + 19) = 192k^2 + 168k + 38 .$$

Let  $c = n - a - b$ . Thus,

$$(a, b, c) = (72k^2 + 63k + 14, 96k^2 + 84k + 18, 120k^2 + 105k + 23) .$$

From the first part of the solution, we see that  $(a, b, c)$  is a triangular triple.

We observe that  $(a, b, c) = (T(59), T(77), T(83)) = (1770, 3003, 3486)$  is a triangular triple.

Check:

$$\begin{aligned} 1770 \times 1771 + 3003 \times 3004 &= (2 \times 3 \times 7 \times 11)(295 \times 23 + 13 \times 1502) \\ &= (2 \times 3 \times 7 \times 11)(26311) = 2 \times 3 \times 7 \times 11 \times 83 \times 317 \\ &= (2 \times 3 \times 7 \times 83)(11 \times 317) = 3486 \times 3487 . \end{aligned}$$

However, the sum of the numbers is  $1770 + 3003 + 3486 = 8259$ , which exceeds  $8256 = T(128)$  by only 3.

*Comment.* David Rhee observed by drawing diagrams that for any triangular triple  $(a, b, c)$ ,  $T(a+b-c) = (c-b)(c-a)$ . This can be verified directly. Checking increasing values of  $a+b-c$  and factoring  $T(a+b-c)$  led to the smallest triangular triple.